A Harris-Kesten theorem for confetti percolation

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June 19, 2013

Abstract

Percolation properties of the dead leaves model, also known as confetti percolation, are considered. More precisely, we prove that the critical probability for confetti percolation with square-shaped leaves is $1/2$. This result is related to a question of Benjamini and Schramm concerning disk-shaped leaves and can be seen as a variant of the Harris-Kesten theorem for bond percolation. The proof is based on techniques developed by Bollobás and Riordan to determine the critical probability for Voronoi and Johnson-Mehl percolation.

1 Introduction

In recent years much progress has been made to determine the critical value of various two-dimensional configuration models in percolation theory and statistical mechanics that exhibit a more complex dependency structure than classical Bernoulli percolation, see e.g. [1, 2, 5, 7, 20, 21]. In this paper we consider a spatial percolation process based on the so-called *dead leaves model* which is popular in stochastic geometry, see [15]. This model describes the coloring of $\mathbb{R}^2$ observed when covering the plane by black and white leaves according to a space-time Poisson process. A precise definition is given in Section 2. In percolation literature this process is also known under the name of *confetti percolation* and Benjamini and Schramm have conjectured in [4, Problem 5] that $p_c = 1/2$ for the case of disk-shaped leaves. We will show how the techniques from [5] and [8] can be used to prove that the critical probability for square-shaped confetti percolation is precisely $1/2$.

Let us give a rough outline of the main ideas. As in Bernoulli percolation the part $p_c \geq 1/2$ follows from Zhang's elegant proof of $\theta(1/2) = 0$. Indeed to apply his method we need to check positive correlation of black-increasing events (this is standard, see e.g. [6, 17]) and the uniqueness of the infinite cluster. For the latter part one may use the geometric method of [12] which has the advantage that no additional discretization is needed.

Proving $p_c \leq 1/2$ is less canonical and we follow the framework developed by Bollobás and Riordan in [5, 8]. Although one might hope at first that the problem of confetti percolation is considerably simpler than the problem of Voronoi percolation (for instance since the range of dependence is finite), still a fair amount of work needs to be done to resolve discretization issues.

In Section 2 we give detailed definitions of planar confetti-type percolation models and introduce further useful notation. The proof of $\theta(1/2) = 0$ is provided in Section 3. The proof of $p_c \leq 1/2$ can be subdivided into two steps that are presented in Sections 4 and 5. In Section 4 we show that the assumptions of the general RSW-type theorem of Bollobás and Riordan (see e.g. [5]) are satisfied in the confetti model. Together with the sharp-threshold theorem [9,

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Lemma 1] (which itself is a variant of the sharp-threshold result [11, Theorem 2.1]) this result is used to complete the proof of $p_c \le 1/2$ in Section 5 (except for the proof of a rather complicated elementary geometric lemma which is postponed to Section 6).

We believe that a similar approach could be used to consider disk-shaped leaves, although starting from the current article this seems not completely straightforward. Furthermore, we would be very interested in considering generalizations using different shapes of sufficiently symmetric leaves or where the leaves are rotated at random. A very interesting idea to make the current argument less dependent on the specific shape of the leaf was suggested by an anonymous referee in Remark 2. However, this approach depends on a non-trivial estimate on the tail behavior of the number of visible leaves of the dead leaves model in a bounded sampling window.

2 Notations and basic definitions

For $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space and $\{A_s\}_{s\in[0,\infty)}$ a family of events, we say that A_s holds with high probability (short whp) if $\mathbb{P}(A_s) \to 1$ as $s \to \infty$. For $\rho > 0$, $u \in \mathbb{R}^2$ we denote by $Q_\rho(u) = u + \rho[-1/2, 1/2]^2$ the square of side length ρ centered at u and write $Q(u) = Q_1(u)$.

For $\varphi = \{x_n\}_{n\ge1} = \{(z_n, t_n, \sigma_n)\}_{n\ge1} \subset \mathbb{R}^2 \times [-1, \infty) \times \{\pm1\}$ locally finite we will often use the notation $y_n = (z_n, t_n)$ to denote the space-time coordinates of the element (z_n, t_n, σ_n). Furthermore, we denote by $A \subset \mathbb{R}^2$ a leaf which is compact, path-connected, contains the origin o in its interior, satisfies $A = -A$ and whose boundary is ν_1-rectifiable, where ν_1 denotes the one-dimensional Hausdorff measure. For a precise definition of ν_1-rectifiability, see e.g. [18, Chapter 14]. Then the dead leaves process describes a sequence of colored leaves falling onto the plane according to the space-time process φ. To be more precise, at time t_n a z_n-centered leaf appears that is of shape A and color σ_n (say black if $\sigma_n = 1$ and white if $\sigma_n = -1$). This yields a coloring of the plane by defining the color of a point $u \in \mathbb{R}^2$ to be the color of the first leaf covering u (or undefined if there is either no such leaf or if the color is non-unique). Note that we observe the configuration of leaves *from below* in order to obtain a static coloring of the entire plane.

To each such locally finite φ we can associate a function $height_\varphi : \mathbb{R}^2 \to \mathbb{R}$ mapping a point $u \in \mathbb{R}^2$ to the time the leaf visible at u had arrived. Formally, we put

$$height_\varphi(u) = \min(\{t_m : x_m = (z_m, t_m, \sigma_m) \in \varphi, u - z_m \in A\})$$

if this set is non-empty and all points x_m assuming this minimum are of the same color, and $height_\varphi(u) = -2$ otherwise. To each point of $\mathbb{R}^2$ we assign a number from $\{0, \pm1\}$ according to the function $\psi_\varphi : \mathbb{R}^2 \to \{0, \pm1\}$ defined by $\psi(u) = \psi_\varphi(u) = \sigma_n$, where the index n is chosen so that $height_\varphi(u) = t_n$ and $\psi_\varphi(u) = 0$ if $height_\varphi(u) = -2$. Sometimes we also write *ψ-black* to describe the attribute of being black in the coloring ψ. For instance, the connected components of $\psi^{-1}(1)$ are called *ψ-black connected components*. Furthermore, for colorings $\psi_1, \psi_2 : \mathbb{R}^2 \to \{\pm1\}$ we say ψ_1 *black-dominates* ψ_2 if $\psi_1(x) \ge \psi_2(x)$ holds for all $x \in \mathbb{R}^2$.

One can add a probabilistic flavor to this model by replacing the locally finite set φ by an independently $\{\pm1\}$-marked homogeneous Poisson point process $X \subset \mathbb{R}^2 \times [0, \infty)$. Furthermore, we write $p = \mathbb{P}(\sigma_n = 1)$ for the probability that a fixed leaf is colored black in X. It is easy to see that in the coloring ψ_X with probability 1 all points of $\mathbb{R}^2$ are colored either black or white. We write $\theta(p, A)$ for the probability that the origin is contained in an unbounded ψ-black component. In case that the leaf $A = Q(o)$ is the unit square centered at the origin we also write $\theta(p)$ for $\theta(p, A)$. Furthermore, we use the standard definition of *critical probability for percolation*, namely $p_{c,A} = \inf\{p > 0 : \theta(p, A) > 0\}$ and write p_c in the special case $A = Q(o)$. Note also that it is straightforward to extend the definition of the confetti process so as to allow leaves of randomly varying size and shape.

3 $\theta(1/2) = 0$

3.1 Harris's inequality

The basic statement of Harris's inequality is that black-increasing events are positively correlated. Although the classical Harris inequality is stated in a lattice setting, some extra technical work makes it possible to adapt it to the situation of confetti percolation. Similar technical adjustments are explained in detail by Bollobás and Riordan in [6] for the case of Voronoi percolation and we follow their presentation.

An event E defined in terms of two independent Poisson processes (X^+, X^-) is called *black-increasing* if for every configuration $\omega_1 = (X_1^+, X_1^-)$ and $\omega_2 = (X_2^+, X_2^-)$ with $X_1^+ \subset X_2^+$, $X_1^- \supset X_2^-$ and $\omega_1 \in E$ we have $\omega_2 \in E$. Similarly, one defines the notion of black-increasing functions.

First, let us consider Harris's inequality when only the process X^+ is involved. Let $(\Omega_1, \mathcal{A}_1, \mathbb{P}_1)$ be the canonical probability space of the random variable X^+. Denote by Σ_k the σ-algebra generated by the following information. Set $n = 2^k$ and divide $[-n, n]^2 \times [0, n]$ in $4n^6$ cubes of side length $1/n$. Decide for each of them whether it contains at least one point of X^+ or not. The local finiteness of X^+ then implies that $\{\Sigma_k\}_{k \ge 1}$ forms a filtration of $(\Omega_1, \mathcal{A}_1, \mathbb{P}_1)$.

Let g_1, g_2 be increasing, bounded and measurable functions. As $\mathbb{E}(g_i|\Sigma_k)$ is an increasing function on the discrete product space determined by Σ_k, the lattice version of Harris's inequality implies $\mathbb{E}(\mathbb{E}(g_1|\Sigma_k)\mathbb{E}(g_2|\Sigma_k)) \ge \mathbb{E}(g_1)\mathbb{E}(g_2)$. As g_1, g_2 are bounded, the martingale convergence theorem implies $\mathbb{E}(g_i|\Sigma_k) \xrightarrow{k\to\infty} g_i$ and dominated convergence yields $\mathbb{E}(g_1 g_2) \ge \mathbb{E}(g_1)\mathbb{E}(g_2)$. Using these notions we can state a version of Harris's inequality that can be applied to confetti percolation.

Lemma 1. *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be the canonical product space of (X^+, X^-) and let $A, B \in \mathcal{A}$ be two increasing events. Then $\mathbb{P}(A \cap B) \ge \mathbb{P}(A)\mathbb{P}(B)$.*

Proof. Let $f_1 = 1_A$ and $f_2 = 1_B$. Fixing X^-, we obtain $\mathbb{E}(f_1 f_2|X^-) \ge \mathbb{E}(f_1|X^-)\mathbb{E}(f_2|X^-)$ and taking expectations yields $\mathbb{E}(f_1 f_2) \ge \mathbb{E}(\mathbb{E}(f_1|X^-)\mathbb{E}(f_2|X^-))$. Now observe that $g_i = \mathbb{E}(f_i|X^-)$ is decreasing in X^-. This yields $\mathbb{E}(\mathbb{E}(f_1|X^-)\mathbb{E}(f_2|X^-)) \ge \mathbb{E}(f_1)\mathbb{E}(f_2)$. Combining these two inequalities completes the proof of the lemma. □

In particular, if $X \subset \mathbb{R}^2 \times [0, \infty) \times \{\pm 1\}$ is an independently $\{\pm 1\}$-marked homogeneous Poisson point process, then Lemma 1 can be applied to $X^+ = \{y_n : \sigma_n = 1\}$ and $X^- = \{y_n : \sigma_n = -1\}$.

3.2 Uniqueness of the unbounded black connected component

Let $A \subset \mathbb{R}^2$ denote a leaf as described in Section 2 and which is additionally invariant under rotations by $\pi/2$ and under reflections at the coordinate axes. In this subsection we consider the uniqueness of the unbounded black connected component for confetti percolation. To be more precise, we show the following result.

Proposition 2. *Let $p \in (0, 1)$ and denote by N the (random) number of unbounded black connected components. Then $\mathbb{P}_p(N = 1) = 1$ or $\mathbb{P}_p(N = 0) = 1$.*

The proof of Proposition 2 is a slight variation to an argument developed by Gandolfi, Keane and Russo in [12]. Indeed, their method is based purely on geometric properties of $\mathbb{R}^2$ (such as the Jordan curve theorem) and works just as well in continuous situations. A similar (but in fact more complicated) adaptation was considered in [14]. The proof depends on the following properties of the percolation model:

(A0) The origin lies in the interior of a connected component (i.e., either black connected or white connected) a.s. and it lies in the interior of a white connected component with positive probability.

(A1) $\mathbb{P}$ is invariant under horizontal and vertical translations, under rotations by $\pi/2$, and under reflections at the coordinate axes.

(A2) $\mathbb{P}$ is ergodic with respect to (discrete) horizontal translations and, separately, with respect to (discrete) vertical translations.

(A3) Black-increasing events are positively correlated.

(A4) There exists at least one unbounded connected component with positive probability.

Note that confetti percolation satisfies the first four of these items. For ergodicity, this is a consequence of the mixing property of the three-dimensional homogeneous Poisson point process, see [10, Chapter 12.3]. In the following, we proceed closely along the original presentation in [12].

3.2.1 Preliminaries

Before we begin with the proof of Proposition 2 we collect some preliminary results. For readability we write $\mathbb{P}$ for $\mathbb{P}_p$ in the following. Furthermore, when dealing with confetti percolation it suffices to consider self-avoiding piecewise linear paths consisting of line segments between points of rational coordinates. We say that a path is *closed* if it starts and ends at the same point. First, we note that it suffices to prove a uniform lower bound for the probability that squares are surrounded by black paths.

Lemma 3. *Assume there exists $\delta > 0$ such that for all $m \geq 1$,*

$$\mathbb{P}\left(\textit{there exists a closed black path surrounding } Q_m(o)\right) \geq \delta. \tag{1}$$

Then with probability 1 every bounded subset of $\mathbb{R}^2$ is surrounded by a closed black path. In particular, $\mathbb{P}(N = 1) = 1$.

Proof. If we denote by A the event that for every $m \geq 1$ the set $Q_m(o)$ is surrounded by a closed black path, then ergodicity implies $\mathbb{P}(A) \in \{0, 1\}$. Furthermore, as we assumed the existence of $\delta > 0$ such that (1) holds uniformly for all $m \geq 1$, we conclude that $\mathbb{P}(A) > 0$. Hence, the event A occurs with probability 1 and every bounded subset of $\mathbb{R}^2$ is surrounded by a closed black path. □

For $m \geq 1$ we denote by $H_m = \left\{(z_1, z_2) \in \mathbb{R}^2 : |z_2| \leq m\right\}$ the horizontal strip of height $2m$, centered at the x-axis. Furthermore, for Borel sets $A, B, C \subset \mathbb{R}^2$ with $A \cup B \subset C$ we say that $[A, B; C]$ occurs if there exists a black path in W that starts in A and ends in C. Similarly, we say that $[A, \infty; C]$ occurs if A has non-empty intersection with an unbounded connected component of black points in C. Then it is useful to note that percolation cannot occur in horizontal strips.

Lemma 4. *Let $m \geq 1$ be arbitrary. Then $\mathbb{P}([z, \infty; H_m]) = 0$ for all $z \in H_m$.*

Proof. From Assumption (A0) we conclude that there exists $\varepsilon > 0$ such that with positive probability the segment $\{0\} \times [-\varepsilon, \varepsilon]$ is colored white. Stationarity and (A3) therefore imply that with positive probability the entire segment $\{0\} \times [-m, m]$ is white. Hence, denoting by A the event that there exist arbitrarily large $k \geq 1$ such that $\{k\} \times [-m, m]$ is white and arbitrarily large $k \geq 1$ such that $\{-k\} \times [-m, m]$ is white, the ergodicity assumption implies that $\mathbb{P}(A) = 1$. The proof is completed by noting that A is contained in the complement of $[z, \infty; H_m]$. □

Denote by $S, T : \mathbb{R}^2 \to \mathbb{R}^2$, $S(z) = z + (1,0)$ and $T(z) = z + (0,1)$ the horizontal and vertical translation, respectively. Next, we recall an important ergodic result whose proof can be found in [12].

Lemma 5 (Multiple Ergodic Lemma)**.** *Let A_0, A_1 and A_2 be monotonic (i.e., either increasing or decreasing) events. Then*

$$D\text{-}lim_{N\to\infty}\mathbb{P}\left(A_0 \cap S^{-N}A_1 \cap S^{-2N}A_2\right) = \mathbb{P}\left(A_0\right)\mathbb{P}\left(A_1\right)\mathbb{P}\left(A_2\right),$$

where we write $D\text{-}lim_{N\to\infty}\alpha_N = \alpha$ if there exists a subsequence of density 1 converging to α.

As an important consequence of Lemma 5 we obtain lower bounds for percolation probabilities outside bounded Borel sets that are far away from the origin.

