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BERRY–ESSEEN BOUNDS AND CRAMÉR-TYPE LARGE DEVIATIONS FOR THE VOLUME DISTRIBUTION OF POISSON CYLINDER PROCESSES

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Dedicated to the memory of Vytautas Statulevičius

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Abstract. A stationary Poisson cylinder process $\Pi_{\mathrm{cyl}}^{(d,k)}$ is composed of a stationary Poisson process of k-flats in $\mathbb{R}^d$ that are dilated by i.i.d. random compact cylinder bases taken from the corresponding orthogonal complement. We study the accuracy of normal approximation of the d-volume $V_\varrho^{(d,k)}$ of the union set of $\Pi_{\mathrm{cyl}}^{(d,k)}$ that covers ϱW as the scaling factor ϱ becomes large. Here W is some fixed compact star-shaped set containing the origin as an inner point. We give lower and upper bounds of the variance of $V_\varrho^{(d,k)}$ that exhibit long-range dependence within the union set of cylinders. Our main results are sharp estimates of the higher-order cumulants of $V_\varrho^{(d,k)}$ under the assumption that the $(d-k)$-volume of the typical cylinder base possesses a finite exponential moment. These estimates enable us to apply the celebrated "Lemma on large deviations" of Statulevičius.

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1 INTRODUCTION AND PRELIMINARIES

In integral and stochastic geometry, a *cylinder* in $\mathbb{R}^d$ is an unbounded set of the form $L \oplus B$ with *direction space* $L \in \mathbb{G}(d,k)$ (= the Grassmannian of k-dimensional linear subspaces of $\mathbb{R}^d$), $k = 1, \dots, d-1$, and a convex, compact subset B of the orthogonal complement $L^\perp$ called *base* of the cylinder (see, e.g., [12, 16, 20] for details). Throughout this paper, the orientation of the direction space L is suppressed, and the restriction of convexity of B is dropped. The general notion of a point process of cylinders (briefly *cylinder process*, subsequently abbreviated CP) was first considered in [20]. In order to find explicit formulas for numerical characteristics of union sets of CPs, such as the volume fraction, covariance, etc., one needs specific distributional assumptions determining shape, direction, and position of the random cylinders. In order to describe various real-life random set structures, it is quite natural to assume that the sizes and spatial positions of cylinders are governed by an independently marked Poisson process. Following the concept of Poisson

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processes defined on the space of cylinders with bases in the convex ring, Poisson cylinder processes (briefly PCPs) were studied in [17] with applications in modeling materials consisting of long thick fibres or thick membranes.

To be precise in describing our problem, we first introduce some notation and give a rigorous definition of a stationary PCP (which slightly differs from that in [17]). For this, let $\{e_1,\dots,e_d\}$ denote the usual orthonormal basis of $\mathbb{R}^d$ defining the orthogonal subspaces $E_k = \operatorname{span}\{e_{d-k+1},\dots,e_d\}$ and $E_k^\perp = \operatorname{span}\{e_1,\dots,e_{d-k}\}$, where $k\in\{1,\dots,d-1\}$ is fixed in what follows. It is well known from differential geometry that, for any given $L\in\mathbb{G}(d,k)$, there exists an equivalence class $\mathbf{O}_L\in\mathbb{SO}_d/\mathbb{S}(\mathbb{O}_{d-k}\times\mathbb{O}_k)$ of orthogonal matrices $O\in\mathbb{R}^{d\times d}$ with $\det(O)=1$ such that $OE_k=L$. In other words, two matrices $O,\widehat{O}\in\mathbb{SO}_d$ belong to $\mathbf{O}_L$ iff $OE_k=\widehat{O}E_k=L$ and $O^{-1}\widehat{O}\in\mathbb{S}(\mathbb{O}_{d-k}\times\mathbb{O}_k)$, where $\mathbb{S}(\mathbb{O}_{d-k}\times\mathbb{O}_k)$ coincides with the set

$$\left\{\begin{pmatrix}A & 0\\ 0 & B\end{pmatrix}\colon\ A\in\mathbb{R}^{(d-k)\times(d-k)},\ B\in\mathbb{R}^{k\times k},\ A^T=A^{-1},\ B^T=B^{-1},\ \det(A)\det(B)=1\right\}.$$

We identify each equivalence class $\mathbf{O}_L$ with a single representative $O_L\in\mathbf{O}_L$ and write somewhat loosely $O_L\in\mathbb{SO}_d/\mathbb{S}(\mathbb{O}_{d-k}\times\mathbb{O}_k)$. Further, since $\dim\mathbb{G}(d,k)=(d-k)k$ (see Chap. 16.11 in [2]), there exists a measurable mapping from a bounded Borel parameter set $\Theta_{d,k}\subset\mathbb{R}^{(d-k)k}$ onto $\{O_L\colon L\in\mathbb{G}(d,k)\}$. Note that an explicit form of this mapping seems to be known only for special cases, e.g., for $d=2$, $k=1$ or $d=3$, $k=1$:

$$O_L(\theta)=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\quad\text{or}\quad O_L(\theta_1,\theta_2)=\begin{pmatrix}\sin\theta_1 & \cos\theta_1\cos\theta_2 & \cos\theta_1\sin\theta_2\\ -\cos\theta_1 & \sin\theta_1\cos\theta_2 & \sin\theta_1\sin\theta_2\\ 0 & -\sin\theta_2 & \cos\theta_2\end{pmatrix}$$

for $\theta\in\Theta_{2,1}=[0,\pi)$ and $(\theta_1,\theta_2)\in\Theta_{3,1}=[0,2\pi)\times[0,\frac{\pi}{2})$, respectively.

In this way, a random subspace $L\in\mathbb{G}(d,k)$ and the corresponding random matrix $O_L\in\mathbb{SO}_d/\mathbb{S}(\mathbb{O}_{d-k}\times\mathbb{O}_k)$ can simply be described by the distribution of a random vector in $\Theta_{d,k}$. Throughout this paper, all random elements are defined on a common probability space $[\Omega,\mathfrak{F},\mathsf{P}]$, and E (respectively Var) denotes the expectation (respectively variance) w.r.t. P. In particular, let (O_0,Ξ_0) be a measurable mapping from $[\Omega,\mathfrak{F},\mathsf{P}]$ into the product space $\Omega_{d,k}=\mathbb{SO}_d/\mathbb{S}(\mathbb{O}_{d-k}\times\mathbb{O}_k)\times\mathcal{K}_{d-k}$, where $\mathcal{K}_{d-k}$ denotes the space of nonvoid compact subsets of $\mathbb{R}^{d-k}$ equipped with the Hausdorff metric. The image measure $Q:=\mathsf{P}\circ(O_0,\Xi_0)^{-1}$ acting on the corresponding Borel product σ-field $\mathfrak{B}(\Omega_{d,k})$ determines the joint distribution of the (not necessarily independent) random elements O_0 and Ξ_0. Now we are in a position to introduce the stationary independently marked Poisson process $\Pi_{\lambda,Q}=\sum_{i\geqslant 1}\delta_{[P_i,(O_i,\Xi_i)]}$ with intensity λ and mark distribution $Q(\cdot)$, i.e., $\Pi_{\lambda,Q}(\cdot)$ is a random locally finite counting measure (shift-invariant in the first component) on the Borel subsets of $\mathbb{R}^{d-k}\times\Omega_{d,k}$ such that the numbers $\Pi_{\lambda,Q}(B\times L)$ are Poisson distributed with mean $\lambda|B|_{d-k}Q(L)$ for any bounded $B\in\mathfrak{B}(\mathbb{R}^{d-k})$ (with Lebesgue measure $|\cdot|_{d-k}$) and $L\in\mathfrak{B}(\Omega_{d,k})$, see [1] for a standard reference on general (Poisson) point processes. This definition implies that the numbers of atoms of the unmarked Poisson process $\Pi_\lambda=\sum_{i\geqslant 1}\delta_{P_i}$ located in disjoint subsets of $\mathbb{R}^{d-k}$ are independent and the marks (O_i,Ξ_i) associated with the atoms P_i are i.i.d. copies of $(O_0,\Xi_0)\sim Q$ independent of Π_λ.

