Exercise sheet 3 (total – 16 points) till November 15, 2017

Exercise 3-1 (3 points)

Prove that if \( \{X(t), t \in \mathbb{R}_+\} \) is a mean square continuous process with càdlàg trajectories then the process \( Y(t) = \int_0^t X(s) \, ds, t \geq 0 \) is mean square differentiable and its derivative in mean square sense \( Y'(t) = X(t), t \in \mathbb{R}_+ \). Find mean value and covariance functions of \( Y \).

Exercise 3-2 (3 points)

A radiation measuring device accumulates radiation with the rate that equals \( a \) Röntgen per hour, right up to the failing moment. Let \( X(t) \) be the reading at point of time \( t \geq 0 \). Find the mean and covariance functions for the process \( X \) if \( X(0) = 0 \), the failing moment has distribution function \( F \), and after the failure the measuring device is fixed (a) at zero point; (b) at the last reading.

Exercise 3-3 (3 points)

Let \( \{X(t), t \geq 0\} \) be a stochastic process with independent increments for all \( t \in \mathbb{R}_+ \) \( \mathbb{E}|X(t)|^2 < \infty \). Prove that its covariance function is equal to \( K_X(t, s) = F(t \wedge s), t, s \in \mathbb{R}_+ \), where \( F \) is some non-decreasing function.

Exercise 3-4 (4 points)

Let \( \{X(t), t \in \mathbb{R}_+\} \) be a real-valued process with stationary and independent increments, and \( \text{Var}(X(1) - X(0)) > 0 \). Prove that the processes \( X \) and \( Y(t) := X(t+1) - X(t), t \geq 0 \) are mean square continuous but not mean square differentiable.

Hint: You may use the solution of Cauchy’s functional equation \( f(x+y) = f(x) + f(y) \). If \( f \) is bounded on any interval then the solution has a form \( f(x) = cx \), where \( c \in \mathbb{R} \).

Exercise 3-5 (3 points)

Let the mean and covariance functions of the process \( X = \{X(t), t \geq 0\} \) be equal to \( \mu_X(t) = \frac{t}{1+t}; t \geq 0, K_X(t, s) = e^{-|t-s|^2}, t, s \geq 0 \). Prove that \( X \) is mean square differentiable. Let \( X' \) be its derivative defined in \( L_2 \) sense. Please find:

1. (1 point) its variogram \( \gamma(t, s), t, s \geq 0 \).

2. (2 points) \( \mathbb{E}[X(t)X'(s)], t, s \geq 0 \).