Corollary 6. *Let $U, V \subset \mathbb{R}^2$ be bounded Borel sets and let $W \subset \mathbb{R}^2$ be an unbounded Borel set. Furthermore, let $z \in W$ be arbitrary. Then there exist arbitrarily large integers N such that*

$$\mathbb{P}\left(\left[z, \infty; W \setminus \left(S^{-N}U \cup S^N V\right)\right]\right) \geq \frac{1}{2}\mathbb{P}\left([z, \infty; W]\right).$$

Proof. Consider the events

$$\begin{aligned}
A_0 &= \{U \text{ is white}\},\\
A_1 &= [z, \infty; W],\\
A_2 &= \{V \text{ is white}\},\\
\widetilde{A_N} &= [z, \infty; W \setminus \left(S^{-N}U \cup S^N V\right)],
\end{aligned}$$

and note that

$$S^N A_0 \cap A_1 \cap S^{-N}A_2 = S^N A_0 \cap \widetilde{A_N} \cap S^{-N}A_2.$$

Hence,

$$\mathbb{P}\left(\widetilde{A_N}\right)\mathbb{P}\left(S^N A_0 \cap S^{-N}A_2\right) \geq \mathbb{P}\left(S^N A_0 \cap A_1 \cap S^{-N}A_2\right).$$

Then Lemma 5 implies

$$D\text{-}\lim_{N\to\infty}\mathbb{P}\left(S^N A_0 \cap A_1 \cap S^{-N}A_2\right) = \mathbb{P}\left(A_0\right)\mathbb{P}\left(A_1\right)\mathbb{P}\left(A_2\right),$$

and

$$D\text{-}\lim_{N\to\infty}\mathbb{P}\left(S^N A_0 \cap S^{-N}A_2\right) = \mathbb{P}\left(A_0\right)\mathbb{P}\left(A_2\right).$$

Combining the latter identities yields

$$\mathbb{P}\left(\widetilde{A_N}\right) \geq \mathbb{P}\left(A_1\right)/2,$$

for arbitrarily large $N \geq 1$, as desired. □

The proof of Proposition 2 proceeds in two steps. The first step considers the case, where there exists an unbounded black connected component in the upper half-plane of $\mathbb{R}^2$.

3.2.2 First case: percolation in the upper half-plane

Denote by $H^+ = \{(z_1, z_2) \in \mathbb{R}^2 : z_2 \geq 0\}$ the upper half-plane in $\mathbb{R}^2$. In the first step of the proof we assume that $\mathbb{P}\left([o, \infty; H^+]\right) > 0$ and denote this probability by q.

Let $m \geq 1$ be arbitrary and put $B = Q_m(o)$. Applying Corollary 6 with $z = o$, $U = \emptyset$, $V = B$, $W = H^+$ and T instead of S yields

$$\mathbb{P}\left(\left[y_{-N}, \infty; T^{-N}H^+ \setminus B\right]\right) = \mathbb{P}\left(\left[o, \infty; H^+ \setminus T^N B\right]\right) \geq q/2,$$

for arbitrarily large integers $N \geq 1$. Lemma 4 implies that $\mathbb{P}([y_{-N}, \infty; H_N]) = 0$, so that

$$\mathbb{P}([y_{-N}, L_N; H_N \setminus B]) \geq q/2,$$

where $L_N = \{(z_1, z_2) \in \mathbb{R}^2 : z_2 = N\} \subset H^+$ denotes the horizontal line at distance N from the origin. Using the decomposition $L_N = L_N^+ \cup L_N^-$ with

$$L_N^\sigma = \{(z_1, z_2) \in L_N : \sigma z_1 \geq 0\}, \quad \sigma \in \{+, -\},$$

we obtain

$$\mathbb{P}\left(\left[y_{-N}, L_N^+; L_N^+ \cup H_N \setminus (B \cup L_N)\right]\right) \geq q/4.$$

Hence, by horizontal reflection,

$$\mathbb{P}\left(\left[y_N, L_{-N}^+; L_{-N}^+ \cup H_N \setminus (B \cup L_{-N})\right]\right) \geq q/4.$$

Defining the event

$$J = \left[y_{-N}, L_N^+; L_N^+ \cup H_N \setminus (B \cup L_N)\right] \cap \left[y_N, L_{-N}^+; L_{-N}^+ \cup H_N \setminus (B \cup L_{-N})\right],$$

we conclude from (A3) that $\mathbb{P}(J) \geq q^2/16$.

Next, we note that $J \subset [y_{-N}, y_N; H_N \setminus B]$. Indeed, denote by Γ a path connecting y_{-N} to L_N^+. Then y_N and L_{-N}^+ are contained in different connected components of $H_N \setminus \Gamma$ and any path Γ' connecting y_N and L_{-N}^+ must intersect Γ. Therefore,

$$\mathbb{P}([y_{-N}, y_N; H_N \setminus B]) \geq q^2/16.$$

Moreover, we note that a path from y_{-N} to y_N in $H_N \setminus B$ partitions H_N into two connected components with the property that for all sufficiently large $n \geq 1$ the sets $[n, \infty) \times [-N, N]$ and $[-n, \infty) \times [-N, N]$ are contained in different connected components. We denote by J^+ the event that there exists a black path Γ in $H_N \setminus B$ that connects L_N and L_{-N} and such that B and $[-n, \infty) \times [-N, N]$ are contained in different connected components of $H_N \setminus \Gamma$ for all sufficiently large $n \geq 1$. The event J^- is defined similarly.

From $\mathbb{P}(J^+) = \mathbb{P}(J^-) \geq q^2/32$ we obtain $\mathbb{P}(J^+ \cap J^-) \geq q^4/1024$, and, moreover, $J^+ \cap J^-$ implies that there exist black paths $\Gamma, \Gamma' \subset H_N \setminus B$ such that B is contained in a bounded connected component of $\mathbb{R}^2 \setminus (\Gamma \cup \Gamma')$. In particular, an application of Lemma 3 completes the proof.

3.2.3 Second case: no percolation in the upper half-plane

In the second part of the proof we consider the case where percolation does not occur in the upper half-plane, i.e., $\mathbb{P}([o, \infty; H^+]) = 0$. We make use of the notion of the *winding number* $i(\Gamma, z)$ of a path $\Gamma \subset \mathbb{R}^2$ around a point $z \in \mathbb{R}^2 \setminus \Gamma$. Intuitively it is defined as $1/(2\pi)$ times the total change of angle of the vector $z' - z$ as z' moves from the starting point of Γ to its end point. We refer the reader to [3, Chapter 7] for a precise definition and elementary properties.

In the following we denote by q the probability that the origin is contained in an unbounded black connected component. Due to assumption (A4) we have $q > 0$. As before let $m \geq 1$ be arbitrary and put $B = Q_m(o)$. Applying Corollary 6 with $z = o$, $U = V = B$ and $W = \mathbb{R}^2$, we see that there exist arbitrarily large $R \geq 1$ such that

$$\mathbb{P}\left(\left[x_{2R}, \infty; \mathbb{R}^2 \setminus \left(S^R B \cup S^{3R} B\right)\right]\right) = \mathbb{P}\left(\left[o, \infty; \mathbb{R}^2 \setminus \left(S^{-R} B \cup S^R B\right)\right]\right) \geq q/2.$$

Similarly, putting $I = [0, 4R] \times \{0\}$ and applying Corollary 6 with $z = x_{2R}$, $U = V = I$ and $W = \mathbb{R}^2 \setminus \left(S^R B \cup S^{3R} B\right)$ yields

$$\mathbb{P}([x_{2R}, \infty; K_{2R}]) \geq q/4,$$

for arbitrarily large $M \geq 1$, where we write

$$K_{2R} = \mathbb{R}^2 \setminus \left(S^R B \cup S^{3R} B \cup S^M I \cup S^{-M} I\right).$$

Putting $K = \mathbb{R}^2 \setminus \left(S^R B \cup S^{-R} B\right)$, we also note that the assumption $\mathbb{P}\left([o, \infty; H^+]\right) = 0$ implies that every unbounded black path starting from the origin must intersect the union $S^M L_0^+ \cup S^{-M} L_0^-$ infinitely often. Hence, we obtain

$$\mathbb{P}\left(\left[o, S^M L_0^+; K \setminus S^{-M} L_0^-\right]\right) \geq q/4, \tag{2}$$

and

$$\mathbb{P}\left(\left[x_{2R}, S^{-M} L_0^-; K_{2R} \setminus S^{M+4R} L_0^+\right]\right) \geq q/8.$$

Next, for $\sigma \in \{+, -\}$ denote by A^σ the event that there exists a black path Γ in $K \setminus S^{-M} L_0^-$ that starts in o, ends in $S^M L_0^+$, and satisfies $\sigma \cdot i\left(\Gamma, x_{2R}\right) > 0$. Using

$$A^+ \cup A^- = \left[o, S^M L_0^+; K \setminus S^{-M} L_0^-\right],$$

we conclude from (2) that $\mu\left(A^+\right) = \mu\left(A^-\right) \geq q/8$. In particular,

$$\mathbb{P}\left(A^+ \cap A^- \cap \left[x_{2R}, S^{-M} L_0^-; K_{2R} \setminus S^{M+4R} L_0^+\right]\right) \geq q^3/512.$$

To conclude the proof, it suffices to prove that

$$A^+ \cap A^- \cap \left[x_{2R}, S^{-M} L_0^-; K_{2R} \setminus S^{M+4R} L_0^+\right] \subset \left[o, x_{2R}; \mathbb{R}^2 \setminus S^R B\right]. \tag{3}$$

Indeed, then similar to the arguments at the end of the previous subsection we can show that the probability that B is surrounded by a closed black path is bounded from below by $\left(q^3/1024\right)^2$. Hence, Lemma 3 implies the claim.

To prove (3) we note that the intersection on the left hand side of (3) implies the existence of black paths $\Gamma_+, \Gamma_- \subset \mathbb{R}^2 \setminus \left(S^R B \cup S^{-M} L_0^-\right)$ and $\widetilde{\Gamma} \subset \mathbb{R}^2 \setminus \left(S^R B \cup S^M L_0^+\right)$ with the following properties:

(i) Γ_+ and Γ_- begin at o and end in $S^M L_0^+$;

(ii) $i\left(\Gamma_+, x_{2R}\right) > 0$ and $i\left(\Gamma_-, x_{2R}\right) < 0$;

(iii) $\widetilde{\Gamma}$ begins at x_{2R} and ends in $S^{-M} L_0^-$.

In particular, we can define a (not-necessarily black) closed path $\overline{\Gamma}$ by first using Γ_+ to get from o to $S^M L_0^+$, then moving along $S^M L_0^+$ to the endpoint of Γ_- and finally returning to o by traversing Γ_- in the reverse direction. Additivity of the winding number then implies

$$i\left(\overline{\Gamma}, x_{2R}\right) = i\left(\Gamma_+, x_{2R}\right) - i\left(\Gamma_-, x_{2R}\right) \geq 1,$$

where we also used that the winding number of a closed path is an integer. Hence, x_{2R} is contained in a bounded connected component of $\mathbb{R}^2 \setminus \overline{\Gamma}$, while $S^{-M} L_0^-$ is contained in the unbounded connected component of $\mathbb{R}^2 \setminus \overline{\Gamma}$. Since the path $\widetilde{\Gamma} \subset \mathbb{R}^2 \setminus \left(S^R B \cup S^{-M} L_0^+\right)$ connects x_{2R} and $S^{-M} L_0^-$, it must intersect $\overline{\Gamma} \setminus S^{-M} L_0^+ \subset \Gamma_+ \cup \Gamma_-$. In particular, x_{2R} and o can be connected by a black path in $\mathbb{R}^2 \setminus S^R B$, so that the event $\left[o, x_{2R}; \mathbb{R}^2 \setminus S^R B\right]$ occurs.

3.3 Zhang's theorem

Zhang's famous proof of $\theta(1/2) = 0$ for Bernoulli bond percolation on $\mathbb{Z}^2$ is based on rather general arguments and can be generalized to many two-dimensional percolation models with a more complex dependency structure. We provide an explicit proof for the confetti percolation model, closely following the exposition in [13].

Proposition 7. *Let A denote a fixed leaf that is invariant under rotations by $\pi/2$ and under reflections at the coordinate axes. Then $\theta(1/2, A) = 0$.*

Proof. Suppose the contrary, i.e., that $\theta(1/2, A) > 0$. First, for any $n \geq 1$ we denote by $A^l(n)$ the event that the left vertical boundary segment of the square $Q_n(o)$ intersects an unbounded black connected component. By $A^b(n)$, $A^r(n)$ and $A^t(n)$ we denote the analogous events corresponding to the bottom, right and top side of $Q_n(o)$, respectively. From ergodicity and our assumption $\theta(1/2, A) > 0$, we conclude that there exists an unbounded black connected component with probability 1. In particular,

$$\lim_{n\to\infty} \mathbb{P}\left(A^l(n) \cup A^b(n) \cup A^r(n) \cup A^t(n)\right) = 1.$$

As the events $A^u(n)$, $u \in \{l, b, r, t\}$ are decreasing events, we conclude from Lemma 1 that

$$\mathbb{P}\left(\left(A^l(n) \cup A^b(n) \cup A^r(n) \cup A^t(n)\right)^c\right) \geq \mathbb{P}\left(\left(A^l(n)\right)^c\right)^4,$$

so that also $\lim_{n\to\infty} \mathbb{P}\left(A^l(n)\right) = 1$. Therefore, we can choose $n_0 \geq 1$ such that $\mathbb{P}\left(A^l(n_0)\right) > 7/8$. As $p = 1/2$ the latter inequality is equivalent to $\mathbb{P}\left(A^l_w(n_0)\right) > 7/8$, where for $u \in \{l, b, r, t\}$ the event $A^u_w(n_0)$ is defined analogously to the event $A^u(n_0)$, just replacing "black connected component" by "white connected component". Next, consider the event

$$A = A^l(n_0) \cap A^r(n_0) \cap A^t_w(n_0) \cap A^b_w(n_0),$$

and note that

$$\mathbb{P}(A^c) \leq \mathbb{P}\left(A^l(n_0)^c\right) + \mathbb{P}\left(A^r(n_0)^c\right) + \mathbb{P}\left(A^t_w(n_0)^c\right) + \mathbb{P}\left(A^b_w(n_0)^c\right) < 1/2. \tag{4}$$

The event A guarantees the existence of at least two unbounded black connected components and two unbounded white connected components in $\mathbb{R}^2 \setminus Q_{n_0}(o)$. Both of the unbounded black connected components in $\mathbb{R}^2 \setminus Q_{n_0}(o)$ are contained in the unique unbounded connected component C of black points in $\mathbb{R}^2$. Planarity implies that the two unbounded white connected components of $\mathbb{R}^2 \setminus Q_{n_0}(o)$ whose existence is guaranteed by A are contained in different connected components of $\mathbb{R}^2 \setminus C$. Hence (4) contradicts Proposition 2 that guarantees the existence of a unique white connected component with probability 1. □

4 An RSW theorem for confetti percolation

4.1 Statement of the result

Again let $A \subset \mathbb{R}^2$ denote a leaf as described in Section 2 which is additionally invariant under rotations by $\pi/2$ and under reflections at the coordinate axes. The first step in the proof of $p_c \leq 1/2$ is to check that a general RSW-type theorem as developed by Bollobás and Riordan for Voronoi percolation in [5] (see also [7, Section 5]) is true for confetti percolation, too. For $R \subset \mathbb{R}^2$ a rectangle we denote by $H(R)$ the event that there exists a black horizontal crossing of R, i.e., a path connecting the left with the right vertical boundary of R that is contained in the intersection of R with the set of black points. Note that for confetti percolation it is no restriction of generality to consider only piecewise linear paths. The goal of this section is to prove the following result.

Proposition 8. *Let $p \in (0, 1)$, $\rho > 1$ be arbitrary and suppose that there exists $c > 0$ such that $\mathbb{P}(H([0, s] \times [0, s])) \geq c$ holds for all sufficiently large $s > 0$. Then there exists $c' > 0$ such that for all $s_0 > 0$ we can find $s \geq s_0$ with $\mathbb{P}_p(H([0, \rho s] \times [0, s])) \geq c'$.*

Remark 1. Putting $n = \lceil s \rceil$, the relation $[0, \rho n] \subset [0, 2\rho s]$ holds for all sufficiently large $s \geq 1$, so that Proposition 8 also yields the existence of arbitrarily large integers $s \geq 1$ with $\mathbb{P}_p(H([0, \rho s] \times [0, s])) \geq c'$.

As explained in [7, Section 5] the proof of Proposition 8 depends on the following properties of the percolation model:

(B1) Black increasing events are positively correlated (we have seen this in Section 3.1).

(B2) The model has the symmetries of $\mathbb{Z}^2$ (true, since our leaves have this property).

(B3) Disjoint regions are asymptotically independent (in fact for the dead leaves model the range of dependence is finite)

(B4) For every fixed rectangle R there exists $C > 0$ such that the probability that $H(sR)$ holds but the shortest black horizontal crossing of sR has length at least $Cs^{2.5}$ tends to 0 as $s \to \infty$ (this is not clear *a priori* – the remainder of this subsection is devoted to this item).

To prove useful asymptotics for the length of a shortest black horizontal crossing we observe that the length of such a crossing is bounded from above by the perimeter of the black connected component in which it is contained. Since the boundary of this component is a subset of the union of the boundary of R and the union of the boundaries of (partly) visible leaves intersecting R, it suffices to bound the latter.