Furthermore, we need two important formulas for $\Pi_{\lambda,Q}$, each of them characterizing the distribution of $\Pi_{\lambda,Q}$: The *probability generating functional* $G_{\lambda,Q}[v]=\mathsf{E}(\prod_{i\geqslant 1}v(P_i,O_i,\Xi_i))$ of $\Pi_{\lambda,Q}$ takes the form

$$G_{\lambda,Q}[v]=\exp\left\{-\lambda\int\limits_{\mathbb{R}^{d-k}}\int\limits_{\Omega_{d,k}}\big(1-v(x,O,K)\big)\,Q(\mathrm{d}(O,K))\,\mathrm{d}x\right\}\tag{1.1}$$

for any measurable function $v:\mathbb{R}^{d-k}\times\Omega_{d,k}\mapsto[0,1]$ such that $1-v(\cdot,O,K)$ has bounded support for

$(O,K) \in \Omega_{d,k}$, whereas the nth-order *Campbell formula* reads for any $n \in \mathbb{N}$ as follows:

$$\mathsf{E}\Biggl(\sum_{i_1,\dots,i_n\geqslant 1}^{*} \prod_{j=1}^{n} f_j(P_{i_j}, O_{i_j}, \Xi_{i_j})\Biggr) = \lambda^n \prod_{j=1}^{n} \int_{\mathbb{R}^{d-k}} \int_{\Omega_{d,k}} f_j(x,O,K)\, Q(\mathrm{d}(O,K))\, \mathrm{d}x \tag{1.2}$$

for nonnegative measurable functions $f_1,\dots,f_n : \mathbb{R}^{d-k} \times \Omega_{d,k} \mapsto \mathbb{R}^1$, where the sum $\sum^*$ on the left-hand side of (1.2) runs over all n-tuples of pairwise distinct indices $i_1,\dots,i_n \geqslant 1$ (see [1] or [16, 19]).

DEFINITION. Given an independently marked Poisson process $\Pi_{\lambda,Q} = \sum_{i\geqslant 1} \delta_{[P_i,(O_i,\Xi_i)]}$ satisfying the above assumptions, by a stationary PCP we understand a countable family of cylinders

$$\Pi_{\mathrm{cyl}}^{(d,k)}(\lambda,Q) := \{O_i((\Xi_i' + P_i') \oplus E_k),\ i \geqslant 1\} = \{O_i((\Xi_i + P_i) \times \mathbb{R}^k),\ i \geqslant 1\} \tag{1.3}$$

with $\Xi_i' + P_i' = \{(x + P_i, 0,\dots,0) \colon x \in \Xi_i\} \subset E_k^{\perp}$ for $i \geqslant 1$.

In this paper, we are mainly interested in the random union set

$$\Xi_{\mathrm{cyl}}^{(d,k)}(\lambda,Q) = \bigcup_{i\geqslant 1} O_i((\Xi_i + P_i) \times \mathbb{R}^k) \tag{1.4}$$

derived from (1.3) and, in particular, in the asymptotic behavior (after centering and scaling) of the random d-volume $V_\varrho^{(d,k)} = |\Xi_{\mathrm{cyl}}^{(d,k)}(\lambda,Q) \cap \varrho W|_d$ as $\varrho \to \infty$ for a fixed set $W \in \mathcal{K}_d$ chosen star-shaped (w.r.t. the origin $\mathbf{o}$) and such that $b(\mathbf{o},\delta_W) \subseteq W \subseteq b(\mathbf{o},1)$ for some $\delta_W > 0$. Here $b(x,r)$ is the closed ball in $\mathbb{R}^d$ with radius $r \geqslant 0$ and center $x \in \mathbb{R}^d$.

Remark 1. In the degenerate case $k = 0$ (where $E_0 = \{\mathbf{o}\}$ and $O_0 =$ unit matrix), the union set (1.3) coincides with the well-studied Boolean (or Poisson grain, Poisson blob, Swiss cheese) model in $\mathbb{R}^d$ with typical grain Ξ_0 (see [12, 19]).

Remark 2. The union set $\Xi_{\mathrm{cyl}}^{(d,k)}(\lambda,Q)$ is (P-a.s.) closed if $\mathsf{E}|\Xi_0 \oplus \pi_{d-k}(b(\mathbf{o},\varepsilon))|_{d-k} < \infty$ for some $\varepsilon > 0$, where $\pi_{d-k}(y)$ denotes the projection of the vector $y \in \mathbb{R}^d$ on its first $d-k$ components in $\mathbb{R}^{d-k}$. In this case, the hitting probability $\mathsf{P}(\Xi_{\mathrm{cyl}}^{(d,k)}(\lambda,Q) \cap C \neq \emptyset)$ can be calculated for any $C \in \mathcal{K}_d$ by applying (1.1), see Appendix.

A realization of the union set (1.4) for $d = 2$, $k = 1$ and $d = 3$, $k = 1$ is shown in Fig. 1 and Fig. 2, respectively.

In the next section, we state the announced sharp estimates of the higher-order cumulants $\mathsf{Cum}_n(V_\varrho^{(d,k)})$ of the d-volume $V_\varrho^{(d,k)}$ under the exponential moment condition

$$m_a = \mathsf{E}\exp\{a|\Xi_0|_{d-k}\} < \infty \quad \text{for some } a > 0. \tag{1.5}$$

This condition is by no means sufficient to imply the closedness of $\Xi_{\mathrm{cyl}}^{(d,k)}(\lambda,Q)$ with probability 1. There exist simple counter-examples with $\mathsf{P}(\Xi_{\mathrm{cyl}}^{(d,k)}(\lambda,Q) \text{ is closed}) = 0$. For instance, if the typical cylinder base is defined by $\Xi_0 = \bigcup_{1\leqslant i_1,\dots,i_{d-k}\leqslant N} \times_{j=1}^{d-k} [i_j, i_j + \frac{1}{N}]$ for some positive random integer N satisfying $\mathsf{E}N = \infty$, then the union set is closed with probability 0, no matter which distribution O_0 has.

