**Stochastics II**
WS 2017/2018
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# Exercise sheet 11 (total – 20 points) till January 24, 2018

## Exercise 11-1 (3 points)

Let $\{A_n\}$ be a partition of $\Omega$ such that $A_i \cap A_j = \varnothing$, $i \neq j$ and $\mathbf{P}(A_i) > 0$. Let $\mathcal{F} = \sigma(A_n : n \in \mathbb{N})$. Prove that for any $X \in L^1(\Omega, \mathcal{A}, \mathbf{P})$,

$$\mathbf{E}[X|\mathcal{F}] = \sum_{n \geq 0} \mathbb{I}_{A_n} \frac{\mathbf{E}[X\mathbb{I}_{A_n}]}{\mathbf{P}(A_n)}$$

## Exercise 11-2 (4 points)

Let $X$ and $Y$ be random variables on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. The conditional variance is defined by $\mathbf{Var}(Y|X) = \mathbf{E}((Y - \mathbf{E}(Y|X))^2|X)$. Show that:

1. (2 points) $\mathbf{E}Y = \mathbf{E}(\mathbf{E}(Y|X))$,
2. (2 points) $\mathbf{Var}Y = \mathbf{E}(\mathbf{Var}(Y|X)) + \mathbf{Var}(\mathbf{E}(Y|X))$.

## Exercise 11-3 (7 points)

For a stopping time $\tau$ define the stopped $\sigma$-algebra $\mathcal{F}_\tau$ as follows:

$$\mathcal{F}_\tau = \{B \in \mathcal{F} : B \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for arbitrary } t \geq 0\}.$$

Let now $\sigma$ and $\tau$ be stopping times w.r.t. the filtration $\{\mathcal{F}_t, t \geq 0\}$.

1. (3 points) Prove that $\mathcal{F}_\tau$ is a $\sigma$-algebra.
2. (2 points) Show that $A \cap \{\sigma \leq \tau\} \in \mathcal{F}_\tau \ \forall A \in \mathcal{F}_\sigma$.
3. (2 points) Prove that $\mathcal{F}_{\min\{\sigma,\tau\}} = \mathcal{F}_\sigma \cap \mathcal{F}_\tau$.

## Exercise 11-4 (3 points)

Let $\xi_1, \xi_2, \ldots$ be a sequence of independent $N(0,1)-$distributed random variables. Let $S_n = \xi_1 + \cdots + \xi_n$. Prove that the sequence $\{X_n, n \geq 1\}$ given by

$$X_n = \frac{1}{\sqrt{n+1}} \exp\left(\frac{S_n^2}{2(n+1)}\right)$$

is a martingale w.r.t. the filtration $\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n), n \geq 1$.

## Exercise 11-5 (3 points)

Let $\xi_1, \xi_2, \ldots$ be a sequence of random variables with finite means and satisfying

$$\mathbf{E}(\xi_{n+1}|\xi_0, \ldots, \xi_n) = a\xi_n + b\xi_{n-1}, n \geq 1,$$

where $0 < a, b < 1$ and $a + b = 1$. Find a value of $\alpha$ for which $X_n = \alpha\xi_n + \xi_{n-1}, n \geq 1.$ defines a martingale with respect to the filtration generated by sequence $\{\xi_n, n \geq 1\}$.