Choose $r_1, r_2 > 0$ such that $Q_{r_1}(o) \subset A \subset Q_{r_2}(o)$. The first step is to give a more economical construction of the dead leaves process on a rectangle of the form $R = [0, as] \times [0, bs]$ for $a, b, s > 0$. Define $R^{\mathrm{ext}} = [-r_2, as+r_2] \times [-r_2, bs+r_2]$ and denote by c a positive constant to be determined in the following paragraph. Furthermore, let $X = \{(y_n, \sigma_n)\}_{1 \leq n \leq N} \subset R^{\mathrm{ext}} \times [0, c \log s] \times \{\pm 1\}$ be an independently $\{\pm 1\}$-marked homogeneous Poisson point process on $R^{\mathrm{ext}} \times [0, c \log s]$ with intensity 1 whose marks are iid, have constant shape A and satisfy $\mathbb{P}(\sigma_n = 1) = p$. Furthermore, denote by $\pi_1 : R^{\mathrm{ext}} \times [0, c \log s] \times \{\pm 1\} \to R^{\mathrm{ext}}$ the projection onto the first coordinate. Then $Z = \pi_1(X) \subset R^{\mathrm{ext}}$ is a homogeneous Poisson point process on R^{ext} with intensity $\lambda = c \log s$. Note that to obtain from X a realization of the dead leaves process on the finite rectangle R it suffices to simulate a further $\{\pm 1\}$-marked homogeneous Poisson process $X' \subset R^{\mathrm{ext}} \times [c \log s, \infty) \times \{\pm 1\}$ and consider the coloring of R induced by the superposition of X and X'.

Denote by B_s the event $R \subset \bigcup_{z \in Z} \{z + A\}$. Observe that if B_s occurs then the process X' is not needed to determine the coloring of R induced by $X \cup X'$. We claim that $\mathbb{P}(B_s) \to 1$ as $s \to \infty$. Using the notation $M = \{v \in (r_1/4)\mathbb{Z}^2 : \big(v + [0, r_1/4]^2\big) \cap R \neq \emptyset\}$, we see that B_s is implied by the occurrence of the event $Z(v + [0, r_1/4]^2) \geq 1$ for all $v \in M$. However, the probability of its complement can be bounded from above by $|M| \exp(-\lambda r_1^2/16) \leq 16 r_1^{-2}(as + 2r_2)(bs + 2r_2)\exp(-\lambda r_1^2/16)$. Choosing a suitable c for which $s^2 \exp(-\lambda r_1^2/16) \in O(s^{-1})$ (e.g., for $A = Q(o)$ we choose $c = 50$), we see that this expression tends to 0 as $s \to \infty$. In particular, we conclude that the number of visible squares intersecting R is of order at most $O(s^{2.5})$ whp, thereby verifying the final assumption (B4).

4.2 Proof of Proposition 8

For the proof of Proposition 8 we closely follow the presentations in [5] and [6], only accomodating the arguments to the case of finite range of dependence. In the present subsection, for $\rho, s > 0$ it is convenient to write $f(\rho, s) = \mathbb{P}_p(H([0, \rho s] \times [0, s]))$. We assume for a contradiction that Proposition 8 was false, i.e., that there exists $\rho > 1$ with $\lim_{s \to \infty} f(\rho, s) = 0$. First, note that this implies

$$\lim_{s \to \infty} f(1 + \varepsilon, s) = 0, \tag{5}$$

for all $\varepsilon > 0$. Indeed, as explained in [6, Chapter 8.3] positive correlation of increasing events yields

$$f(a_1 + a_2 - 1, s) \geq f(a_1, s) f(a_2, s) f(1, s), \tag{6}$$

for all $a_1, a_2 > 1$, so that choosing $k \geq (\rho - 1)/\varepsilon$ and applying (6) several times (k times, to be more precise) implies

$$f(\rho, s) \geq f(1 + k\varepsilon, s) \geq f(1 + \varepsilon, s) f(1 + (k-1)\varepsilon, s) f(1, s) \geq (f(1 + \varepsilon, s) f(1, s))^k f(1, s).$$

Since $f(\rho, s)$ tends to 0 as $s \to \infty$, so does $f(1 + \varepsilon, s)$.

We note that (5) has important implications for the shape of black horizontal crossings in a square. Indeed, for any $\varepsilon > 0$, whp all horizontal black crossings of a square with side length s must pass within distance εs of the top and bottom sides. Otherwise, we would obtain a black horizontal crossing in an $(s \times (1 - \varepsilon)s)$-rectangle.

In order to illustrate the strength of (5), we consider the following result taken from [5, Claim 4.3].

Claim 1. *Let $\varepsilon > 0$ be arbitrary and assume that* (5) *holds. Then whp no black path in $[0, s] \times \mathbb{R}$ that starts in $\{0\} \times [-\varepsilon s, \varepsilon s]$ leaves $[0, s] \times [-(1/2 + 2\varepsilon)s, (1/2 + 2\varepsilon)s]$.*

Proof. Denoting by E the event that there exists a black path Γ in $S' = [0, s] \times [-(1/2 + 2\varepsilon)s, (1/2 + 2\varepsilon)s]$ that starts in $\{0\} \times [-\varepsilon s, \varepsilon s]$ and touches the top side of S', it suffices (by symmetry) to show that the complement of E occurs whp.

Furthermore, denote by E_1 the event that there exists such a path that is contained in the rectangle $R = [0, s] \times [-s/2, s/2 + 2\varepsilon s]$. Observe that the event $E \setminus E_1$ implies the existence of a black vertical crossing in the rectangle $[0, s] \times [-s/2, (1/2 + 2\varepsilon)s]$ and by the observation preceding Claim 1 this is an event whose complement occurs whp. Hence, it suffices to show that $\lim_{s \to \infty} \mathbb{P}(E_1) = 0$.

Finally, denote by E_2 the event that there exists a black path in R starting from an element in $\{0\} \times [\varepsilon s/2, 3\varepsilon s/2]$ and ending at $[0, s] \times \{-s/2\}$. By symmetry we note that $\mathbb{P}(E_1) = \mathbb{P}(E_2)$. As increasing events are positively correlated, we see that $\mathbb{P}(E_1 \cap E_2) \geq \mathbb{P}(E_1)^2$. Furthermore, if both E_1 and E_2 occur, then the black paths Γ_1 and Γ_2 guaranteed by the events E_1 and E_2 must intersect and define a black vertical crossing of $[0, s] \times [-s/2, s/2 + 2\varepsilon]$. In particular, this yields a contradiction to the discussion preceding the claim. □

Further refinements of such arguments can be used to show that whp every black horizontal crossing of a rectangle contains 16 disjoint subpaths that define black horizontal crossings of slightly shrunken rectangles. To be more precise, we use the following claim whose proof can be found in [5, Claim 4.6].

Claim 2. *Fix $C > 1/2$ and put $R = R_s = [0, s] \times [-Cs, Cs]$. Moreover, for $j \in \{0, \ldots, 4\}$ put $R_j = [js/100, (j + 96)s/100] \times [-Cs, Cs]$ and assume that* (5) *holds. Then whp every black horizontal crossing Γ of R contains 16 disjoint subpaths $\Gamma_1, \ldots, \Gamma_{16}$ with the property that for every $i \in \{1, \ldots, 16\}$ there exists $j \in \{0, \ldots, 4\}$ such that Γ_i crosses R_j horizontally.*

In the final step, we show that the shape of black paths imposed by the claim contradicts our assumption (B4) on the length of shortest black horizontal crossings. Indeed, for R_s an $(s \times 2s)$-rectangle denote by $L(R_s)$ the length of the shortest black horizontal crossing of R_s provided that such a crossing exists (otherwise, we put $L(R_s) = \infty$). Then, for $\eta > 0$ a (small) constant, it is convenient to consider the function

$$g(s) = \sup \{x \geq 0 : \mathbb{P}(L(R_s) < x) \leq \eta\}.$$

Note that the probability that $L(R_s) < \infty$, i.e., that there exists a black horizontal crossing in R_s is bounded from below by $f(1, s)$, so that by choosing $\eta > 0$ sufficiently small we can guarantee that $g(s) \in (0, \infty)$ for all sufficiently large $s > 0$.

Our next goal is to show $\mathbb{P}\left(L(R_s) < 16g(0.47s)\right) < \eta$, i.e.,

$$g(s) \geq 16g(0.47s), \tag{7}$$

for all sufficiently large $s > 0$. Before we show (7), let us explain how this inequality can be used in order to derive at the desired contradiction. Iterating (7) yields $g((1/0.47)^n s) \geq 16^n g(s)$. In particular, there are arbitrarily large $s > 0$ such that $g(s) > s^3$. Hence, when $\eta > 0$ is chosen such that $\eta < \liminf_{s\to\infty} \mathbb{P}\left(L(R_s) < \infty\right)/2$, then $\limsup_{s\to\infty} \mathbb{P}\left(L(R_s) \in (s^3, \infty)\right) > 0$, contradicting assumption (B4).

The first step in proving (7) consists of establishing the inequality

$$\mathbb{P}\left(L(R_s) < g(0.47s)\right) \leq 2 \cdot 10^4 \eta^2 \tag{8}$$

for all sufficiently large $s > 0$, where $R_s = [0, 0.96s] \times [-s, s]$. Subdivide both the left vertical boundary and the right vertical boundary of R_s into 100 line segments $\{L_i\}_{1\leq i\leq 100}$ and $\{L'_i\}_{1\leq i\leq 100}$, respectively, where each of these segments is of length $0.02s$. For fixed $i, j \in \{1, \ldots, 100\}$, we say that a black horizontal crossing Γ of R is *eligible* if Γ starts from a point of L_i and ends in a point of L'_j. As the two families of line segments cover the vertical boundaries of R_s, it suffices to show $\mathbb{P}\left(B_{i,j}\right) \leq 2\eta^2$, where $B_{i,j}$ denotes the event that there exists an eligible path Γ with $\nu_1(\Gamma) \leq g(0.47s)$.

Consider an eligible path Γ starting from $(0, y_0)$ and ending at $(0.96s, y_1)$. Moreover, denote by Γ_0 the subpath of Γ starting from $(0, y_0)$ and ending at the first point where Γ touches the vertical line $x = 0.47s$. Then, we conclude from Claim 1 that Γ_0 is contained in $R_0 = [0, 0.47s] \times [y - 0.47s, y + 0.47s]$ whp, where y denotes the midpoint of L_i. Similarly, denoting by Γ_1 the subpath of Γ starting from the last point where Γ touches the vertical line $x = 0.49s$ and ending at $(0.96s, y_1)$, we see that Γ_1 is contained in $R_1 = [0.49s, 0.96s] \times [y' - 0.47s, y' + 0.47s]$ whp, where y' denotes the midpoint of L'_j. In particular, we have $L(R_0) + L(R_1) \leq L(R_s)$ whp. As confetti percolation exhibits a finite range of dependence, we conclude for all sufficiently large $s > 0$ that

$$\begin{aligned}
\mathbb{P}\left(L(R_0) + L(R_1) < g(0.47s)\right) &\leq \mathbb{P}\left(\max\left(L(R_0), L(R_1)\right) < g(0.47s)\right) \\
&= \mathbb{P}\left(L(R_0) < g(0.47s)\right)\mathbb{P}\left(L(R_1) < g(0.47s)\right) \\
&\leq \eta^2,
\end{aligned}$$

which completes the proof of (8).

Having established these preliminaries, we can now complete the proof of (7). Using the notation of Claim 2, we put $R = [0, s] \times [-s, s]$ and $R_j = [js/100, (j + 96)s/100] \times [-s, s]$, $j \in \{0, \ldots, 4\}$. First, note that (8) implies

$$\mathbb{P}\left(\min_{j\in\{0,\ldots,4\}} L(R_j) \geq g(0.47s)\right) \geq 1 - 10^5\eta^2.$$

Second, from Claim 2 we conclude that $L(R) \geq 16 \min_{j\in\{0,\ldots,4\}} L(R_j)$ whp, so that for all sufficiently large $s > 0$,

$$\mathbb{P}\left(L(R) \geq 16g(0.47s)\right) \geq 1 - 10^6\eta^2,$$

and the latter expression is bounded from below by $1 - \eta$, provided that η was chosen sufficiently small. In particular, $16g(0.47s) \leq g(s)$, as desired.

5 $p_c \leq 1/2$

In the following we restrict our attention to square-shaped leaves, i.e., $A = Q(o)$. In this section we show how the RSW-type theorem of Section 4 in conjunction with a suitable sharp-threshold result can be used to obtain $\theta(p) > 0$ for $p > 1/2$. As in [6, Section 8.3], we see that in order to apply this sharp threshold result, we need to work with a mesh-size $\delta(s)$ of the form $\delta(s) \sim s^{-\gamma}$ for some constant $\gamma > 0$. Since γ may be quite small, we have to expect the occurrence of discretization defects.

For simplicity of exposition, we assume from now on that $s > 0$ is an integer. We denote by $\mathbb{T}^2 = \mathbb{T}^2_{10s}$ the two-dimensional torus obtained by the standard glueing of the boundaries of $[0, 10s]^2$. Furthermore, it is no problem to construct a confetti process on the torus in the same way as we did for the case of $\mathbb{R}^2$. Even the more economical construction of the process can be transferred. Indeed, write

$$\lambda = \lambda(s) = 50\lfloor \log s \rfloor$$

and let $X \subset \mathbb{T}^2 \times [0, \lambda] \times \{\pm 1\}$ be an independently $\{\pm 1\}$-marked homogeneous Poisson point process with intensity 1 whose marks have constant shape $A = Q(o)$ and satisfy $\mathbb{P}(\sigma_n = 1) = p$. Furthermore, denote by $\pi_1 : \mathbb{T}^2 \times [0, \lambda] \times \{\pm 1\} \to \mathbb{T}^2$ and $\pi_{1,2} : \mathbb{T}^2 \times [0, \lambda] \times \{\pm 1\} \to \mathbb{T}^2 \times [0, \lambda]$ the projection onto the first coordinate and the first two coordinates, respectively. Then $Z = \pi_1(X) \subset \mathbb{T}^2$ is a homogeneous Poisson point process on $\mathbb{T}^2$ with intensity λ and $Y = \pi_{1,2}(X)$ is a homogeneous Poisson point process on $\mathbb{T}^2 \times [0, \lambda]$ with intensity 1.

Note that to obtain from X a realization of the confetti process on $\mathbb{T}^2$ it suffices to independently simulate a further $\{\pm 1\}$-marked homogeneous Poisson process $X' \subset \mathbb{T}^2 \times [\lambda, \infty) \times \{\pm 1\}$ and consider the coloring of $\mathbb{T}^2$ induced by the superposition of X and X'. We denote by $C_s^{(1)}$ the event $\mathbb{T}^2 \subset \bigcup_{z \in Z} Q_{1/2}(z)$. In this case the process X' is not needed to determine the coloring of $\mathbb{T}^2$ induced by $X \cup X'$ (more precisely, X' is still not needed after perturbing the squares in X by at most $1/4$ in each direction). Just as in Section 4 one proves that $C_s^{(1)}$ occurs whp.

Since the notation $\delta = \delta(s) \sim s^{-\gamma}$ is reserved for the mesh-size corresponding to the discretization we are about to construct, we use the notation δ_0 for a temporary positive variable that could take the value of $\delta(s)$ or some related quantity. We want to formalize the notion of unstable configuration of squares whose connectivity properties can be altered by small perturbations in the space or time dimension. For $\delta_0 > 0$ and $z, z' \in Z$, $z \neq z'$ we say that $\{z, z'\}$ forms a *spatially* δ_0*-unstable pair* if and only if $z - z' \in A_{\delta_0}$, where we write

$$A_{\delta_0} = \{z \in \mathbb{T}^2 : z \in Q_{2\delta_0}(o) \oplus ((\{0, \pm 1\} \times [-2, 2]) \cup ([-2, 2] \times \{0, \pm 1\}))\},$$

and where for any $A, B \subset \mathbb{R}^2$ we put $A \oplus B = \{a + b : a \in A, b \in B\}$. See Figure 1 for an illustration of the set A_{δ_0}.

Similarly, we can define instabilities with respect to time. For $t, t' > 0$, $t \neq t'$, we say that $\{t, t'\}$ forms a *temporally* δ_0*-unstable pair* if $|t - t'| < \delta_0$.

Finally, for $y = (z, t)$, $y' = (z', t') \in \mathbb{T}^2 \times [0, \lambda]$ we say that $\{y, y'\}$ forms a δ_0*-unstable pair* if the following two properties hold:

(i) $|y - y'|_\infty \leq 2 + \delta_0$ and

(ii) $\{z, z'\}$ forms a spatially δ_0-unstable pair or $\{t, t'\}$ forms a temporally δ_0-unstable pair.

As already mentioned above, due to the coarseness of the discretization we *cannot* exclude the existence of δ-unstable pairs whp. We first note that the total number of δ-unstable pairs in a $\log s$-square is bounded from above by a constant whp.