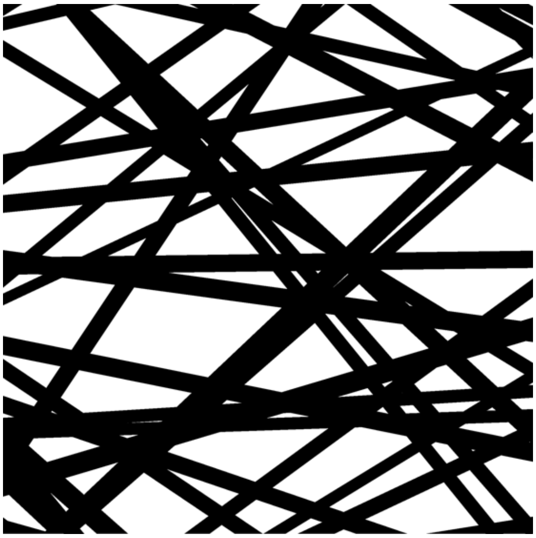

Figure 1. Planar anisotropic PCP in a square.

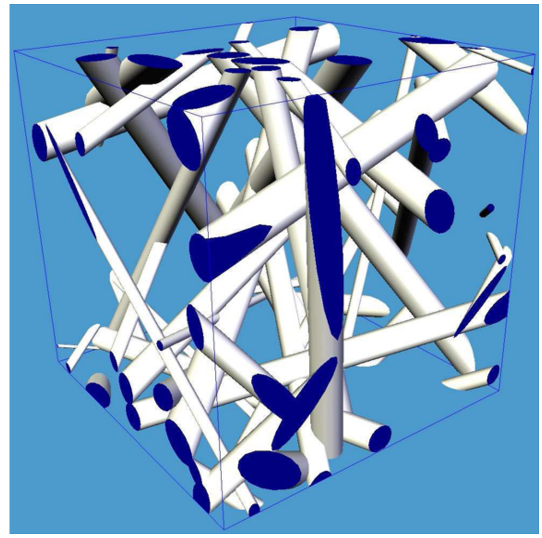

Figure 2. Spatial isotropic PCP in a cube.

2 MAIN RESULTS

For notational ease, we will mostly use the abbreviation $\varXi$ instead of $\varXi_{\text{cyl}}^{(d,k)}(\lambda, Q)$. We first recall the fact that the probability space $[\varOmega, \mathfrak{F}, \mathsf{P}]$ on which the marked Poisson process $\varPi_{\lambda,Q} = \sum_{i\geqslant 1} \delta_{[P_i,(O_i,\varXi_i)]}$ is defined can be chosen in such a way that the mapping $\mathbb{R}^d \times \varOmega \ni (x,\omega) \mapsto \mathbf{1}_{\varXi(\omega)}(x) \in \{0,1\}$ is measurable w.r.t. the product-σ-field $\mathfrak{B}(\mathbb{R}^d)\otimes\mathfrak{F}$ (see Appendix in [4]). This enables us to apply Fubini's theorem to the 0–1-valued random field $\{\mathbf{1}_{\varXi}(x),\ x\in\mathbb{R}^d\}$ and implies, among others, that its nth-order mixed moments (also called n-point probabilities of $\varXi$)

$$\mathbb{R}^{dn} \ni (x_1,\dots,x_n) \mapsto p_{\varXi}^{(n)}(x_1,\dots,x_n) := \mathsf{E}\left(\prod_{i=1}^{n} \mathbf{1}_{\varXi}(x_i)\right) = \mathsf{P}(x_1\in\varXi,\ \dots,\ x_n\in\varXi)$$

are $\mathfrak{B}(\mathbb{R}^{dn})$-measurable for any $n\in\mathbb{N}$ and that $p_{\varXi^c}^{(n)}(x_1,\dots,x_n)$ takes the following explicit form:

$$p_{\varXi^c}^{(n)}(x_1,\dots,x_n) := \mathsf{E}\left(\prod_{i=1}^{n}(1-\mathbf{1}_{\varXi}(x_i))\right) = \exp\left\{-\lambda\mathsf{E}\left|\bigcup_{i=1}^{n}(\varXi_0 - \pi_{d-k}(O_0^T x_i))\right|_{d-k}\right\}, \tag{2.1}$$

see Appendix. Likewise, the nth-order mixed cumulants $c_{\varXi^c}^{(n)}(x_1,\dots,x_n)$ of $\{1-\mathbf{1}_{\varXi}(x),\ x\in\mathbb{R}^d\}$ are Borel-measurable functions leading to the following integral representation of the nth-order cumulant of $V_{\varrho}^{(d,k)} = |\varXi\cap\varrho W|_d$ (see (5.2) in Appendix):

$$\mathsf{Cum}_n(V_{\varrho}^{(d,k)}) = (-1)^n \int_{(\varrho W)^n} c_{\varXi^c}^{(n)}(x_1,\dots,x_n)\,\mathrm{d}(x_1,\dots,x_n) \quad \text{for } n\geqslant 2. \tag{2.2}$$

We are now in a position to formulate our main results.

Theorem 1. *Let* $\varXi$ *be the union set* (1.4) *of the stationary* PCP $\varPi_{\text{cyl}}^{(d,k)}(\lambda,Q)$ *with compact typical cylinder base* $\varXi_0\subset\mathbb{R}^{d-k}$ *satisfying* (1.5) *and* $M_1 = \mathsf{E}|\varXi_0|_{d-k} > 0$. *Further, let* $W\subset\mathbb{R}^d$ *be compact and star-shaped* *w.r.t.* $\mathbf{o}$ *satisfying* $b(\mathbf{o},\delta_W)\subseteq W\subseteq b(\mathbf{o},1)$ *for some* $\delta_W\in(0,1)$*. Then*

$$\big|\,\mathsf{Cum}_n\big(V_\varrho^{(d,k)}\big)\big|\leqslant \varrho^{(n-1)k+d}(n-1)!H_a\Delta_a^{n-2}\quad \textit{for } n\geqslant 2,\ \varrho>0, \tag{2.3}$$

where $H_a=2^{2k+1}|W|_d\lambda m_a(1+\exp\{\lambda M_1\})/a^2$ *and* $\Delta_a=2^{2k+3}(a+\lambda m_a)(1+\exp\{\lambda M_1\})/a^2$*.*

The next Theorem 2 states Cramér's large-deviation relations for the random d-volume $V_\varrho^{(d,k)}$ and an optimal Berry–Esseen bound of the distance between the distribution functions

$$F_\varrho(x)=\mathsf{P}\bigg(\frac{V_\varrho^{(d,k)}-\varrho^d|W|_d(1-\exp\{-\lambda\mathsf{E}|\Xi_0|_{d-k}\})}{\sigma_\varrho\varrho^{(d+k)/2}}\leqslant x\bigg)\quad\text{and}\quad \Phi(x)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{x}\mathrm{e}^{-t^2/2}\,\mathrm{d}t,$$

where $\mathsf{P}(\mathbf{o}\in\Xi)=\mathsf{E}|\Xi\cap[0,1]^d|_d=1-\exp\{-\lambda\mathsf{E}|\Xi_0|_{d-k}\}$ is just the *volume fraction* of the stationary random set (1.4), the normalized variance σ_ϱ^2 of $V_\varrho^{(d,k)}$ satisfies the estimate

$$0<c_1\leqslant\sigma_\varrho^2\leqslant c_2<\infty\quad\text{for all } \varrho\geqslant 1 \text{ with } \sigma_\varrho^2=\mathsf{Var}\big(V_\varrho^{(d,k)}\big)/\varrho^{d+k}, \tag{2.4}$$

and c_1,c_2 are constants independent of $\varrho\geqslant 1$ (see Lemma 1 below).

Theorem 2. *Let the assumptions of Theorem* 1 *be satisfied. Then the following asymptotic relations hold in the interval* $0\leqslant x\leqslant\sigma_\varrho\varrho^{(d-k)/2}/2\Delta_a(1+4H_{a,\varrho})$ *with* $H_{a,\varrho}=H_a/2\sigma_\varrho^2$*:*

$$\frac{1-F_\varrho(x)}{1-\Phi(x)}=\exp\bigg\{\frac{x^3}{\sigma_\varrho\varrho^{(d+k)/2}}\sum_{s=0}^{\infty}\mu_s^{(\varrho)}\Big(\frac{x}{\sigma_\varrho\varrho^{(d+k)/2}}\Big)^s\bigg\}\bigg(1+\mathcal{O}\Big(\frac{1+x}{\varrho^{(d-k)/2}}\Big)\bigg) \tag{2.5}$$

and

$$\frac{F_\varrho(-x)}{\Phi(-x)}=\exp\bigg\{\frac{-x^3}{\sigma_\varrho\varrho^{(d+k)/2}}\sum_{s=0}^{\infty}\mu_s^{(\varrho)}\Big(\frac{-x}{\sigma_\varrho\varrho^{(d+k)/2}}\Big)^s\bigg\}\bigg(1+\mathcal{O}\Big(\frac{1+x}{\varrho^{(d-k)/2}}\Big)\bigg) \tag{2.6}$$

as $\varrho\to\infty$*, where the coefficients* $\mu_s^{(\varrho)}$ *are defined by*

$$\mu_s^{(\varrho)}=\frac{1}{(s+2)(s+3)}\sum_{j=1}^{s+1}(-1)^{j-1}\binom{s+j+1}{j}\sum_{s_1+\cdots+s_j=s+1}^{>}\prod_{i=1}^{j}\frac{\mathsf{Cum}_{s_i+2}\big(V_\varrho^{(d,k)}\big)}{\mathsf{Var}\big(V_\varrho^{(d,k)}\big)(s_i+1)!}. \tag{2.7}$$

Here the sum $\sum^>$ *runs over the* j*-tuples of positive integers, and the series in* (2.5) *and* (2.6) *converges absolutely due to the estimate* $|\mu_s^{(\varrho)}|\leqslant 4H_{a,\varrho}\Delta_a\varrho^{k(s+1)}(2\Delta_a(1+4H_{a,\varrho}))^s/(s+2)(s+3)$ *for all* $s\geqslant 0$*.*

Furthermore, there exists some constant $c_3>0$ (*depending on* a,λ,m_a*, and* c_1,c_2) *such that*

$$\sup_{x\in\mathbb{R}^1}\big|F_\varrho(x)-\Phi(x)\big|\leqslant c_3\varrho^{-(d-k)/2}\quad \textit{for all } \varrho\geqslant 1. \tag{2.8}$$

Theorem 2 is derived from (2.3) combined with a general lemma on large deviations for a single random variable with mean 0 and variance 1 due to Statulevičius [18] (see also Lemma 2.3 in the monograph [15]). Relations (2.5) and (2.6) are of particular interest at $x=\varepsilon|W|_d\varrho^{(d-k)/2}/\sigma_\varrho$ for small $\varepsilon>0$.

It is an open question whether the Berry–Esseen estimate (2.8) can be obtained under weaker conditions on the cylinder base. Perhaps, it suffices to require $\mathsf{E}|\Xi_0|_{d-k}^3<\infty$ as one would expect from the CLT for independent random variables. In [8], the authors prove the central limit theorem $F_\varrho(x)\underset{\varrho\to\infty}{\longrightarrow}\Phi(x)$ for $x\in\mathbb{R}^1$ under $\mathsf{E}|\Xi_0|_{d-k}^2<\infty$ without rates of convergence.

We further mention that the above theorems can be extended to analogous results for estimators of the *covariance* $C_{\Xi^c}(u) = \mathsf{P}(\mathbf{o} \in \Xi^c,\ u \in \Xi^c)$ for fixed $u \in \mathbb{R}^d$ (see, e.g., [12, 19] and [17]). This is seen from the obvious relation $C_{\Xi^c}(u) = 1 - \mathsf{P}(\mathbf{o} \in \Xi \cup (\Xi - u))$ and the fact that the union $\Xi \cup (\Xi - u)$ takes the form (1.4) with typical base $\Xi_0 \cup (\Xi_0 - \pi_{d-k}(O_0^T u))$.

The rest of this paper is organized as follows: In Section 3, we derive bounds for the variance of the volume $V_\varrho^{(d,k)}$, and Section 4 contains a rather technical proof of the cumulant estimates (2.3). At the end of Section 4, we show how to apply the large deviations lemma in [18] to our situation. In the Appendix, we recall some basic facts on mixed moments and cumulants connected with random set (1.4) and the random 0–1-field $\{\mathbf{1}_\Xi(x),\ x \in \mathbb{R}^d\}$. Finally, a criterion for (non)closedness of $\Xi_{\text{cyl}}^{(d,k)}(\lambda, Q)$ is given in analogy to that in [4] for Boolean models.

3 LOWER AND UPPER BOUNDS FOR THE VARIANCE

In this section, we derive a lower and an upper bound for the variance of $V_\varrho^{(d,k)} = |\Xi \cap \varrho W|_d$, provided that the second moment of $|\Xi_0|_{d-k}$ exists. To this end, we first derive a closed-term expression of the variance $\mathsf{Var}(|\Xi \cap B|_d)$ for any bounded Borel set $B \in \mathfrak{B}(\mathbb{R}^d)$ using the above formulae for $p_{\Xi^c}^{(2)}(\mathbf{o}, x)$ and $p_{\Xi^c}^{(1)}(\mathbf{o})$.

By using the very definition of the one- and two-point probabilities $p_\Xi^{(n)}$, $n = 1, 2$, and the shift-invariance and additivity of the Lebesgue measure $|\cdot|_{d-k}$, we deduce from (2.1) that

$$\begin{aligned}
&p_\Xi^{(2)}(x_1, x_2) - p_\Xi^{(1)}(x_1) p_\Xi^{(1)}(x_2) \\
&\quad = p_{\Xi^c}^{(2)}(\mathbf{o}, x_2 - x_1) - p_{\Xi^c}^{(1)}(\mathbf{o}) p_{\Xi^c}^{(1)}(\mathbf{o}) \\
&\quad = \exp\{-\lambda \mathsf{E}|\Xi_0 \cup (\Xi_0 - \pi_{d-k}(O_0^T(x_2 - x_1)))|_{d-k}\} - \exp\{-2\lambda \mathsf{E}|\Xi_0|_{d-k}\} \\
&\quad = \mathrm{e}^{-2\lambda M_1}\big(\exp\{\lambda \mathsf{E}|\Xi_0 \cap (\Xi_0 - \pi_{d-k}(O_0^T(x_2 - x_1)))|_{d-k}\} - 1\big) \quad \text{for } x_1, x_2 \in \mathbb{R}^d.
\end{aligned}$$