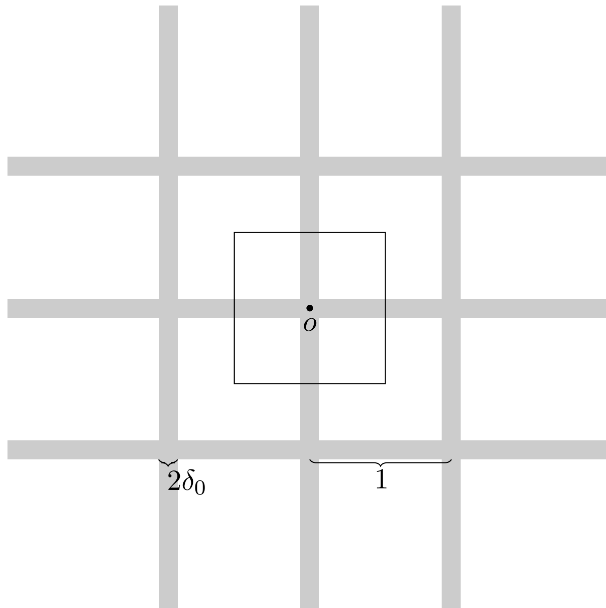

Figure 1: The square $Q(o)$ (black) and the set A_{δ_0} (gray)

Lemma 9. *Let X be as above, let $\gamma_0 > 0$ be arbitrary and let $\delta_0 = \delta_0(s) \le s^{-\gamma_0}$. Then there exists $K > 0$ such that the probability that the total number of δ_0-unstable pairs contained in $Q_{\log s}(o) \times [0, \lambda]$ is larger than K is of order $O\left(s^{-3}\right)$.*

Proof. By the Poisson concentration inequality (see e.g. [16, Lemma 1.2]) there exist constants $c_1, K_1 > 0$ such that for $M = K_1 (\log s)^3$ we have $\mathbb{P}\left(Z(Q_{\log s}(o)) \ge M\right) \le c_1 s^{-3}$. Let $y_1, \ldots, y_M$ be iid points placed uniformly in $Q_{\log s}(o) \times [0, \lambda]$. Next, denote by

$$A' = \bigcup_{j=1}^{M} \{|\{k \in \{1, \ldots, M\} : \{y_j, y_k\} \text{ is } \delta_0\text{-unstable}\}| > K_2\}$$

the event that there exists $j \in \{1, \ldots, M\}$ such that y_j is contained in more than K_2 pairs which are δ_0-unstable, where $K_2 > 0$ is a constant to be specified. Then we can choose $K_2 > 0$ with $\mathbb{P}(A') \le c_2 s^{-3}$ for some constant $c_2 > 0$. Indeed, for any pairwise distinct $j_1, \ldots, j_{K_2} \in \{2, \ldots, M\}$ the probability that all pairs $\{y_1, y_{j_1}\}, \ldots, \{y_1, y_{K_2}\}$ are δ_0-unstable can be bounded from above by

$$\left(\left(\lambda\nu_2\left(A_{\delta_0}\right) + 2\delta_0 \left(\log s\right)^2\right) / \left(\lambda \left(\log s\right)^2\right)\right)^{K_2 - 1},$$

where ν_2 denotes the Lebesgue measure in $\mathbb{R}^2$. The latter probability is in $O\left(s^{-4}\right)$, provided that $K_2 > 0$ is chosen sufficiently large, while the number of possible choices of pairwise distinct $\{j_1, \ldots, j_{K_2}\}$ is in $O\left(M^{K_2}\right)$. Hence, $\mathbb{P}(A') \le c_2 s^{-3}$ for some constant $c_2 > 0$, as desired.

Furthermore, we claim the existence of $c_3, K_3 > 0$ (independent of s) such that the probability that the number of δ_0-unstable pairs formed by the points $y_1, \ldots, y_M$ placed uniformly in $Q_{\log s}(o) \times [0, \lambda]$ exceeds K_3 is bounded from above by $c_3 s^{-3}$. Indeed, suppose that for $i \ge 1$ the points $y_1, \ldots, y_i$ are already placed and denote by A_i the event that y_{i+1} forms a δ_0-unstable pair with one of $y_1, \ldots, y_i$. Then the conditional probability of A_i given the location of the points $y_1, \ldots, y_i$ can be bounded from above by $c_4' M \delta_0 \le c_4 \left(\log s\right)^3 \delta_0$ for suitable constants $c_4, c_4' > 0$. Now observe that if A' does not occur and there exist more than K_3 pairs which are

δ_0-unstable, then the event A_i occurs for at least K_3/K_2 values of i. Thus, the probability of obtaining more than K_3 unstable pairs is bounded from above by

$$(c_1 + c_2)s^{-3} + M^{K_3/K_2} c_4^{K_3/K_2} (\log s)^{3K_3/K_2} \delta_0{}^{K_3/K_2},$$

so that it suffices to choose any K_3 with $\gamma_0 K_3/K_2 > 3$. □

It is easy to visualize that a small perturbation of squares centered at the nodes of a δ_0-unstable pair could also change the connectivity properties of neighboring squares. For instance, a small perturbation of the spatially δ-unstable white squares in Figure 2 could lead to the disconnectedness of the two black squares even if these are not spatially δ-unstable. A naive approach would be to say $\{y, y', y''\}$ forms a δ_0-unstable triple if $\{y, y'\}$ constitutes a δ_0-unstable pair and $\min(|z - z''|_\infty, |z' - z''|_\infty) \le 2$, where we recall our convention that z, z' and z'' denote the spatial coordinates of y, y' and y'', respectively. However, as $\pi_1(Y) \subset \mathbb{T}^2$ forms a homogeneous Poisson process with intensity $\lambda = 50\lfloor \log s \rfloor$, at least heuristically we would expect that for a fixed δ_0-unstable pair there exist $\asymp \log s$ nodes y'' such that $\{y, y', y''\}$ forms a (naively) δ_0-unstable triple. A direct adaptation of the coupling trick of [5] will not work if the number of unstable triples is that large.

Therefore, we take only those neighbors into account whose connectivity properties could be destroyed by a small perturbation of the unstable pair. To be more precise, we formalize the notion of leaves visible at locations close to the boundary of another leaf. Let $\varphi = \{x_n\}_{1 \le n \le N} = \{(z_n, t_n, \sigma_n)\}_{1 \le n \le N}$ and for $P \in \mathbb{T}^2$ denote by φ_P the set of leaves whose interior covers the point P, i.e.,

$$\varphi_P = \{x_n \in \varphi : P \in Q(z_n) \setminus \partial Q(z_n)\},$$

where $\partial Q(z_n)$ denotes the topological boundary of $Q(z_n)$ in $\mathbb{R}^2$. For $x = (z, t, \sigma)$, $x' = (z', t', \sigma') \in \varphi$ we say that *x is boundary-visible from x' in φ* (or also that *$y = (z, t)$ is boundary-visible from $y' = (z', t')$ in φ*) if there exists a corner P_0 of $\partial Q(z')$ and a point $P \in \partial Q(z') \cap \partial Q(z)$ such that $P_0 \in Q(z)$ and $t = \mathit{height}_{\varphi_{P_0}}(P)$. An illustration of this definition is shown in Figure 3, where only the right-most white square is boundary-visible from the black square. First, we note that for given x_0 the number of $x \in \varphi$ which are boundary-visible from x_0 is rather small.

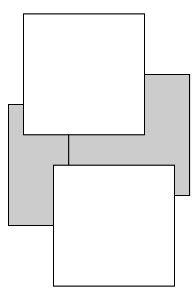

Figure 2: Configuration sensitive to small perturbation of white squares

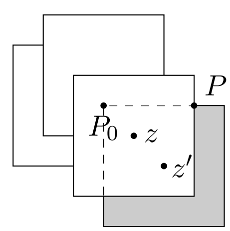

Figure 3: y is boundary-visible from y'

Lemma 10. *Let $x_0 \in \mathbb{T}^2 \times [0, \lambda] \times \{\pm 1\}$ be arbitrary and let X be as above. Then there exists a constant $c > 0$ such that*

$$\mathbb{P}\left(\#\{x \in X : x \text{ is boundary-visible from } x_0 \text{ in } X' \cup \{x_0\}\} \ge c \log s / \log\log s\right) \le s^{-3},$$

where $X' \subset X$ denotes an arbitrary thinning of X.

Proof. Without loss of generality, we may assume $z_0 = o$, so that $x_0 = (o, t_0)$. Furthermore, it suffices to prove a corresponding bound for the number N' of $x \in X'$ such that $P_0 = (-1/2, -1/2) \in Q(o)$ and such that for $P = [P_0, P_0 + (1, 0)] \cap \partial Q(z)$ we have $t = height_{X'_{P_0}}(P)$. Observe that for $n \geq 1$ if $N' \geq n$ then there exist $x_1, x_2, \ldots, x_n \in X \cap Q_3(o) \times [0, \lambda]$ satisfying

$$t_n \leq t_{n-1} \leq \cdots \leq t_1 \quad \text{and} \quad z_n^{(1)} \leq z_{n-1}^{(1)} \leq \cdots \leq z_1^{(1)},$$

where $z_i^{(1)}$ denotes the first coordinate of z_i. Therefore, using the Slivnyak-Mecke formula we compute

$$\begin{aligned}
\mathbb{P}(N' \geq n) &\leq 3^n \left(\int_{-3/2}^{3/2} \int_{-3/2}^{z_1^{(1)}} \cdots \int_{-3/2}^{z_{n-1}^{(1)}} 1 dz_n^{(1)} \cdots dz_2^{(1)} dz_1^{(1)} \right) \left(\int_0^\lambda \int_0^{t_1} \cdots \int_0^{t_{n-1}} 1 dt_n \cdots dt_2 dt_1 \right) \\
&= 3^n \left(\int_0^3 \int_0^{\xi_1} \cdots \int_0^{\xi_{n-1}} 1 d\xi_n \cdots d\xi_2 d\xi_1 \right) \frac{\lambda^n}{n!} \\
&= \frac{9^n \lambda^n}{n!^2}.
\end{aligned}$$

Using Stirling's formula we conclude that there exists $c_1 > 0$ such that the latter expression is bounded from above by $c_1^n \lambda^n / n^{2n}$ for all sufficiently large n. In particular, choosing $n = c \log s / \log \log s$ then for all s sufficiently large we compute

$$\begin{aligned}
\mathbb{P}(N_1 \geq n) &\leq c_1^n \left(\frac{50 (\log \log s)^2}{c^2 \log s} \right)^n \\
&= \exp\left(n \left(-\log \log s + \log\left(50 c_1 / c^2\right) + 2 \log \log \log s\right)\right) \\
&\leq \exp\left(-\frac{n \log \log s}{2}\right) \\
&= \exp\left(-\frac{c \log s}{2}\right).
\end{aligned}$$

In particular, choosing $c = 6$ proves the claim. $\square$

For $\delta_0 > 0$ and $y \in \pi_{1,2}(\varphi)$ define $\varphi_{\delta_0}^y$ to be the set of all y' in $\pi_{1,2}(\varphi)$ such that $\{y, y'\}$ does not form a δ_0-unstable pair. For $x, x', x'' \in \varphi$ we say that $\{x, x', x''\}$ forms a δ_0*-unstable triple with respect to* φ (or also that $\{y, y', y''\}$ forms a δ_0*-unstable triple with respect to* φ) if $\{y, y'\}$ forms a δ_0-unstable pair and y'' is boundary-visible from y in $\varphi_{\delta_0}^y$. If the locally finite set φ is understood, we also just say δ_0*-unstable triple* instead of δ_0-unstable triple with respect to φ.

By a δ_0*-bad component* of $\varphi = \{x_n\}_{1 \leq n \leq N}$ we denote a connected component of the graph with vertex set φ and where an edge is drawn between $x_1, x_2 \in \varphi$ if $\{x_1, x_2\}$ forms a δ_0-unstable pair or if there exists a δ_0-unstable triple containing x_1 and x_2. As in [5, Lemma 6.4] we first prove that δ_0-bad components are rather small (to be more precise, of sub-logarithmic size).

Lemma 11. *Let X be as above, let $\eta, \gamma_0 > 0$ be arbitrary and let $\delta_0 = \delta_0(s) \leq s^{-\gamma_0}$. For $\eta > 0$ denote by $C^{(2)}_{s,\gamma_0,\eta}$ the event that no δ_0-bad component of X consists of more than $\eta \log s$ vertices. Then for all $\eta, \gamma_0 > 0$ the event $C^{(2)}_{s,\gamma_0,\eta}$ occurs whp.*

Proof. By Palm calculus it suffices to prove the existence of a constant $c_1 > 0$ such that for any fixed $(z_0, t_0, \sigma_0) \in \mathbb{T}^2 \times [0, \lambda] \times \{\pm 1\}$ the probability that the δ_0-bad component of $X \cup \{(z_0, t_0, \sigma_0)\}$ containing (z_0, t_0, σ_0) consists of more than $\eta \log s$ vertices is in $O(s^{-5/2})$. Next, observe that to prove this statement it suffices to consider the confetti process on a $\log s$-square centered at z_0. Indeed, for all $\eta > 0$ sufficiently small, if the δ_0-bad component containing x consists of at most $\eta \log s$ vertices then its d_∞-diameter is bounded from above by $\log s$.

By Lemma 9 there exist constants $c_1, K > 0$ such that for all sufficiently large $s > 0$ the number of δ_0-unstable pairs in $X \cap (Q_{\log s}(z_0) \times [0, \lambda])$ is bounded from above by K with probability at least $1 - c_1 s^{-3}$. Furthermore, by Lemma 10 and Palm calculus there exist $c_2, c_3 > 0$ such that for all s sufficiently large the probability that for all $x \in X \cap (Q_{\log s}(z_0) \times [0, \lambda])$ there exist at most $c_2 \log s / \log \log s$ elements $x' \in X$ such that x' is boundary-visible from x in $X^x_{\delta_0}$ is bounded from below by $1 - c_3 s^{-5/2}$. Combining the two observations completes the proof of the lemma. □

Remark 2. The above argument could be used to show that the δ_0-bad components are of order $O(\log s / \log \log s)$. This approach was motivated by a suggestion of an anonymous referee. Originally she/he noted that although the number of leaves falling into the cube $Q(o) \times [0, \lambda]$ is $\asymp \log s$, the number of leaves that are visible in the window $Q(o)$ (and not only at its boundary) should be much smaller. For instance if one could show that the probability that more than k leaves are visible decays as k^{-ck} then we could deduce that whp only $O(\log s / \log \log s)$ are relevant. This tail behavior is rather easily established in dimension 1 due to the linear order of leaves. However, it seems not completely trivial to prove the analog in dimension 2, since configurations of visible square-shaped leaves do not form a linear but a tree-like structure. In particular, if the tree structure is very balanced and binary, naive estimates do not seem strong enough in order to obtain tail estimates strictly stronger than exponential. As hinted by the referee if this estimate could be made rigorous, it would lead to substantial simplifications of the proof.

After these preparations we recall the coupling trick introduced by Bollobás and Riordan in [5, Section 6.3]. Let $p_1, p_2 \in (0, 1)$ be such that $p_1 < p_2$ and for $i \in \{1, 2\}$ denote by $X_i = \{(z_{i,n}, t_{i,n}, \sigma_{i,n})\}_{1 \le n \le N_i} \subset \mathbb{T}^2 \times [0, \lambda] \times \{\pm 1\}$ two $\{\pm 1\}$-marked homogeneous Poisson processes with intensity 1 and $\mathbb{P}(\sigma_{i,n} = 1) = p_i$. Write $\delta = \delta(s) = (4\lceil s^\gamma / 4 \rceil)^{-1}$ for some $\gamma > 0$ to be specified in the proof of Theorem 16. Furthermore, let $\delta_1 = \lceil \delta^{-1/2} \rceil^{-1}$ and write $R_1 = [0.25s, 8.75s] \times [-0.25s, 1.25s]$ and $R_2 = [0.5s, 8.5s] \times [-0.5s, 1.5s]$. For $\delta_0 > 0$ we construct from $\varphi = \{(z_n, t_n, \sigma_n)\}_{1 \le n \le N} \subset \mathbb{T}^2 \times \mathbb{R} \times \{\pm 1\}$ a $\{\pm 1\}$-marked set φ^{δ_0} with the property that black squares are delayed and shrunk, while white ones are advanced and enlarged. More precisely, write $\varphi^{\delta_0} = \{(z_n, t_n + \sigma_n \delta_0, \sigma_n)\}_{1 \le n \le N}$, where the leaf associated with $(z_n, t_n + \sigma_n \delta_0)$ is given by $Q_{1 - 2\sigma_n \delta_0}(o)$. Furthermore, defining the coloring $\psi^{\delta_0} = \psi_{\varphi^{\delta_0}}$, we prove the following result.