Here and below, we use the abbreviation $M_s = \mathsf{E}|\Xi_0|_{d-k}^s$ for $s \in \mathbb{N}$. Hence, by multiple application of Fubini's theorem we get that, for any bounded $B \in \mathfrak{B}(\mathbb{R}^d)$,

$$\begin{aligned}
\mathsf{Var}(|\Xi \cap B|_d) &= \mathsf{E}\int_B \int_B \mathbf{1}_\Xi(x_1) \mathbf{1}_\Xi(x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 - \Big(\mathsf{E}\int_B \mathbf{1}_\Xi(x)\,\mathrm{d}x\Big)^2 \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbf{1}_B(x_1)\mathbf{1}_B(x_2)\big(p_\Xi^{(2)}(x_1, x_2) - p_\Xi^{(1)}(x_1)p_\Xi^{(1)}(x_2)\big)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&= \mathrm{e}^{-2\lambda M_1} \int_{\mathbb{R}^d} |B \cap (B - x)|_d \big(\exp\{\lambda \mathsf{E}|\Xi_0 \cap (\Xi_0 - \pi_{d-k}(O_0^T x))|_{d-k}\} - 1\big)\,\mathrm{d}x.
\end{aligned}$$

Now we replace B by the star-shaped set ϱW that increases when ϱ does. In view of the relation $\{x \in \mathbb{R}^d\colon \varrho W \cap (\varrho W - x) \neq \emptyset\} = \varrho(W \oplus (-W)) \subseteq b(\mathbf{o}, 2\varrho)$ and the inequality $\mathrm{e}^y - 1 \leqslant y\mathrm{e}^y$ for $y \geqslant 0$, we may write

$$\begin{aligned}
\mathsf{Var}(V_\varrho^{(d,k)}) &\leqslant \lambda \mathrm{e}^{-\lambda M_1} |\varrho W|_d \int_{\varrho(W \oplus (-W))} \mathsf{E}|\Xi_0 \cap (\Xi_0 + \pi_{d-k}(O_0^T x))|_{d-k}\,\mathrm{d}x \\
&\leqslant \lambda |W|_d \mathrm{e}^{-\lambda M_1} \varrho^d \mathsf{E} \int_{b(\mathbf{o}, 2\varrho)} |\Xi_0 \cap (\Xi_0 + \pi_{d-k}(x))|_{d-k}\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \lambda |W|_d \mathrm{e}^{-\lambda M_1} \varrho^d \mathsf{E} \int\limits_{[-2\varrho,2\varrho]^k} \int\limits_{\mathbb{R}^{d-k}} \left|\Xi_0 \cap (\Xi_0 + y_1)\right|_{d-k} \mathrm{d}y_1\, \mathrm{d}y_2 \\
&= \lambda |W|_d \mathrm{e}^{-\lambda M_1} 4^k \mathsf{E}|\Xi_0|_{d-k}^2 \varrho^{k+d} \\
&\leqslant \lambda M_2 \mathrm{e}^{-\lambda M_1} 2^{d+2k} \varrho^{k+d} \quad \text{for any } \varrho > 0.
\end{aligned} \tag{3.1}
$$

To find a positive lower bound of the ratio σ_ϱ^2, we make use of $b(\mathbf{o}, \delta_W) \subseteq W$, which implies that $\varrho W \cap (\varrho W - x) \supseteq b(\mathbf{o}, \delta_W \varrho) \cap b(-x, \delta_W \varrho)$ and $\varrho(W \oplus (-W)) \subseteq b(\mathbf{o}, 2\delta_W \varrho)$. This, combined with $\mathrm{e}^y - 1 \geqslant y$ for $y \geqslant 0$, implies

$$
\begin{aligned}
\mathsf{Var}(V_\varrho^{(d,k)}) &\geqslant \lambda \mathrm{e}^{-2\lambda M_1} \int\limits_{b(\mathbf{o},2\delta_W\varrho)} \left|b(\mathbf{o}, \delta_W \varrho) \cap b(x, \delta_W \varrho)\right|_d \mathsf{E}\left|\Xi_0 \cap (\Xi_0 + \pi_{d-k}(O_0^T x))\right|_{d-k} \mathrm{d}x \\
&\geqslant \lambda \mathrm{e}^{-2\lambda M_1} \int\limits_{b(\mathbf{o},\delta_W\varrho)} \left|b(\mathbf{o}, \delta_W \varrho) \cap b(x, \delta_W \varrho)\right|_d \mathsf{E}\left|\Xi_0 \cap (\Xi_0 + \pi_{d-k}(x))\right|_{d-k} \mathrm{d}x \\
&\geqslant \lambda \mathrm{e}^{-2\lambda M_1} c(d) (\varrho \delta_W)^d \int\limits_{b(\mathbf{o},\delta_W\varrho)} \mathsf{E}\left|\Xi_0 \cap (\Xi_0 + \pi_{d-k}(x))\right|_{d-k} \mathrm{d}x \\
&\geqslant \lambda \mathrm{e}^{-2\lambda M_1} c(d) (\varrho \delta_W)^d \int\limits_{[-\varrho\delta_W/\sqrt{d},\varrho\delta_W/\sqrt{d}]^d} \mathsf{E}\left|\Xi_0 \cap (\Xi_0 + \pi_{d-k}(x))\right|_{d-k} \mathrm{d}x \\
&= \lambda 2^k d^{-k/2} \mathrm{e}^{-2\lambda M_1} (\varrho \delta_W)^{d+k} c(d) I_{d,k}(\varrho)
\end{aligned} \tag{3.2}
$$

with

$$
c(d) = \left|b(\mathbf{o}, 1) \cap b(e_1, 1)\right|_d > 0 \quad \text{and} \quad I_{d,k}(\varrho) = \int\limits_{[-\varrho\delta_W/\sqrt{d},\varrho\delta_W/\sqrt{d}]^{d-k}} \mathsf{E}\left|\Xi_0 \cap (\Xi_0 + y)\right|_{d-k} \mathrm{d}y.
$$

Making use of $\mathsf{P}(|\Xi_0|_{d-k} > 0) > 0$ and standard measure-theoretic arguments, it follows that $I_{d,k}(\varrho) > 0$ for any $\varrho > 0$ and $I_{d,k}(\varrho)$ increases with $\varrho \uparrow \infty$ to the limit $\mathsf{E}|\Xi_0|_{d-k}^2$. In this way, we confirm estimate (2.4) with constants

$$
c_1 = \lambda 2^k d^{-k/2} \mathrm{e}^{-2\lambda M_1} \delta_W^{d+k} c(d) I_{d,k}(1) \quad \text{and} \quad c_2 = \lambda 2^{d+2k} \mathrm{e}^{-\lambda M_1} M_2.
$$

Another consequence of the above estimates is stated in the following:

Lemma 1. *Let Ξ be the union set* (1.4) *of the stationary* PCP $\Pi_{\mathrm{cyl}}^{(d,k)}(\lambda, Q)$ *with compact cylinder base $\Xi_0 \subset \mathbb{R}^{d-k}$ satisfying $0 < M_2 = \mathsf{E}|\Xi_0|_{d-k}^2 < \infty$. Further, let $W \subset R^d$ be a compact set satisfying $b(\mathbf{o}, \delta_W) \subseteq W \subseteq b(\mathbf{o}, 1)$ for some $\delta_W > 0$. Then we have*

$$
\frac{c_1 M_2}{I_{d,k}(1)} \leqslant \liminf_{\varrho\to\infty} \frac{\mathsf{Var}(|\Xi \cap W\varrho|_d)}{\varrho^{d+k}} \leqslant \limsup_{\varrho\to\infty} \frac{\mathsf{Var}(|\Xi \cap W\varrho|_d)}{\varrho^{d+k}} \leqslant c_2. \tag{3.3}
$$

Remark 3. Lemma 1 reveals that the variance of $V_\varrho^{(d,k)}$ grows proportional to the power $|\varrho W|_d^{1+k/d}$ of the window volume that expresses long-range dependence within the random set (1.4). The same effect could be observed in investigating the asymptotic behavior of the total $(d-k)$-volume of intersection of $(d-k)$-flats generated by Poisson hyperplane processes in $b(\mathbf{o}, \varrho)$, respectively ϱW (for convex W), as $1 \leqslant \varrho \uparrow \infty$ (see [7], respectively [6]). The existence of the limit of the ratio σ_ϱ^2 as $\varrho \to \infty$ seems to be difficult to prove. So far, only in the simplest case $d = 2$, $k = 1$ can we give a positive answer (see [8]).

4 PROOFS OF THEOREMS 1 AND 2

The main part of this section consists of a combination of recursive estimation procedures carried out in several steps, which finally result in estimate (2.3). This proving idea was developed in [4] to obtain a similar estimate for Boolean models. However, the techniques used there had to be extended to unbounded cylinders, which cause long-range dependence in contrast to the classical Boolean model. To begin with, using the shift-invariance $c^{(n)}_{\Xi^c}(x_1,\dots,x_n) = c^{(n)}_{\Xi^c}(\mathbf{o},y_1,\dots,y_{n-1})$ for $y_i = x_{i+1}-x_1$, $i=1,\dots,n-1$, we rewrite (2.2) as follows:

$$\mathsf{Cum}_n(V_\varrho^{(d,k)}) = (-1)^n \int\limits_{(\varrho(W\oplus(-W)))^{n-1}} \Big| \bigcap_{y\in Y_{n-1}\cup\{\mathbf{o}\}} (\varrho W + y)\Big|_d \, c^{(n)}_{\Xi^c}(Y_{n-1}\cup\{\mathbf{o}\})\,\mathrm{d}Y_{n-1} \qquad (4.1)$$

for any integer $n \geqslant 2$. Here and in what follows, we denote by $X_m = \{x_1,\dots,x_m\}$ and $Y_n = \{y_1,\dots,y_n\}$ the (unordered) sets of distinct points $x_1,\dots,x_m \in \mathbb{R}^d$ and $y_1,\dots,y_n\in\mathbb{R}^d$, respectively. $|Y|$ gives the number of elements of any finite set $Y\subset\mathbb{R}^d$. For notational simplicity, we put $p(Y) = p^{(|Y|)}_{\Xi^c}(Y)$ and $c(Y) = c^{(|Y|)}_{\Xi^c}(Y)$, so that, in view of (2.1), we may write

$$p(Y) = \exp\{-\lambda \mathsf{E}|\Xi_0(Y)|_{d-k}\} \quad \text{with } \Xi_0(Y) := \bigcup_{y\in Y} (\Xi_0 - \pi_{d-k}(O_0^T y)). \qquad (4.2)$$

Further, write $\Xi_0^c(Y)$ for the complement of $\Xi_0(Y)$ in $\mathbb{R}^{d-k}$ and put $\Xi_0(\emptyset)=\emptyset$, $\Xi_0^c(\emptyset) = \mathbb{R}^{d-k}$, $p(\emptyset)=1$, and $c(\emptyset)=0$. Note that $c(\{y\}) = 1 - c^{(1)}_{\Xi}(y) = p(\{y\}) = \exp\{-\lambda M_1\}$ for any $y\in\mathbb{R}^d$. Since $W\oplus(-W)\subseteq b(\mathbf{o},2)$ as consequence of $W\subseteq b(\mathbf{o},1)$, it follows from (4.1) that

$$\big|\,\mathsf{Cum}_{n+1}(|\Xi\cap\varrho W|_d)\big| \leqslant \varrho^d |W|_d \int\limits_{(b(\mathbf{o},2\varrho))^n} \big|c(\{\mathbf{o}\}\cup Y_n)\big|\,\mathrm{d}Y_n. \qquad (4.3)$$

The (mixed) cumulant functions $c(Y)$ are connected with the (mixed) moment functions $p(U)$, $\emptyset\neq U\subseteq Y$, of the random field $\{\mathbf{1}_{\Xi^c}(x),\ x\in\mathbb{R}^d\}$ by

$$c(Y) = \sum_{j=1}^{|Y|} (-1)^{j-1}(j-1)! \sum_{U_1\cup\cdots\cup U_j = Y} p(U_1)\cdots p(U_j) \quad \text{for any finite } Y\subset\mathbb{R}^d,$$

where the inner sum runs over all decompositions of Y into pairwise disjoint, nonempty subsets $U_1,\dots,U_j$. This formula follows directly by calculating the derivatives in (5.1). The equivalent relationships $c(Y) = p(Y) - \sum_{\emptyset\subset X\subset Y} c(X)p(Y\setminus X)$ or

$$c(\{x\}\cup Y_n) = p(\{x\}\cup Y_n) - \sum_{\emptyset\subseteq Y\subset Y_n} c(\{x\}\cup Y)p(Y_n\setminus Y) \quad \text{for } x\in\mathbb{R}^d\setminus Y_n$$

do not really help to establish sharp upper bounds of the integral on the rhs of (4.3). Rather than this, we introduce the more general functions $X_m\times Y_n \mapsto c(X_m,Y_n)$ for arbitrary $m\geqslant 1$ and $n\geqslant 1$ (with $X_m\cap Y_n=\emptyset$) by using the recursive relation

$$p(X_m\cup Y_n) = \sum_{\emptyset\subseteq Y\subseteq Y_n} c(X_m,Y)p(Y_n\setminus Y) \quad \text{with } c(X_m,\emptyset) = p(X_m). \qquad (4.4)$$

Obviously, $c(X_m,Y_n)$ is symmetric in $x_1,\dots,x_m$ and in $y_1,\dots,y_n$, but the x_i's and y_j's cannot be interchanged. Furthermore, we have $c(\{x\},Y_n) = c(\{x\}\cup Y_n)$ for $x\notin Y_n$ and $n\geqslant 0$.

As an immediate consequence of (4.4), the recursive relation

$$c(X_m, Y_n) = p(X_m \cup Y_n) - \sum_{\emptyset \subseteq Y \subset Y_n} c(X_m, Y) p(Y_n \setminus Y)$$

reveals that $c(X_m, Y_n)$ coincides with the $(n+1)$st-order mixed cumulant of the 0–1-valued random variables $\prod_{i=1}^{m} \mathbf{1}_{\Xi^c}(x_i)$ and $\mathbf{1}_{\Xi^c}(y_j)$, $j = 1, \dots, n$, which means, formally written, that $c(X_m, Y_n) = \mathsf{Cum}_{n+1}(\mathbf{1}\{\Xi \cap X_m = \emptyset\}, \mathbf{1}_{\Xi^c}(y_1), \dots, \mathbf{1}_{\Xi^c}(y_n))$.