Lemma 12. *Fix any $p_1, p_2 \in (0, 1)$ such that $p_1 < p_2$. There exists a coupling between X_1 and X_2 with $\mathbb{P}(E_g) \to 1$ as $s \to \infty$, where E_g denotes the intersection of the following events.*

(i) *$N_1 = N_2$ and $|y_{1,i} - y_{2,i}|_\infty \le \delta_1$ for all $i \in \{1, \dots, N_1\}$,*

(ii) *$\sigma_{2,i} \ge \sigma_{1,i}$ for all $i \in \{1, \dots, N_1\}$,*

(iii) *if there exists a black horizontal crossing of R_1 in the dead leaves process induced by X_1, then there exists a $\psi^{2\delta}$-black horizontal crossing of R_2 in the dead leaves process induced by X_2.*

First, we describe the construction of this coupling. Recall from the beginning of the present section that we assume s to be integer. Define $K_1 = 100 s^2 \lambda \delta_1^{-3}$ and subdivide $\mathbb{T}^2 \times [0, \lambda]$ into K_1 cubes Q_i of the form $Q_i = Q_{\delta_1}(z) \times [(k-1)\delta_1, k\delta_1]$ for some $z \in \delta_1 \mathbb{Z}^2$ and $k \in \{1, \dots, \lambda/\delta_1\}$. Denote by N a Poisson random variable with mean $100 s^2 \lambda$ and independently for each $j \in \{1, \dots, N\}$ choose an index s_j uniformly from $\{1, \dots, K_1\}$. Then for $i \in \{1, 2\}$ the unmarked point process $Y_i = \{y_{i,1}, \dots, y_{i,N}\} \subset \mathbb{T}^2 \times [0, \lambda]$ corresponding to X_i is constructed by choosing $y_{i,j}$ uniformly from Q_{s_j}. Our task is to provide a suitable coupling of $(y_{1,j}, \sigma_{1,j})$ and $(y_{2,j}, \sigma_{2,j})$ for all $j \in \{1, \dots, N\}$.

Write $\delta_2 = \sqrt{\delta_1}$. For $a, b \in \{1, \dots, N\}$ we say that $\{a, b\}$ forms a *potentially δ_2-unstable pair* if it is possible to find $\widetilde{y}_a \in Q_{s_a}$ and $\widetilde{y}_b \in Q_{s_b}$ such that $\{\widetilde{y}_a, \widetilde{y}_b\}$ forms a δ_2-unstable pair (here we do not necessarily assume $\widetilde{y_a}, \widetilde{y_b} \in Y_1$ or $\widetilde{y_a}, \widetilde{y_b} \in Y_2$). Similarly for $a, b, c \in \{1, \dots, N\}$ we say that $\{a, b, c\}$ forms a *potentially δ_2-unstable triple* if it is possible to find $\widetilde{y_j} \in Q_{s_j}$, $j \in \{1, \dots, N\}$ such that $\{\widetilde{y}_a, \widetilde{y}_b, \widetilde{y}_c\}$ forms a δ_2-unstable triple with respect to $\{\widetilde{y_j}\}_{1\le j\le N}$. Finally, we say that $C \subset \{1, \dots, N\}$ forms a *potentially δ_2-bad component* if C is a connected component in the graph with vertex set $\{1, \dots, N\}$ and where for $a, b \in \{1, \dots, N\}$ a and b are connected by an edge if $\{a, b\}$ forms a potentially δ_2-unstable pair or if there exists a potentially δ_2-unstable triple containing both a and b.

Lemma 13. *For $\eta > 0$ denote by $C^{(3)}_{s,\eta}$ the event of the absence of potentially δ_2-bad components consisting of more than $\eta \log s$ vertices. Then for every $\eta > 0$ the event $C^{(3)}_{s,\eta}$ occurs whp.*

Proof. First, observe that if $\{y_a, y_b\}$ forms a δ_2-unstable pair, then for all $\widetilde{y_a} \in Q_{s_a}$, $\widetilde{y_b} \in Q_{s_b}$ the pair $\{\widetilde{y_a}, \widetilde{y_b}\}$ forms a $(\delta_2 + 2\delta_1)$-unstable pair. Furthermore, suppose that $\{y_a, y_b, y_c\}$ forms a δ_2-unstable triple. In particular, assume that y_c is boundary-visible from y_a in $(Y_1)^{y_a}_{\delta_2}$ and for $j \in \{1, \dots, N\}$ let $\widetilde{y_j} \in Q_{s_j}$ be arbitrary. Then we claim $\widetilde{y_a}$ and $\widetilde{y_c}$ are contained in the same $(\delta_2+2\delta_1)$-bad component with respect to $\{\widetilde{y_j}\}_{1\le j\le N}$. Using Lemma 11, this will show that whp no potentially δ_2-bad component contains more than $\eta \log s$ vertices.

Without loss of generality we may assume that z_c lies to the north-west of z_a and that a point $P \in \partial Q(z_a) \cap \partial Q(z_c)$ satisfying the assumption in the definition of the boundary-visibility property lies on the upper horizontal boundary of $Q(z_a)$. Furthermore, write $\widetilde{P} = P + \pi_1(\widetilde{z_c} - z_c)e_1 + \pi_2(\widetilde{z_a} - z_a)e_2$, where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. First, we assert $\left|\widetilde{z_c} - \widetilde{P}\right|_\infty \le 1/2$ (and similarly for $\widetilde{z_a}$). Indeed, as $\left|\pi_1(\widetilde{P} - \widetilde{z_c})\right| = 1/2$, we have

$$
\begin{aligned}
\left|\widetilde{z_c} - \widetilde{P}\right|_\infty > 1/2 &\iff \left|\pi_2(\widetilde{P} - \widetilde{z_c})\right| > 1/2 \\
&\iff |\pi_2(P - z_c) + \pi_2(z_c - \widetilde{z_c}) + \pi_2(\widetilde{z_a} - z_a)| > 1/2.
\end{aligned}
$$

In particular, $\left|\widetilde{z_c} - \widetilde{P}\right|_\infty > 1/2$ implies $1/2 - 2\delta_1 \le |\pi_2(P - z_c)| \le 1/2$, so that $\{z_a, z_c\}$ forms a spatially $2\delta_1$-unstable pair, thereby contradicting $y_c \in (Y_1)^{y_a}_{\delta_2}$. This completes the proof of the assertion and we may henceforth use that $\left|\widetilde{z_c} - \widetilde{P}\right|_\infty = \left|\widetilde{z_a} - \widetilde{P}\right|_\infty = 1/2$.

Denote by $\widetilde{P_0}$ the upper-left corner of $Q(\widetilde{z_a})$. Under these assumptions, note that if $\{\widetilde{y_a}, \widetilde{y_c}\}$ does not form a $(\delta_2 + 2\delta_1)$-unstable pair, then the set of all $d \in \{1, \dots, N\}$, $d \ne a$ with

(i) $\widetilde{y_d} \in (\{\widetilde{y_j}\}_{1\le j\le N})^{\widetilde{y_a}}_{\delta_2+2\delta_1}$,

(ii) $\left|\widetilde{z_d} - \widetilde{P_0}\right|_\infty \le 1/2$,

(iii) $\widetilde{t_d} \le \widetilde{t_c}$ and

(iv) $\pi_1(\widetilde{z_d}) \ge \pi_1(\widetilde{z_c})$

is non-empty, as $d = c$ is an admissible choice. Among all these values choose d such that $\widetilde{t_d}$ is minimal. See Figure 4 for an illustration of the configuration. In particular, $\widetilde{y_d}$ is boundary-visible from $\widetilde{y_a}$ so that $\widetilde{y_a}$ and $\widetilde{y_d}$ are contained in the same $(\delta_2 + 2\delta_1)$-bad component. This completes the proof of the claim if $d = c$, so we may assume from now that $d \ne c$.

If $\{\widetilde{y_c}, \widetilde{y_d}\}$ forms a $2\delta_1$-unstable pair, then $\widetilde{y_c}$ and $\widetilde{y_d}$ (and thereby also $\widetilde{y_a}$ and $\widetilde{y_c}$) are contained in the same $(\delta_2 + 2\delta_1)$-bad component with respect to $\{\widetilde{y_j}\}_{1\le j\le N}$. Hence, we may assume $\widetilde{t_d} + 2\delta_1 \le \widetilde{t_c}$ and $\pi_1(\widetilde{y_d}) \ge \pi_1(\widetilde{y_c}) + 2\delta_1$. But then $\pi_1(y_d) \ge \pi_1(y_c)$ and $t_d \le t_c$ contradicting the assumption y_c is boundary-visible from y_a. This proves the claim that $\widetilde{y_a}$ and $\widetilde{y_c}$ are contained in the same $(\delta_2 + 2\delta_1)$-bad component with respect to $\{\widetilde{y_j}\}_{1\le j\le N}$.

□

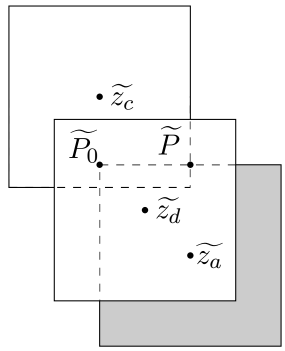

Figure 4: Configuration in the proof of Lemma 13

For C a potentially δ_2-bad component, we say that $B(C)$ occurs for $\{(y_{1,j}, \sigma_{1,j})\}_{1\le j\le N}$ if there exist $a, b \in C$ such that $\{y_{1,a}, y_{1,b}\}$ forms a 16δ-unstable pair. To prove Lemma 12 we need the following probabilistic result.

Lemma 14. *There exists a coupling between $\{(y_{1,j}, \sigma_{1,j})\}_{1\le j\le N}$ and $\{(y_{2,j}, \sigma_{2,j})\}_{1\le j\le N}$ such that the probability of occurrence of the following events tends to 1 as $s \to \infty$.*

(i) $|y_{1,i} - y_{2,i}|_\infty \le \delta_1$ *for all* $i \in \{1, \dots, N\}$,

(ii) $\sigma_{2,i} \ge \sigma_{1,i}$ *for all* $i \in \{1, \dots, N\}$,

(iii) if $C \subset \{1, \dots, N\}$ is a potentially δ_2-bad component and if there exist $a, b \in C$ with $\sigma_{1,a} = 1$ and $\sigma_{2,b} = -1$, then $B(C)$ does not occur and we have $y_{1,j} = y_{2,j}$ for all $j \in C$.

Proof. Choose $\eta > 0$ such that $2s^{-\gamma/3} \le s^{\eta \log(p_2 - p_1)}$ holds for all sufficiently large $s > 0$, with γ the constant used in the definition of $\delta = \delta(s)$ in the paragraph preceding Lemma 12. Also assume that $C^{(3)}_{s,\eta}$ holds, where we recall from Lemma 13 that this happens whp. To construct the desired coupling between $\{(y_{1,j}, \sigma_{1,j})\}_{1\le j\le N}$ and $\{(y_{2,j}, \sigma_{2,j})\}_{1\le j\le N}$ we first define a preliminary version that may be considered as *natural coupling*. This natural coupling is constructed simply by choosing $y_{1,j} = y_{2,j}$ to be uniformly distributed in Q_{s_j}. Furthermore, we choose $\sigma_{1,j} = 1$ with probability p_1 and the value of $\sigma_{2,j}$ is determined conditionally on the value of $\sigma_{1,j}$. If $\sigma_{1,j} = 1$ then we also put $\sigma_{2,j} = 1$, but if $\sigma_{1,j} = -1$ then we put $\sigma_{2,j} = 1$ with probability $(p_2 - p_1)/(1 - p_1)$. Starting from this simple coupling we construct the final coupling on each of the potentially δ_2-bad components separately.

Let $C \subset \{1, \dots, N\}$ be a potentially δ_2-bad component. For any $\delta_0 > 0$ we denote by $F_{\delta_0} \subset \mathbb{T}^2 \times \mathbb{R}$ the set of all $(z, t) \in \mathbb{T}^2 \times \mathbb{R}$ such that $\{(z, t), (o, 0)\}$ forms a δ_0-unstable pair. It is easy to check that there exists a constant $c_1 > 0$ such that for all $j \in \{1, \dots, K_1\}$ and all $\widetilde{y} \in Q_j$ we have $\nu_3((\widetilde{y} + F_{16\delta}) \cap Q_j) \le c_1 \delta_1^2 \delta$, where ν_3 denotes the Lebesgue measure in $\mathbb{R}^3$ and where we recall that $K_1 = 100 s^2 \lambda \delta_1^{-3}$. As C consists of at most $\eta \log s$ vertices, we see that the expected number of 16δ-unstable pairs in C is bounded from above by $(\eta \log s)^2 \cdot c_1 \delta_1^2 \delta / \delta_1^3 \le s^{-\gamma/3}$ for all sufficiently large values of s.

On the other hand, denote by $G(C)$ the event that in the natural coupling we have $\sigma_{2,j} = 1$ and $\sigma_{1,j} = -1$ for all $j \in C$, so that

$$\mathbb{P}(G(C)) \ge (p_2 - p_1)^{\eta \log s} = s^{\eta \log(p_2 - p_1)} \ge 2s^{-\gamma/3}.$$

In particular, we obtain $\mathbb{P}(G(C) \setminus B(C)) \ge \mathbb{P}(B(C))$. The final coupling is now constructed by using the cross-over coupling described in [5]. For the convenience of the reader we briefly recall this technique.

Let X_1^*, X_2^* be random variables with marginals distributed as X_1, X_2 and that are coupled according to the natural coupling described above. Choose $G'(C) \subset G(C) \setminus B(C)$ with $\mathbb{P}(G'(C)) = \mathbb{P}(B(C))$ and a measure-preserving bijection f_C that maps $B(C) \cup G'(C)$ to itself and where $B(C)$ is mapped into $G'(C)$ and vice versa.The existence of f_C and $G'(C)$ is a consequence of the observation that our probability space is a non-atomic standard probability space (see, e.g. [19, pp. 42-43]). Then we put $X_1 = X_1^*$ and define X_2 by

$$X_2(\omega) = \begin{cases} X_2^*(\omega) & \text{if } \omega \notin B(C) \cup G'(C), \\ X_2^*(f_C(\omega)) & \text{if } \omega \in B(C) \cup G'(C). \end{cases}$$

In particular, the final coupling has the properties

(i) If $\omega \notin B(C) \cup G'(C)$, then $y_{1,j} = y_{2,j}$ for all $j \in C$ and there do not exist 16δ-unstable pairs in C.

(ii) If $\omega \in B(C)$ then $\sigma_{2,j} = 1$ for all $j \in C$.

(iii) If $\omega \in G'(C)$ then $\sigma_{1,j} = -1$ for all $j \in C$.

As the δ_2-bad components define a partition of $\{1, \dots, N\}$ and the cubes Q_{s_j}, $j \in \{1, \dots, N\}$ form a subdivision of $\mathbb{T}^2 \times [0, \lambda]$ the above construction yields the desired coupling. $\square$

The second step in the proof of Lemma 12 is to show that given a coupling as described in Lemma 14, the occurrence of a ψ_{X_1}-black horizontal crossing of R_1 implies the existence of a $\psi^{2\delta}$-black horizontal crossing of R_2 in the coloring associated with X_2. After having constructed the explicit coupling this is a completely elementary geometric (i.e., deterministic) problem. Unfortunately, its proof is nevertheless rather intricate so that we postpone it to Section 6. We now explain how Lemma 12 and the Friedgut-Kalai sharp-threshold result can be used to obtain a proof of $\theta(p) > 0$ for $p > 1/2$.

We begin by recalling the following variant (that appears in [9, Lemma 1]) of a sharp-threshold result due to Friedgut and Kalai, see [11, Theorem 2.1]. For $m, n \geq 1$ we say that an event $E \subset \{0, \pm 1\}^n$ has *symmetry of order* m if it is invariant under the action of a subgroup $\Gamma \subset \Sigma_n$ with the property that all of its orbits consist of at least m elements (here Σ_n denotes the symmetric group on n symbols). Furthermore, for $p_b, p_w \in (0,1)$ with $p_b + p_w < 1$ we write $\mathbb{P}_{p_b,p_w}$ for the product measure on the space $(\{0, \pm 1\}^n, \mathcal{P}(\{0, \pm 1\}^n))$ determined by the marginals $\mathbb{P}_{p_b,p_w}(\omega \in \{1\} \times \{0, \pm 1\}^{n-1}) = p_b$ and $\mathbb{P}_{p_b,p_w}(\omega \in \{-1\} \times \{0, \pm 1\}^{n-1}) = p_w$. Here $\mathcal{P}(\{0, \pm 1\}^n)$ denotes the family of all subsets of $\{0, \pm 1\}^n$.

Proposition 15. *There is an absolute constant $c_3 > 0$ such that if $0 < p_b < q_b < 1/e$ and $0 < q_w < p_w < 1/e$, if $E \subset \{0, \pm 1\}^n$ is increasing and has symmetry of order m, and if $\mathbb{P}_{p_b,p_w}(E) > \eta$, then $\mathbb{P}_{q_b,q_w}(E) > 1 - \eta$ whenever*

$$\min\{q_b - p_b, p_w - q_w\} \geq c_3 \log(1/\eta) p_{\max} \log(1/p_{\max}) / \log m,$$

where $p_{\max} = \max\{q_b, p_w\}$.

We apply this result in the following situation. Let $X = (z_n, t_n, \sigma_n)_{1 \leq n \leq N}$ be an independently $\{\pm 1\}$-marked homogeneous Poisson point process on $\mathbb{T}^2 \times [0, \lambda]$ with intensity 1 and $\mathbb{P}(\sigma_n = 1) = p$. Furthermore, write $Y_b = \{(z_n, t_n) : \sigma_n = 1\}_{(z_n,t_n,\sigma_n) \in X}$ and $Y_w = \{(z_n, t_n) : \sigma_n = -1\}_{(z_n,t_n,\sigma_n) \in X}$. Discretize $\mathbb{T}^2 \times [0, \lambda]$ into $n_0 = n_0(s) = 100 s^2 \lambda \delta^{-3}$ cubes Q_i of the form $Q_i = Q_\delta(z) \times [(k-1)\delta, k\delta]$ where $z \in \mathbb{T}^2 \cap \delta \mathbb{Z}^2$ and $k \in \{1, \dots, \lambda \delta^{-1}\}$. We call Q_i *white* if $Q_i \cap Y_w \neq \emptyset$ and *black* if $Q_i \cap Y_b \neq \emptyset$ and $Q_i \cap Y_w = \emptyset$. Otherwise (i.e., if $Q_i \cap Y = \emptyset$) we say that Q_i is *neutral.* Then the states of different cubes are independent and the probabilities that a fixed cube has a specified state are given by

$$\begin{aligned} p_{\text{white}} &= 1 - \exp(\delta^3(1-p)) \\ p_{\text{black}} &= \exp(-\delta^3(1-p))(1 - \exp(-\delta^3 p)) \\ p_{\text{neutral}} &= \exp(-\delta^3). \end{aligned}$$

In other words, this probability measure is a product measure on $\{0, \pm 1\}^{n_0}$ and our goal is to apply Proposition 15. The relevant subgroup $\Gamma \subset \Sigma_{n_0}$ is generated by the translations $(x, y, z) \mapsto (x + \delta, y, z)$ and $(x, y, z) \mapsto (x, y + \delta, z)$ of $(\mathbb{T}^2 \cap \delta\mathbb{Z}^2) \times ((0, \lambda] \cap \delta\mathbb{Z})$.

We claim that if $\omega \in \{0, \pm 1\}^{n_0}$ is a configuration that results from a realization $\varphi = \{(z_n, t_n, \sigma_n)\}_{1 \le n \le N}$ of X having the property that R_2 has a $\psi^{2\delta}$-black horizontal crossing, then there is a $\psi_{\varphi'}$-black horizontal crossing of R_2 for *any* $\varphi' \subset \mathbb{T}^2 \times [0, \lambda] \times \{\pm 1\}$ that induces the same discrete configuration $\omega \in \{0, \pm 1\}^{n_0}$ as φ. This may be seen as follows. Any discrete configuration $\omega \in \{0, \pm 1\}^{n_0}$ determines a coloring ψ_ω. Namely define

$$\varphi_\omega = \{(z, t + (\omega(z,t) - 1)\delta/2, \omega(z,t))\}_{(z,t) \in (\delta\mathbb{Z}^2 \cap \mathbb{T}^2) \times (\delta\mathbb{Z} \cap (0,\lambda]), \omega(z,t) \in \{\pm 1\}},$$

where the mark of $(z, t + (\omega(z,t) - 1)\delta/2, \omega(z,t))$ is given by $Q_{1 - 2\omega(z,t)\delta}(o))$ (and also put $\psi_\omega = \psi_{\varphi_\omega}$). We claim that

(i) $\psi_{\varphi'}$ black-dominates ψ_ω and that

(ii) ψ_ω black-dominates $\psi_{\varphi^{2\delta}}$.

To prove the first claim, let $\zeta \in \mathbb{R}^2$ with $\psi_{\varphi'}(\zeta) = -1$ be arbitrary. Next, choose $(z'_{n_1}, t'_{n_1}, -1) \in \varphi'$ with $\zeta \in Q\left(z'_{n_1}\right)$ and such that t'_{n_1} is minimal with this property. Also choose $(z, t) \in \delta\mathbb{Z}^2 \times (\delta\mathbb{Z} \cap (0, \lambda])$ with $\left(z'_{n_1}, t'_{n_1}\right) \in Q_\delta(z) \times [t - \delta, t]$. Now suppose that $\left(\widetilde{z}, \widetilde{t}\right) \in \delta\mathbb{Z}^2 \times (\delta\mathbb{Z} \cap (0, \lambda])$ is such that $\omega\left(\widetilde{z}, \widetilde{t}\right) = 1$ and such that $\zeta \in Q_{1-2\delta}(\widetilde{z})$. Then there exists $\left(z'_{n_2}, t'_{n_2}, 1\right) \in \varphi'$ with $\left(z'_{n_2}, t'_{n_2}\right) \in Q_\delta\left(\widetilde{z}\right) \times \left[\widetilde{t} - \delta, \widetilde{t}\right]$. From $\zeta \in Q\left(z'_{n_2}\right)$ we conclude $t'_{n_2} > t'_{n_1}$ so that $t - \delta \le t'_{n_1} < t'_{n_2} \le \widetilde{t}$.

We proceed similarly for the second claim. Let $\zeta \in \mathbb{R}^2$ with $\psi_{\varphi^{2\delta}}(\zeta) = 1$ be arbitrary. Next, choose $(z_{n_1}, t_{n_1}, 1) \in \varphi'$ with $\zeta \in Q_{1-4\delta}\left(z_{n_1}\right)$ and such that t_{n_1} is minimal with this property. Also choose $(z, t) \in \delta\mathbb{Z}^2 \times (\delta\mathbb{Z} \cap (0, \lambda])$ with $(z_{n_1}, t_{n_1}) \in Q_\delta(z) \times [t - \delta, t]$. Now suppose that $\left(\widetilde{z}, \widetilde{t}\right) \in \delta\mathbb{Z}^2 \times (\delta\mathbb{Z} \cap (0, \lambda])$ is such that $\omega\left(\widetilde{z}, \widetilde{t}\right) = -1$ and such that $\zeta \in Q_{1+2\delta}(\widetilde{z})$. Then there exists $(z_{n_2}, t_{n_2}, -1) \in \varphi$ with $(z_{n_2}, t_{n_2}) \in Q_\delta\left(\widetilde{z}\right) \times \left[\widetilde{t} - \delta, \widetilde{t}\right]$. From $\zeta \in Q_{1+4\delta}\left(z_{n_2}\right)$ we conclude $t_{n_2} - 2\delta > t_{n_1} + 2\delta$ so that $t \le t_{n_1} + 2\delta < t_{n_2} - 2\delta \le \widetilde{t} - \delta$.

Therefore, the existence of a $\psi^{2\delta}$-black horizontal crossing of R with respect to the coloring defined by φ implies the existence of a black horizontal crossing of R in the coloring defined by φ'. After these preliminaries the proof of $\theta(p) > 0$ for $p > 1/2$ is now very similar to [5, Theorem 1.1] and [6, Theorem 17] and we only present the main ideas.

Theorem 16. *Let $p > 1/2$ be arbitrary. Then $\theta(p) > 0$.*

Proof. Let $p > 1/2$ be arbitrary and consider confetti percolation on $\mathbb{R}^2$ with parameter p. As in [6, Theorem 17] it suffices to prove that for all $s_0 > 0$ there exists $s \ge s_0$ such that the probability of obtaining a black horizontal crossing in a fixed $(3s \times s)$-rectangle is at least 0.99.

Let $c > 0$ be an absolute (large) constant whose precise value will be specified later and write $\gamma = (p - 1/2)/c$. By Proposition 8 there exists $c_0 > 0$ with $\mathbb{P}_p(H([0, 9s] \times [0, s])) \ge c_0$ for arbitrarily high values of s. We write $R_0 = [0, 9s] \times [0, s]$ for a fixed $(9s \times s)$-rectangle and E_1 for the event that a ψ-black horizontal crossing occurs in R_0. Embedding R_0 in $\mathbb{T}^2$ yields $\mathbb{P}^{\mathbb{T}^2}_{1/2}(E_1) \ge c_0$. Note that the event E_1 implies the existence of a ψ-black horizontal crossing of R_1. We denote by E_2 the event that there exists a $\psi^{2\delta}$-black horizontal crossing of R_2 and write $p' = (p + 1/2)/2$. Then we may apply Lemma 12 with $p = 1/2$ and p'. As the global event E_g described in this lemma holds whp, we see that the following bound is true for all sufficiently large values of s:

$$\mathbb{P}^{\mathbb{T}^2}_{p'}(E_2) \ge \mathbb{P}^{\mathbb{T}^2}_{1/2}(E_1) - \mathbb{P}_{1/2}(E_g^c) \ge c_0/2. \tag{9}$$

Furthermore, we denote by E_3 the event in the probability space $(\{0, \pm 1\}^n, \mathcal{P}(\{0, \pm 1\}^n))$ that *some* $(8s \times 2s)$-rectangle in the discrete torus $\delta\mathbb{Z}^2 \cap \mathbb{T}^2$ admits a ψ_ω-black horizontal crossing.

By the discussion preceding the theorem, (9) implies $\mathbb{P}_{p'}^{\mathbb{T}^2}(E_3) \ge c_0/2$. Note that E_3 is a black-increasing event that has symmetries of order $m = (10s/\delta)^2$. Let $\varepsilon = \min\{c_0/2, 10^{-100}\}$. Then similar to [5, Theorem 17] for $\delta = (4\lceil s^\gamma/4\rceil)^{-1}$ with $\gamma > 0$ sufficiently small, one may apply Proposition 15 to obtain

$$\mathbb{P}_p(E_3) \ge 1 - \varepsilon \ge 1 - 10^{-100}.$$

If we denote by E_4 the event that some $(8s \times 2s)$-rectangle in $\mathbb{T}^2$ has a black horizontal crossing, then we obtain from the discussion preceding Theorem 16 that $E_3 \subset E_4$. Now we can follow [5, Theorem 17] verbatim to see that the occurrence of a black horizontal crossing of a fixed $(12s \times 4s)$-rectangle has probability at least 0.99. □

6 Proof of Lemma 12

In this section we provide a proof for the elementary geometric claim in Lemma 12 that was postponed in Section 5. The situation is as follows. For $i \in \{1, 2\}$ we are given finite sets $\{(y_{i,n}, \sigma_{i,n})\}_{1\le n\le N} \subset \mathbb{T}^2 \times [0, \lambda] \times \{\pm 1\}$.

(i) $|y_{1,j} - y_{2,j}|_\infty \le \delta_1$ for all $j \in \{1, \dots, N\}$,

(ii) $\sigma_{1,j} \le \sigma_{2,j}$ for all $j \in \{1, \dots, N\}$,

(iii) for all potentially δ_2-bad components $C \subset \{1, \dots, N\}$ the existence of $a, b \in C$ with $\sigma_{1,a} = 1$ and $\sigma_{2,b} = -1$ implies that $B(C)$ does not occur and that $y_{1,j} = y_{2,j}$ holds for all $j \in C$.

Furthermore, without loss of generality, we may assume $C_s^{(1)}$, where we recall from Section 5 that this event is defined by $\mathbb{T}^2 \subset \bigcup_{z\in Z} Q_{1/2}(z)$. Our goal is to prove that properties (i)-(iii) above imply that if there exists a black horizontal crossing of $R_1 = [0.25s, 8.75s] \times [-0.25s, 1.25s]$ in the confetti process induced by $\varphi_1 = \{(y_{1,n}, \sigma_{1,n})\}_{1\le n\le N}$, then there exists a $\psi^{2\delta}$-black horizontal crossing of $R_2 = [0.5s, 8.5s] \times [-0.5s, 1.5s]$ in the confetti process induced by $\varphi_2 = \{(y_{2,n}, \sigma_{2,n})\}_{1\le n\le N}$.

For $C \subset \{1, 2, \dots, N\}$ we denote by $D(C)$ the event that there exist $n_1, n_2 \in C$ with $\sigma_{1,n_1} = 1$ and $\sigma_{2,n_2} = -1$. Define a sequence $\Delta = \{\delta^{(n)}\}_{1\le n\le N}$ by $\delta^{(n)} = \delta_1 + 4\delta$ if n is contained in a potentially δ_2-bad component C such that $D(C)$ does not occur and put $\delta^{(n)} = 4\delta$ otherwise. By Lemma 14 the occurrence of $D(C)$ implies the non-existence of $m_1, m_2 \in C$ such that $\{y_{1,m_1}, y_{1,m_2}\}$ forms a spatially 16δ-unstable pair. Next, we consider two results formalizing the intuition that $\delta^{(n)}$-instabilities can only occur under severe restrictions on the leaf colors.

Lemma 17. *Let $n_1, n_2 \in \{1, \dots, N\}$ and suppose that $\{z_{1,n_1}, z_{1,n_2}\}$ forms a spatially $(2\delta^{(n_1)} + 2\delta^{(n_2)})$-unstable pair. Denote by C the potentially δ_2-bad component containing n_1 and n_2. Then $D(C)$ does not occur. In particular, $\sigma_{1,n_1} = \sigma_{2,n_2}$.*

Proof. Assume the contrary, i.e., that $D(C)$ occurs. Then $\delta^{(n_1)} = \delta^{(n_2)} = 4\delta$, so that $\{z_{1,n_1}, z_{1,n_2}\}$ forms a spatially 16δ-unstable pair contradicting the occurrence of $D(C)$. □

Lemma 18. *Let $m, n \in \{1, \dots, N\}$ be such that $|z_{1,m} - z_{1,n}|_\infty \le 2$ and $t_{1,m} - \delta^{(m)} < t_{1,n} + \delta^{(n)}$. Denote by $\widetilde{C} \subset \{1 \dots, N\}$ the union of the potentially δ_2-bad connected components containing m and n. Furthermore, assume the existence of $m', n' \in \widetilde{C}$ with $\sigma_{1,m'} = 1$ and $\sigma_{2,n'} = -1$. Then $t_{1,m} < t_{1,n}$.*

Proof. Assume the contrary. Then $|t_{1,m} - t_{1,n}| \le \delta^{(m)} + \delta^{(n)} \le \delta_2$ so that m and n are contained in the same potentially δ_2-bad component $C \subset \{1, \dots, N\}$. In particular, the existence of $m', n' \in C$ with $\sigma_{1,m'} = 1$ and $\sigma_{2,n'} = -1$ implies $D(C)$. But then $\delta^{(m)} = \delta^{(n)} = 4\delta$, so that $\{t_{1,m}, t_{1,n}\}$ forms a temporally 8δ-unstable pair. This contradicts the discussion preceding Lemma 17. □

To obtain a $\psi_{\varphi_2^{2\delta}}$-black horizontal crossing of R_2 we construct a further coloring ψ^Δ that is black dominated by $\psi_{\varphi_2^{2\delta}}$ and show the existence of a black horizontal crossing of R_2 in this new coloring. To be more precise, we define

$$\varphi^\Delta = \{\widetilde{x_n}\}_{1 \le n \le N} = \{(z_{1,n}, t_{1,n} + \delta^{(n)}\sigma_{2,n}, \sigma_{2,n})\}_{1 \le n \le N},$$

where the mark of $(z_{1,n}, t_{1,n}+\delta^{(n)}\sigma_{2,n}, \sigma_{2,n})$ is given by $Q_{1-2\sigma_{2,n}\delta^{(n)}}(o)$ (and also put $\psi^\Delta = \psi_{\varphi^\Delta}$). In other words, the shrinking/expansion and delay/advancement is more pronounced for leaves contained in a potentially δ_2-bad connected component C for which the event $D(C)$ does not occur.

We claim that the coloring $\psi_{\varphi_2^{2\delta}}$ black-dominates the coloring ψ^Δ, so that we only need to establish the existence of a ψ^Δ-black horizontal crossing of R_2. To prove this claim, it suffices to show that for every $n \in \{1, \dots, N\}$,

(i) $Q_{1-2\delta^{(n)}}(z_{1,n}) \subset Q_{1-4\delta}(z_{2,n})$ and $t_{1,n} + \delta^{(n)} \ge t_{2,n} + 2\delta$ if $\sigma_{2,n} = 1$ and

(ii) $Q_{1+2\delta^{(n)}}(z_{1,n}) \supset Q_{1+4\delta}(z_{2,n})$ and $t_{1,n} - \delta^{(n)} \le t_{2,n} - 2\delta$ if $\sigma_{2,n} = -1$.

Denote by C the potentially δ_2-bad connected component containing n. To prove the first claim, we distinguish two cases. If $D(C)$ occurs, then $\delta^{(n)} = 4\delta$ and $y_{1,n} = y_{2,n}$, so that the claim follows from the obvious relations $Q_{1-8\delta}(z_{1,n}) \subset Q_{1-4\delta}(z_{1,n})$ and $t_{1,n} + 4\delta \ge t_{1,n} + 2\delta$. Similarly, if $D(C)$ does not occur, then $\delta^{(n)} = \delta_1 + 4\delta$ and $|y_{1,n} - y_{2,n}|_\infty \le \delta_1$, so that the claim follows from $Q_{1-2\delta_1-8\delta}(z_{1,n}) \subset Q_{1-4\delta}(z_{2,n})$ and $t_{1,n} + \delta_1 + 4\delta \ge t_{2,n} + 4\delta$. The second claim can be proven by analogous arguments.