The relation

$$c(X_m, Y_n) = \sum_{\emptyset \subseteq Y \subseteq Y_n} (-1)^{|Y|} K(X_m, Y) c(X_{m-1} \cup Y, Y_n \setminus Y) \quad \text{for } m + n \geqslant 1, \tag{4.5}$$

where $K(\emptyset, Y) = 0$ for $Y \neq \emptyset$ and

$$K(X_m, Y) = \sum_{\emptyset \subseteq V \subseteq Y} (-1)^{|V|} \frac{p(X_m \cup V)}{p(X_{m-1} \cup V)} \quad \text{for } m, n \geqslant 1,\ \emptyset \subseteq Y \subseteq Y_n,$$

has been shown in [4] by direct computation applying Möbius' inversion formula. Setting

$$p(V \mid U) := \frac{p(U \cup V)}{p(U)} = \mathsf{P}(\Xi \cap V = \emptyset \mid \Xi \cap U = \emptyset),$$

we can rewrite (4.5) in the following way:

$$c(X_m, Y_n) = \frac{p(X_m)}{p(X_{m-1})} \sum_{\emptyset \subseteq Y \subseteq Y_n} (-1)^{|Y|} S(X_m, Y) c(X_{m-1} \cup Y, Y_n \setminus Y), \tag{4.6}$$

where $S(\emptyset, Y) = 0$ for $Y \neq \emptyset$ and

$$S(X_m, Y) := \sum_{\emptyset \subseteq V \subseteq Y} (-1)^{|V|} \frac{p(V \mid X_m)}{p(V \mid X_{m-1})} \quad \text{for } \emptyset \subseteq Y \subseteq Y_n \text{ and } m, n \geqslant 1.$$

For our random set model (1.4), we get with (4.2) that

$$\begin{aligned} \frac{p(X_m \cup V)}{p(X_{m-1} \cup V)} &= \exp\{-\lambda \mathsf{E}|\Xi_0|_{d-k} + \lambda \mathsf{E}\big|(\Xi_0 - \pi_{d-k}(O_0^T x_m)) \cap \Xi_0(V \cup X_{m-1})\big|_{d-k}\} \\ &= \exp\{-\lambda \mathsf{E}\big|(\Xi_0 - \pi_{d-k}(O_0^T x_m)) \cap \Xi_0^c(X_{m-1})\big|_{d-k}\} \exp\{E(X_m, V)\}, \end{aligned}$$

where

$$E(X_m, V) := \lambda \mathsf{E}\big|(\Xi_0 - \pi_{d-k}(O_0^T x_m)) \cap \Xi_0^c(X_{m-1}) \cap \Xi_0(V)\big|_{d-k} \quad \text{for } \emptyset \subset V \subseteq Y_n$$

and $E(X_m, \emptyset) = 0$.

This leads to $p(V \mid X_m)/p(V \mid X_{m-1}) = \exp\{E(X_m, V)\}$, and thus

$$S(X_m, Y) = \sum_{\emptyset \subseteq V \subseteq Y} (-1)^{|V|} \exp\{E(X_m, V)\} \quad \text{for } Y \subseteq Y_n$$

and $S(X_m, \emptyset) = 1$ since $E(X_m, \emptyset) = 0$.

As a simple consequence of (4.6) and $c(X_m, \emptyset) = p(X_m) \leqslant p(X_{m-1}) \leqslant 1$, we get the inequality

$$\int\limits_{(b(\mathbf{o},2\varrho))^n} \left|c(X_m, Y_n)\right| \mathrm{d}Y_n \leqslant \int\limits_{(b(\mathbf{o},2\varrho))^n} \left|c(X_{m-1}, Y_n)\right| \mathrm{d}Y_n + \int\limits_{(b(\mathbf{o},2\varrho))^n} \left|S(X_m, Y_n)\right| \mathrm{d}Y_n$$
$$+ \sum_{\emptyset \subset Y \subset Y_n} \int\limits_{(b(\mathbf{o},2\varrho))^{|Y|}} \left|S(X_m, Y)\right| \mathrm{d}Y$$
$$\times \sup_Y \int\limits_{(b(\mathbf{o},2\varrho))^{n-|Y|}} \left|c(X_{m-1} \cup Y, Y_n \setminus Y)\right| \mathrm{d}(Y_n \setminus Y). \tag{4.7}$$

For any $m \geqslant 1$, we have $c(X_m, \{y\}) = p(X_m \cup \{y\}) - p(X_m)p(\{y\})$ $(\geqslant 0)$ and thus, by (4.2),

$$\begin{aligned} c(X_m, \{y\}) &= \exp\{-\lambda \mathsf{E}|\Xi_0(X_m \cup \{y\})|_{d-k}\} - \exp\{-\lambda \mathsf{E}|\Xi_0(X_m)|_{d-k} - \lambda \mathsf{E}|\Xi_0|_{d-k}\} \\ &= \exp\{-\lambda \mathsf{E}|\Xi_0(X_m \cup \{y\})|_{d-k}\}(1 - \exp\{-\lambda \mathsf{E}|\Xi_0(X_m) \cap (\Xi_0 - \pi_{d-k}(O_0^T y))|_{d-k}\}) \\ &\leqslant \lambda \exp\{-\lambda \mathsf{E}|\Xi_0(X_m)|_{d-k}\} \sum_{i=1}^m \mathsf{E}|(\Xi_0 - \pi_{d-k}(O_0^T x_i)) \cap (\Xi_0 - \pi_{d-k}(O_0^T y))|_{d-k}. \end{aligned}$$

Therefore, since $M_1 = \mathsf{E}|\Xi_0|_{d-k} \leqslant \mathsf{E}|\Xi_0(X_m)|_{d-k}$, we get

$$\int\limits_{b(\mathbf{o},2\varrho)} c(X_m, \{y\}) \,\mathrm{d}y \leqslant \lambda \exp\{-\lambda M_1\} \sum_{i=1}^m \int\limits_{b(\mathbf{o},2\varrho)} \mathsf{E}|\Xi_0 \cap (\Xi_0 - \pi_{d-k}(O_0^T (y - x_i)))|_{d-k} \,\mathrm{d}y.$$

The integrals on the rhs can be bounded from above uniformly in the x_i's. Multiple application of Fubini's theorem, combined with the shift-invariance of the Lebesgue measure in $\mathbb{R}^{d-k}$, yields

$$\begin{aligned} &\int\limits_{b(\mathbf{o},2\varrho)} |\Xi_0 \cap (\Xi_0 - \pi_{d-k}(O_0^T (y - x)))|_{d-k} \,\mathrm{d}y \\ &\quad = \int\limits_{b(\mathbf{o},2\varrho)} |\Xi_0 \cap (\Xi_0 - \pi_{d-k}(y) + \pi_{d-k}(O_0^T x))|_{d-k} \,\mathrm{d}y \\ &\quad \leqslant \int\limits_{[-2\varrho,2\varrho]^k} \int\limits_{\mathbb{R}^{d-k}} |\Xi_0 \cap (\Xi_0 - z_1 + \pi_{d-k}(O_0^T x))|_{d-k} \,\mathrm{d}z_1 \,\mathrm{d}z_2 \\ &\quad = (4\varrho)^k |\Xi_0|^2_{d-k}. \end{aligned}$$