Furthermore, for $n \in \{1, \dots, N\}$ we write $\varphi_1(n) = \{x_{1,n}\} \cup \{x_{1,m} \in \varphi_1 : \sigma_{2,m} = -1,\ t_{1,m} < t_{1,n}\}$ and similarly $\varphi^\Delta(n) = \{\widetilde{x_n}\} \cup \{\widetilde{x_m} \in \varphi^\Delta : \sigma_{2,m} = -1,\ t_{1,m} - \delta^{(m)} < t_{1,n} + \delta^{(n)}\}$ (and also put $\psi^\Delta(n) = \psi_{\varphi^\Delta(n)}$). We need a result to create $\varphi^\Delta(n)$-black paths from $\varphi_1(n)$-black paths.

Lemma 19. *Let $n \in \{1, \dots, N\}$ with $\sigma_{1,n} = 1$ and let $P_1, P_2 \in Q_{1-2\delta^{(n)}}(z_{1,n})$ be such that*

$$height_{\varphi^\Delta(n)}(P_1) = height_{\varphi^\Delta(n)}(P_2) = t_{1,n} + \delta^{(n)}.$$

Assume the existence of $P_1', P_2' \in Q(z_{1,n})$, $m_1, m_2 \in \{1, \dots, N\}$ and $i_1, i_2 \in \{1, 2\}$ such that $\sigma_{1,m_1} = \sigma_{1,m_2} = 1$ and such that the following items hold for all $k \in \{1, 2\}$.

(i) $P_k' \in \partial Q(z_{1,m_k})$ and $|\pi_{i_k}(P_k') - \pi_{i_k}(z_{1,m_k})| = 1/2$,

(ii) $|\pi_{i_k}(P_k') - \pi_{i_k}(P_k)| \le \delta^{(m_k)}$.

Assume furthermore the existence of a $\psi_{\varphi_1(n)}$-black path $\gamma' : [0,1] \to \mathbb{T}^2$ with $\gamma'(0) = P_1'$, $\gamma'(1) = P_2'$. Then there exists a $\psi^\Delta(n)$-black path $\gamma : [0,1] \to \mathbb{T}^2$ with $\gamma(0) = P_1$ and $\gamma(1) = P_2$.

Proof. Without loss of generality we may assume that P_1 lies to the north-west of P_2. Assume the assertion of the lemma was wrong. This implies that there exist two white leaves covering the north-east and the south-west corner of the square $Q(z_{1,n})$ and that prevent the existence of a $\psi^\Delta(n)$-black path connecting P_1 and P_2. To be more precise, we obtain the existence of $m_{ne}, m_{sw} \in \{1, \dots, N\}$ with

(i) $\sigma_{2,m_{ne}} = \sigma_{2,m_{sw}} = -1$,

(ii) $\max(t_{1,m_{ne}} - \delta^{(m_{ne})}, t_{1,m_{sw}} - \delta^{(m_{sw})}) < t_{1,n} + \delta^{(n)}$,

(iii) $\pi_1(z_{1,m_{ne}}) - (\delta^{(m_{ne})} + 1/2) \in (\pi_1(P_1), \pi_1(P_2))$,

(iv) $\pi_2(z_{1,m_{ne}}) - (\delta^{(m_{ne})} + 1/2) \in (\pi_2(P_2), \pi_2(P_1))$,

(v) $\pi_1(z_{1,m_{sw}}) + (\delta^{(m_{sw})} + 1/2) \in (\pi_1(P_1), \pi_1(P_2))$,

(vi) $\pi_2(z_{1,m_{sw}}) + (\delta^{(m_{sw})} + 1/2) \in (\pi_2(P_2), \pi_2(P_1))$,

(vii) $|z_{1,m_{ne}} - z_{1,m_{sw}}|_\infty < 1 + \delta^{(m_{ne})} + \delta^{(m_{sw})}$.

(see also Figure 5, left, after replacing $Q(z_{1,m_{ne}})$ and $Q(z_{1,m_{sw}})$ by $Q_{1+2\delta^{(m_{ne})}}(z_{1,m_{ne}})$ and $Q_{1+2\delta^{(m_{sw})}}(z_{1,m_{sw}})$, respectively). Observe that by Lemma 18 we have $\max(t_{1,m_{ne}}, t_{1,m_{sw}}) < t_{1,n}$. We distinguish two cases.

First, suppose $|z_{1,m_{sw}} - z_{1,m_{ne}}|_\infty < 1$, see Figure 5, left. We claim that P_1 and P_1' are contained in the same connected component of $Q(z_{1,n}) \setminus (Q(z_{1,m_{ne}}) \cup Q(z_{1,m_{sw}}))$ (observe that this set consists of precisely two connected components since we assumed $|z_{1,m_{sw}} - z_{1,m_{ne}}|_\infty < 1$). Let us assume the contrary and furthermore – without loss of generality – that $i_1 = 1$. Since P_1' lies in the same connected component of $Q(z_{1,n}) \setminus (Q(z_{1,m_{ne}}) \cup Q(z_{1,m_{sw}}))$ as P_2 we have

$$\pi_1(P_1') - (\pi_1(z_{1,m_{ne}}) - 1/2) \geq \pi_1(P_1') - (\pi_1(z_{1,m_{sw}}) + 1/2) \geq 0,$$

and also

$$\pi_1(P_1') - (\pi_1(z_{1,m_{ne}}) - 1/2) = \pi_1(P_1') - \pi_1(P_1) + \pi_1(P_1) - (\pi_1(z_{1,m_{ne}}) - 1/2) \leq \delta^{(m_1)} + 0.$$

Hence, $\{z_{1,m_1}, z_{1,m_{ne}}\}$ forms a spatially $\delta^{(m_1)}$-unstable pair, thereby contradicting Lemma 17. Therefore, P_1 and P_1' lie in the same connected component of $Q(z_{1,n}) \setminus (Q(z_{1,m_{ne}}) \cup Q(z_{1,m_{sw}}))$. Of course, the same is true for P_2 and P_2'. But this yields a contradiction to the existence of a $\psi_{\varphi_1(n)}$-black path connecting P_1' and P_2'.

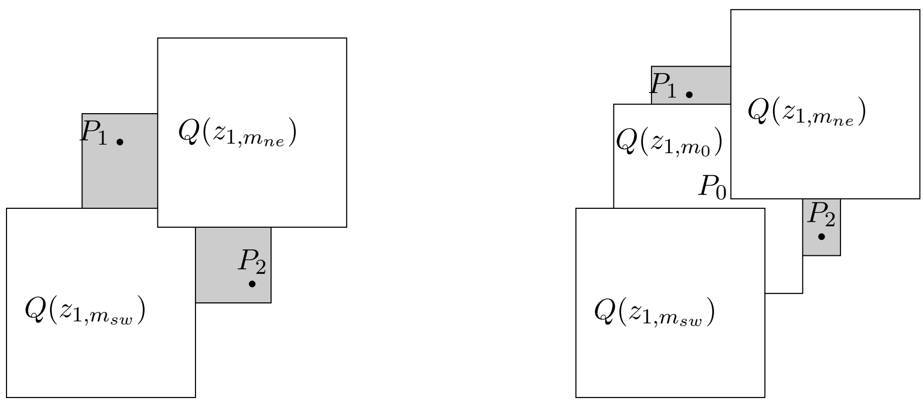

Figure 5: Configurations in the proof of Lemma 19

The reasoning in the case $|z_{1,m_{sw}} - z_{1,m_{ne}}|_\infty \geq 1$ is very similar, but we provide the details for the convenience of the reader. The first important observation is that $\{z_{1,m_{ne}}, z_{1,m_{sw}}\}$ forms a spatially $(\delta^{(m_{ne})} + \delta^{(m_{sw})})$-unstable pair. Next, we denote by P_0 the south-west corner of the square $Q(z_{1,m_{ne}})$. Then there exists a unique $m_0 \in \{1, \ldots, N\}$, such that $t_{1,m_0} = \mathit{height}_{(\varphi_1)_{\delta_2}^{y_{1,m_{ne}}} \setminus \{x_{1,m_{ne}}\}}(P_0)$. In particular, y_{1,m_0} is boundary-visible from $y_{1,m_{ne}}$ (apply the definition of boundary-visibility to the corner P_0 and any point $P \in \partial Q(z_{m_{1,ne}}) \cap \partial Q(z_{1,m_0})$). For an illustration of the situation see Figure 5, right. By Lemma 17 we may assume $\sigma_{1,m_0} = -1$. Note that $Q(z_{1,n}) \setminus (Q(z_{1,m_0}) \cup Q(z_{1,m_{ne}}) \cup Q(z_{1,m_{sw}}))$ consists of at most two connected components and we first assume that P_1' and P_2' both lie in the connected component containing the south-east corner of $Q(z_{1,n})$. Then we have again

$$\pi_1(P_1') - \pi_1(z_{1,m_{ne}}) + 1/2 \geq \pi_1(P_1') - (\min(\pi_1(z_{1,m_{sw}}), \pi_1(z_{1,m_0})) + 1/2) - \delta^{(m_{ne})} - \delta^{(m_{sw})},$$

and the latter expression is bounded from below by $-\delta^{(m_{ne})} - \delta^{(m_{sw})}$. Furthermore, note

$$\pi_1(P_1') - \pi_1(z_{1,m_{ne}}) + 1/2 = \pi_1(P_1') - \pi_1(P_1) + \pi_1(P_1) - (\pi_1(z_{1,m_{ne}}) - 1/2) \leq \delta^{(m_1)} + 0.$$

From $\delta^{(m_{ne})} = \delta^{(m_{sw})}$ we conclude that $\{z_{1,m_{ne}}, z_{1,m_1}\}$ forms a spatially $(2\delta^{(m_1)} + 2\delta^{(m_{ne})})$-unstable pair, thereby contradicting Lemma 17. Since analogous arguments can be used to arrive at a contradiction in the case, where P_1' and P_2' both lie in the connected component containing the north-west corner of $Q(z_{1,n})$, this concludes the proof. $\square$

We finally need a result allowing us to glue together paths obtained from Lemma 19.

Lemma 20. *Let $n_1, n_2 \in \{1, \dots, N\}$ be such that $\sigma_{1,n_1} = \sigma_{1,n_2} = 1$ and $t_{1,n_1} < t_{1,n_2}$. Furthermore, let $P \in \partial Q(z_{1,n_1})$ with $\mathit{height}_{\varphi_1}(P) = t_{1,n_1}$ and $\mathit{height}_{\varphi_1 \setminus \{x_{1,n_1}\}}(P) = t_{1,n_2}$. Then there exist $P_i \in \mathbb{R}^2$, $i \in \{1,2\}$ with the following properties.*

(i) $P_i \in Q_{1-2\delta^{(n_i)}}(z_{1,n_i})$ *for all* $i \in \{1,2\}$,

(ii) P_i *is* $\psi^\Delta(n_i)$*-black for all* $i \in \{1,2\}$,

(iii) $|\pi_k(P_i) - \pi_k(P)| \le \delta^{(n_1)}$ *holds for all* $i \in \{1,2\}$*, where the index* $k \in \{1,2\}$ *is such that* $|\pi_k(P) - \pi_k(z_{1,n_1})| = 1/2$,

(iv) the linear segment $[P_1, P_2]$ *is* ψ^Δ*-black.*

Proof. Without loss of generality we may assume that z_{1,n_2} lies to the south-east of z_{1,n_1} and that P lies on a right vertical boundary of $Q(z_{1,n_1})$. We distinguish 5 cases. The first two cases are devoted to configurations, where we can use arguments exploiting the $(\delta^{(n_1)} + \delta^{(n_2)})$-instability of $\{z_{1,n_1}, z_{1,n_2}\}$ and the remaining cases consider configurations, where this property does not occur necessarily.

Case 1*a*. Denote by $P_0 = z_{1,n_1} + e_1/2 - e_2/2$ the south-eastern corner of the square $Q(z_{1,n_1})$. First, assume that $\{z_{1,n_1}, z_{1,n_2}\}$ forms a spatially $(\delta^{(n_1)} + \delta^{(n_2)})$-unstable pair. Assume additionally that there does *not* exist $n_3 \in \{1, \dots, N\}$ with $t_{1,n_3} < t_{1,n_2}$, $|\pi_1(P) - \pi_1(z_{1,n_3})| \le 1/2$, $\pi_2(z_{1,n_3}) + 1/2 \in (\pi_2(z_{1,n_1}) - 1/2, \pi_2(P))$ and such that $\{z_{1,n_1}, z_{1,n_3}\}$ does *not* form a $(\delta^{(n_1)} + \delta^{(n_3)})$-unstable pair. For instance, the non-existence of n_3 is always satisfied if $\pi_2(z_{1,n_1}) > \pi_2(z_{1,n_2}) + 1 - \delta^{(n_1)} - \delta^{(n_2)}$. In particular, (due to $C_s^{(1)}$) there exists $n_4 \in \{1, \dots, N\}$ with $t_{1,n_4} = \mathit{height}_{(\varphi_1)_{\delta_2}^{y_{1,n_1}} \setminus \{x_{1,n_1}\}}(P_0)$. Hence, y_{1,n_4} is boundary-visible from y_{1,n_1} and $\{z_{1,n_1}, z_{1,n_2}, z_{1,n_4}\}$ are contained in the same potentially δ_2-bad connected component. In particular, we conclude from Lemma 17 that $\sigma_{2,n_4} = 1$. For $j \in \{1,2\}$ define $\xi_j = (1 - 2 \cdot 1_{\pi_j(P_0) > \pi_j(z_{1,n_2})})$. Moreover, we put $P_1 = (\pi_1(P) - \delta^{(n_1)})e_1 + (\pi_2(z_{1,n_1}) - 1/2 + \delta^{(n_1)})e_2$ and $P_2 = (\pi_1(P) + \xi_1 \delta^{(n_1)})e_1 + (\pi_2(z_{1,n_1}) - 1/2 + \xi_2 \delta^{(n_1)})e_2$. See Figure 6 for an illustration.

(i) $P_i \in Q_{1-2\delta^{(n_i)}}(z_{1,n_i})$ is clear.

(ii) To show that P_i is $\psi^\Delta(n_i)$-black assume for the sake of contradiction the existence of $m \in \{1, \dots, N\}$ with $t_{1,m} - \delta^{(m)} < t_{1,n_i} + \delta^{(n_i)}$, $\sigma_{2,m} = -1$ and $P_i \in Q_{1+2\delta^{(m)}}(z_{1,m})$. Observe that by Lemma 18 we have $t_{1,m} < t_{1,n_i}$. By Lemma 17 and the non-existence of n_3, we see that this is impossible.

(iii) $|\pi_1(P) - \pi_1(P_i)| \le \delta^{(n_1)}$ is clear.

(iv) Suppose we could find $P' \in [P_1, P_2]$ and $m \in \{1, \dots, N\}$ with $t_{1,m} - \delta^{(m)} < t_{1,n_4} + \delta^{(n_4)}$, $\sigma_{2,m} = -1$ and such that $P' \in Q_{1+2\delta^{(m)}}(z_{1,m})$. As n_2, n_4 are contained in the same potentially δ_2-bad component we conclude from Lemma 18 that $t_{1,m} < t_{1,n_4}$. But then either $\{z_{1,m}, z_{1,n_1}\}$ forms a spatially δ_2-unstable pair (yielding a contradiction to Lemma 17) or we obtain a contradiction to the assumption $t_{1,n_4} = \mathit{height}_{(\varphi_1)_{\delta_2}^{y_{1,n_1}} \setminus \{x_{1,n_1}\}}(P_0)$.

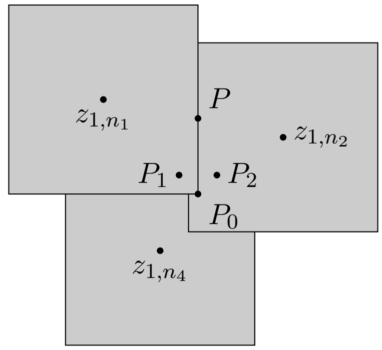

Figure 6: Configuration in case $1a$

Case $1b$. We assume $\pi_1(z_{1,n_2}) - \pi_1(z_{1,n_1}) \in (0, \delta^{(n_1)} + \delta^{(n_2)}) \cup (1 - \delta^{(n_1)} - \delta^{(n_2)}, 1)$. However, now assume additionally the existence of $n_3 \in \{1, \dots, N\}$ with $t_{1,n_3} < t_{1,n_2}$, $|\pi_1(P) - \pi_1(z_{1,n_3})| \le 1/2$, $\pi_2(z_{1,n_3}) + 1/2 \in (\pi_2(z_{1,n_1}) - 1/2, \pi_2(P))$ and such that $\{z_{1,n_1}, z_{1,n_3}\}$ does *not* form a $(\delta^{(n_1)} + \delta^{(n_3)})$-unstable pair. Among all these values choose n_3 with the property that $\pi_2(z_{1,n_3})$ is maximal. Observe that $\{z_{1,n_1}, z_{1,n_3}\}$ either forms a spatially δ_2-unstable pair or $\{y_{1,n_1}, y_{1,n_2}, y_{1,n_3}\}$ forms a δ_2-unstable triple. Therefore, Lemma 17 implies $\sigma_{2,n_3} = 1$. Define $\xi_1 = (1 - 2\cdot 1_{\pi_1(P_0) > \pi_1(z_{1,n_2})})$. Moreover, we put $P_1 = (\pi_1(P) - \delta^{(n_1)})e_1 + (\pi_2(z_{1,n_3}) + 1/2 - \delta^{(n_3)})e_2$ and $P_2 = (\pi_1(P) + \xi_1 \delta^{(n_1)})e_1 + (\pi_2(z_{1,n_3}) + 1/2 - \delta^{(n_3)})e_2$. See Figure 7 for an illustration.

(i) $P_i \in Q_{1-2\delta^{(n_i)}}(z_{1,n_i})$ is clear.

(ii) To show that P_i is $\psi^\Delta(n_i)$-black assume for the sake of contradiction the existence of $m \in \{1, \dots, N\}$ with $t_{1,m} - \delta^{(m)} < t_{1,n_i} + \delta^{(n_i)}$, $\sigma_{2,m} = -1$ and $P_i \in Q_{1+2\delta^{(m)}}(z_{1,m})$. Observe that by Lemma 18 we have $t_{1,m} < t_{1,n_2}$. By Lemma 17 and the choice of n_3, we see that this is only possible if $\pi_2(z_{1,m}) \in (\pi_2(z_{1,n_3}) - \delta^{(m)} - \delta^{(n_3)}, \pi_2(z_{1,n_3}))$. But then again n_1, n_2, n_3 and m are contained in the same potentially δ_2-bad connected component C, thereby yielding a contradiction to Lemma 17.

(iii) $|\pi_1(P) - \pi_1(P_i)| \le \delta^{(n_1)}$ is clear.

(iv) Suppose we could find $P' \in [P_1, P_2]$ and $m \in \{1, \dots, N\}$ with $t_{1,m} - \delta^{(m)} < t_{1,n_3} + \delta^{(n_3)}$, $\sigma_{2,m} = -1$ and $P' \in Q_{1+2\delta^{(m)}}(z_{1,m})$. As n_2, n_3 are contained in the same potentially δ_2-bad component we conclude from Lemma 18 that $t_{1,m} < t_{1,n_3}$. Again by Lemma 17 and the choice of n_3, we see that this is only possible if $\pi_2(z_{1,m}) \in (\pi_2(z_{1,n_3}) - (\delta^{(n_3)} + \delta^{(m)}), \pi_2(z_{1,n_3}))$. But again this would yield a contradiction to Lemma 17, since then n_1, n_2, n_3, m would be contained in the same potentially δ_2-bad component.

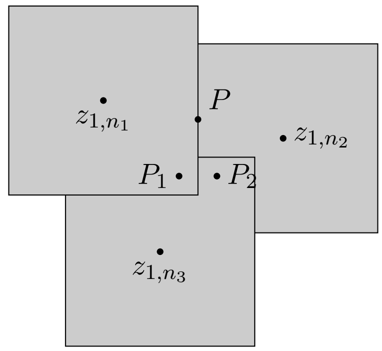

Figure 7: Configuration in case $1b$

Case $2a$**.** Henceforth, we may assume $\pi_1(z_{1,n_2}) - \pi_1(z_{1,n_1}) \in (\delta^{(n_1)} + \delta^{(n_2)}, 1 - \delta^{(n_1)} - \delta^{(n_2)})$ and – due to case $1a$ – that $\pi_2(z_{1,n_1}) - 1/2 < \pi_2(z_{1,n_2}) + 1/2 - \delta^{(n_1)} - \delta^{(n_2)}$. Furthermore, assume that there does *not* exist $n_3 \in \{1, \ldots, N\}$ with $\sigma_{2,n_3} = -1$, $t_{1,n_3} < t_{1,n_2}$, $|\pi_1(P) - \pi_1(z_{1,n_3})| \leq 1/2$, $\pi_2(z_{1,n_3}) + 1/2 \in (\pi_2(z_{1,n_1}) - 1/2, \pi_2(P))$ and such that $\{z_{1,n_1}, z_{1,n_3}\}$ does *not* form a $(\delta^{(n_1)} + \delta^{(n_3)})$-unstable pair. Observe that in contrast to cases $1a$ and $1b$, we require also $\sigma_{2,n_3} = -1$. Then we define $P_1 = (\pi_1(P) - \delta^{(n_1)})e_1 + (\pi_2(z_{1,n_1}) - 1/2 + \delta^{(n_1)})e_2$ and $P_2 = (\pi_1(P) + \delta^{(n_1)})e_1 + (\pi_2(z_{1,n_1}) - 1/2 + \delta^{(n_1)})e_2$. See Figure 8 for an illustration.

(i) $P_1 \in Q_{1-2\delta^{(n_1)}}(z_{1,n_1})$ is clear and $P_2 \in Q_{1-2\delta^{(n_2)}}(z_{1,n_2})$ follows from our assumption $\pi_2(z_{1,n_1}) - 1/2 + \delta^{(n_1)} < \pi_2(z_{1,n_2}) + 1/2 - \delta^{(n_2)}$. For the proof of assertion (iv) below it is also useful to note that $P_1 \in Q_{1-2\delta^{(n_2)}}(z_{1,n_2})$.

(ii) To show that P_i is $\psi^{\Delta}(n_i)$-black assume for the sake of contradiction the existence of $m \in \{1, \ldots, N\}$ with $t_{1,m} - \delta^{(m)} < t_{1,n_i} + \delta^{(n_i)}$, $\sigma_{2,m} = -1$ and $P_i \in Q_{1+2\delta^{(m)}}(z_{1,m})$. Observe that by Lemma 18 we have $t_{1,m} < t_{1,n_2}$. By Lemma 17 and the non-existence of n_3, we see that this is impossible.

(iii) $|\pi_1(P) - \pi_1(P_i)| \leq \delta^{(n_1)}$ is clear.

(iv) Suppose we could find $P' \in [P_1, P_2]$ and $m \in \{1, \ldots, N\}$ with $t_{1,m} - \delta^{(m)} < \max(t_{1,n_1} + \delta^{(n_1)}, t_{1,n_2} + \delta^{(n_2)})$, $\sigma_{2,m} = -1$ and $P' \in Q_{1+2\delta^{(m)}}(z_{1,m})$. We conclude from Lemma 18 that $t_{1,m} < t_{1,n_2}$. Again by Lemma 17 and the non-existence of n_3, we see that this is impossible.

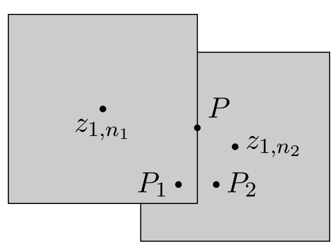

Figure 8: Configuration in case $2a$

Case $2b$**.** Henceforth, we may assume $\pi_1(z_{1,n_2}) - \pi_1(z_{1,n_1}) \in (\delta^{(n_1)} + \delta^{(n_2)}, 1 - \delta^{(n_1)} - \delta^{(n_2)})$ and – due to case $1a$ – that $\pi_2(z_{1,n_1}) - 1/2 < \pi_2(z_{1,n_2}) + 1/2 - \delta^{(n_1)} - \delta^{(n_2)}$. Furthermore, we may assume the existence of $n_3 \in \{1, \ldots, N\}$ with $\sigma_{2,n_3} = -1$, $t_{1,n_3} < t_{1,n_2}$, $|\pi_1(P) - \pi_1(z_{1,n_3})| \leq 1/2$, $\pi_2(z_{1,n_3}) + 1/2 \in (\pi_2(z_{1,n_1}) - 1/2, \pi_2(P))$ and such that $\{z_{1,n_1}, z_{1,n_3}\}$ does *not* form a

$(\delta^{(n_1)} + \delta^{(n_3)})$-unstable pair. Among all those values choose n_3 such that $\pi_2(z_{1,n_3}) + \delta^{(n_3)}$ is maximal. Furthermore, in case $2b$ we also assume that there does *not* exist $n_4 \in \{1,\dots,N\}$ with $\sigma_{2,n_4} = -1$, $t_{1,n_4} < t_{1,n_2}$, $|\pi_1(P) - \pi_1(z_{1,n_4})| \le 1/2$, $\pi_2(z_{1,n_4}) - 1/2 \in (\pi_2(z_{1,n_3}) + 1/2, \pi_2(z_{1,n_3}) + 1/2 + 2\delta^{(n_3)} + 2\delta^{(n_4)})$ and such that $\{z_{1,n_1}, z_{1,n_4}\}$ does *not* form a $(\delta^{(n_1)} + \delta^{(n_4)})$-unstable pair. Then we define $P_1 = (\pi_1(P) - \delta^{(n_1)})e_1 + (\pi_2(z_{1,n_3}) + 1/2 + 2\delta^{(n_3)})e_2$ and $P_2 = (\pi_1(P) + \delta^{(n_1)})e_1 + (\pi_2(z_{1,n_3}) + 1/2 + 2\delta^{(n_3)})e_2$. See Figure 9 for an illustration.

(i) $P_i \in Q_{1-2\delta^{(n_i)}}(z_{1,n_i})$ is clear, as otherwise $\{z_{1,n_2}, z_{1,n_3}\}$ would form a spatially $2\delta^{(n_2)} + 2\delta^{(n_3)}$ unstable pair, contradicting Lemma 17.

(ii) To show that P_i is $\psi^{\Delta}(n_i)$-black assume for the sake of contradiction the existence of $m \in \{1,\dots,N\}$ with $t_{1,m} - \delta^{(m)} < t_{1,n_i} + \delta^{(n_i)}$, $\sigma_{2,m} = -1$ and $P_i \in Q_{1+2\delta^{(m)}}(z_{1,m})$. Observe that by Lemma 18 we have $t_{1,m} < t_{1,n_2}$. By Lemma 17, the choice of n_3 and the non-existence of n_4, we see that this is impossible.

(iii) $|\pi_1(P) - \pi_1(P_i)| \le \delta^{(n_1)}$ is clear.

(iv) Suppose we could find $P' \in [P_1, P_2]$ and $m \in \{1,\dots,N\}$ with $t_{1,m} - \delta^{(m)} < \max(t_{1,n_1} + \delta^{(n_1)}, t_{1,n_2} + \delta^{(n_2)})$, $\sigma_{2,m} = -1$ and $P' \in Q_{1+2\delta^{(m)}}(z_{1,m})$. We conclude from Lemma 18 that $t_{1,m} < t_{1,n_2}$. Again by Lemma 17, the choice of n_3 and the non-existence of n_4, we see that this is impossible.

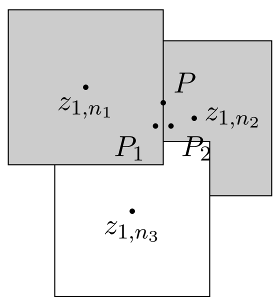

Figure 9: Configuration in case $2b$

Case $2c$. We can now tackle the remaining case. By the previous cases we may work under the following assumptions

- $\pi_1(z_{1,n_2}) - \pi_1(z_{1,n_1}) \in (\delta^{(n_1)} + \delta^{(n_2)}, 1 - \delta^{(n_1)} - \delta^{(n_2)})$
- there exists $n_3 \in \{1,\dots,N\}$ with $t_{1,n_3} < t_{1,n_2}$, $\sigma_{2,n_3} = -1$, $|\pi_1(P) - \pi_1(z_{1,n_3})| \le 1/2$, $\pi_2(z_{1,n_3}) + 1/2 \in (\pi_2(z_{1,n_1}) - 1/2, \pi_2(P))$ and such that $\{z_{1,n_1}, z_{1,n_3}\}$ does *not* form a $(\delta^{(n_1)} + \delta^{(n_3)})$-unstable pair.
- there exists $n_4 \in \{1,\dots,N\}$ with $t_{1,n_4} < t_{1,n_2}$, $\sigma_{2,n_4} = -1$, $|\pi_1(P) - \pi_1(z_{1,n_4})| \le 1/2$, $\pi_2(z_{1,n_4}) - 1/2 \in (\pi_2(P), \pi_2(z_{1,n_2}) + 1/2)$.
- Furthermore, if we choose n_3 such that $\pi_2(z_{1,n_3}) + \delta^{(n_3)}$ is maximal and n_4 such that $\pi_2(z_{1,n_4}) - \delta^{(n_4)}$ is minimal, then $\pi_2(z_{1,n_4}) - \pi_2(z_{1,n_3}) \le 1 + 2\delta^{(n_3)} + 2\delta^{(n_4)}$

See Figure 10 for an illustration. Observe that $\{z_{1,n_3}, z_{1,n_4}\}$ forms a spatially $(2\delta^{(n_3)} + 2\delta^{(n_4)})$-unstable pair. Furthermore, we also make the following definitions

- Choose $n_3' \in \{1,\dots,N\}$ with $t_{1,n_3'} < t_{1,n_2}$, $\left|\pi_1(P) - \pi_1(z_{1,n_3'})\right| \le 1/2$, $\pi_2(z_{1,n_3'}) + 1/2 \in [\pi_2(z_{1,n_3}) + 1/2, \pi_2(P))$ and $\pi_2(z_{1,n_3'})$ is maximal
- Choose $n_4' \in \{1,\dots,N\}$ with $t_{1,n_4'} < t_{1,n_2}$, $\left|\pi_1(P) - \pi_1(z_{1,n_4'})\right| \le 1/2$, $\pi_2(z_{1,n_4'}) - 1/2 \in (\pi_2(P), \pi_2(z_{1,n_4}) - 1/2]$ and $\pi_2(z_{1,n_4'})$ is minimal

As n_3 and n_4 have the properties required in the definition of n_3' respectively n_4', this definition is indeed reasonable, i.e., we are not choosing n_3' or n_4' from an empty set of possible values. Finally, in this situation y_{1,n_3}, $y_{1,n_3'}$, y_{1,n_4}, $y_{1,n_4'}$ and y_{1,n_1} are contained in the same δ_2-bad connected component, contradicting Lemma 17.

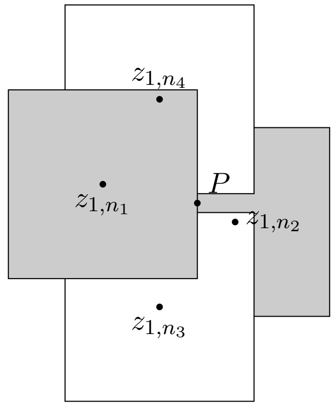

Figure 10: Configuration in case 2c

$\square$

After these preliminaries the existence of ψ^Δ-black crossings of R_2 is rather immediate. Indeed, suppose that Γ is a ψ_{φ_1}-black horizontal crossing of R_1. Subdivide Γ into closed subpaths $\Gamma_1, \ldots, \Gamma_r$ such that if we denote by Γ_i^o the subset Γ_i obtained by deleting its two endpoints, then there exist $z_{n_1}, \ldots, z_{n_r} \in Z$ with the property that $height_{\varphi_1}(\Gamma_i^o) = \{t_{1,n_i}\}$ for all $i \in \{1, \ldots, r\}$, see also Figure 11. Let us write $\{P_i\} = \Gamma_i \cap \Gamma_{i+1}$. Applying Lemma 20 to P_i and choosing $j_0 \in \{0,1\}$ with $height_{\varphi_1}(P_i) = t_{1,n_{i+j_0}}$, we obtain for $j \in \{0,1\}$ points $P_{i,j} \in height^{-1}_{\varphi_1^\Delta}(t_{1,n_{i+j}} + \delta^{(n_{i+j})})$ with the following properties:

- $|\pi_k(P_{i,j}) - \pi_k(P_i)| \leq \delta^{(n_{i+j_0})}$ for all $j \in \{1,2\}$, where the index $k \in \{1,2\}$ is such that $\left|\pi_k(P_i) - \pi_k(z_{1,n_{i+j_0}})\right| = 1/2$.
- The linear segment $[P_{i,1}, P_{i,2}]$ is ψ^Δ-black.

In particular, we may apply Lemma 19 (with $n = i$, $P_1 = P_{i,2}$, $P_2 = P_{i+1,1}$, $P_1' = P_i$, $P_2' = P_{i+1}$) to obtain ψ^Δ-black paths from $P_{i,2}$ to $P_{i+1,1}$. Using the ψ^Δ-blackness of the joining segments $[P_{i,1}, P_{i,2}]$ we can create a ψ^Δ-black path starting from $P_{1,1}$ and ending at $P_{r,2}$.

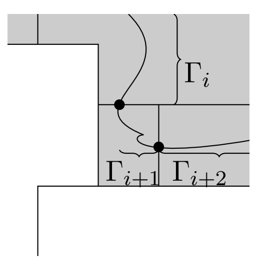

Figure 11: Construction of the paths Γ_i

Acknowledgments

The author is grateful for the detailed reports by all anonymous referees that helped to substantially improve the quality of an earlier version of the manuscript. In particular, one referee proposed the very interesting Remark 2. The author thanks V. Schmidt for interesting discussions and helpful remarks. Furthermore, the author thanks D. Jeulin for his lecture on the dead leaves model during the 2011 stochastic geometry summer school that marked the starting point for the current research. This work has been supported by a research grant from DFG Research Training Group 1100 at Ulm University.

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