Hence,

$$\sup_{x \in \mathbb{R}^d} \int\limits_{b(\mathbf{o},2\varrho)} \mathsf{E}|(\Xi_0 - \pi_{d-k}(O_0^T x)) \cap (\Xi_0 - \pi_{d-k}(O_0^T y))|_{d-k} \,\mathrm{d}y \leqslant (4\varrho)^k \mathsf{E}|\Xi_0|^2_{d-k}\,, \tag{4.8}$$

so that we arrive at the uniform estimate

$$\sup_{X_m} \int\limits_{b(\mathbf{o},2\varrho)} c(X_m, \{y\}) \,\mathrm{d}y \leqslant C_{m,1} \varrho^k \quad \text{with } C_{m,1} = 4^k m \lambda \mathrm{e}^{-\lambda M_1} M_2. \tag{4.9}$$

Let us introduce a further nonnegative function $T(y_n; X_m, Y)$ by

$$T(y_n; X_m, Y) := \sum_{\emptyset \subseteq V \subseteq Y} (-1)^{|V|} \exp\{-E(y_n; X_m, V)\} \quad \text{for } Y \subseteq Y_{n-1},\ n \geqslant 2,$$

where, for $\emptyset \subseteq V \subseteq Y_{n-1}$,

$$E(y_n; X_m, V) := \lambda \mathsf{E}\big|\big(\Xi_0 - \pi_{d-k}(O_0^T x_m)\big) \cap \Xi_0^c(X_{m-1}) \cap \big(\Xi_0 - \pi_{d-k}(O_0^T y_n)\big) \cap \Xi_0(V)\big|_{d-k}.$$

In the next step of our estimation procedure, we determine constants A_n and B_n only depending on n, λ and the first $n+1$ moments $M_1, \dots, M_{n+1}$ of $|\Xi_0|_{d-k}$ such that the uniform estimates

$$\int\limits_{(b(\mathbf{o},2\varrho))^n} \big|S(X_m, Y_n)\big| \, \mathrm{d}Y_n \leqslant A_n \varrho^{kn} \quad \text{and} \quad \int\limits_{(b(\mathbf{o},2\varrho))^n} T(y_n; X_m, Y_{n-1}) \, \mathrm{d}Y_n \leqslant B_n \varrho^{kn} \tag{4.10}$$

hold. The following relations between S- and T-functions can be shown quite analogously to the proof of the corresponding Lemma 4 in [4]:

Lemma 2. *For any $m, n \geqslant 1$, we have*

$$\begin{aligned} S(X_m, Y_n) = {} & S(X_m, Y_{n-1})\big(1 - \exp\{E(X_m, \{y_n\})\}\big) - \exp\{E(X_m, \{y_n\})\} \\ & \times \sum_{\emptyset \subset Y \subseteq Y_{n-1}} T(y_n; X_m, Y) \exp\{E(X_m, Y)\} S(X_m \cup Y, Y_{n-1} \setminus Y). \end{aligned}$$

Combining the inequality $E(X_m, Y) \leqslant \lambda \mathsf{E}|(\Xi_0 - \pi_{d-k}(O_0^T x_m)) \cap \Xi_0(Y)|_{d-k} \leqslant \lambda M_1$ and (4.8) leads to

$$\int\limits_{b(\mathbf{o},2\varrho)} \big|S(X_m, \{y\})\big| \, \mathrm{d}y = \int\limits_{b(\mathbf{o},2\varrho)} \big(\exp\{E(X_m, \{y\})\} - 1\big) \, \mathrm{d}y \leqslant 4^k \lambda \mathrm{e}^{\lambda M_1} M_2 \varrho^k. \tag{4.11}$$

Thus, from Lemma 2 and $S(X_m, \emptyset) = 1$ it follows after obvious arrangements that

$$\begin{aligned} \int\limits_{(b(\mathbf{o},2\varrho))^n} \big|S(X_m, Y_n)\big| \, \mathrm{d}Y_n \leqslant {} & 4^k \lambda \mathrm{e}^{\lambda M_1} M_2 \varrho^k \int\limits_{(b(\mathbf{o},2\varrho))^{n-1}} \big|S(X_m, Y_{n-1})\big| \, \mathrm{d}Y_{n-1} \\ & + \mathrm{e}^{2\lambda M_1} \sum_{j=1}^{n-2} \binom{n-1}{j} \int\limits_{(b(\mathbf{o},2\varrho))^{j+1}} T(y_{j+1}; X_m, Y_j) \, \mathrm{d}Y_{j+1} \\ & \times \sup_{Y_i} \int\limits_{(b(\mathbf{o},2\varrho))^{n-j-1}} \big|S(X_m \cup Y_j, Y_{n-1} \setminus Y_j)\big| \, \mathrm{d}(Y_{n-1} \setminus Y_j) \\ & + \mathrm{e}^{2\lambda M_1} \int\limits_{(b(\mathbf{o},2\varrho))^n} T(y_n; X_m, Y_{n-1}) \, \mathrm{d}Y_n. \end{aligned} \tag{4.12}$$

To make the previous estimate explicit, we need upper bounds for the integrals over $T(y_n; X_m, Y_{n-1})$ w.r.t. the variables $Y_n = \{y_1, \dots, y_{n-1}, y_n\}$ for each $n \geqslant 2$.

Lemma 3. *For fixed $n \geqslant 2$, assume that $M_{n+1} < \infty$. Then, for any $m \geqslant 1$, both estimates in* (4.10) *hold with*

$$B_n = 4^{kn}(n-1)! \sum_{j=1}^{n-1} \frac{\lambda^j}{j!} \sum_{n_1 + \dots + n_j = n-1}^{>} \frac{M_{n_1+2}}{n_1!} \prod_{i=2}^{j} \frac{M_{n_i+1}}{n_i!} \tag{4.13}$$

and

$$A_n = A_{n-1}A_1 + \mathrm{e}^{2\lambda M_1}\sum_{j=0}^{n-2}\binom{n-1}{j}A_j B_{n-j}, \quad A_0 = 1, \quad A_1 = 4^k\lambda\mathrm{e}^{\lambda M_1}M_2. \tag{4.14}$$

Proof. Let X_m and $Y \subseteq Y_{n-1} = \{y_1, \ldots, y_{n-1}\}$ be fixed finite point sets, and let $y_n \in \mathbb{R}^d$. Using the independently marked Poisson process $\Pi_{\lambda,Q}$ with typical mark $(O_0, \Xi_0) \sim Q$, we introduce, in accordance with (1.3) and (1.4), a new stationary PCP and the corresponding stationary random union set $\Xi(y_n; X_m, Y)$ with typical cylinder base $\Xi_0(y_n; X_m, Y) = (\Xi_0 - \pi_{d-k}(O_0^T x_m)) \cap \Xi_0^c(X_{m-1}) \cap (\Xi_0 - \pi_{d-k}(O_0^T y_n)) \cap \Xi_0(Y)$ as follows:

$$\Xi(y_n; X_m, Y) = \bigcup_{i\geqslant 1} O_i\big((\Xi_i(y_n; X_m, Y) + P_i) \times \mathbb{R}^k\big) = \bigcup_{y\in Y} \Xi(y_n; X_m, \{y\}), \tag{4.15}$$

where $\Xi_i(y_n; X_m, Y) = (\Xi_i - \pi_{d-k}(O_i^T x_m)) \cap \Xi_i^c(X_{m-1}) \cap (\Xi_i - \pi_{d-k}(O_i^T y_n)) \cap \Xi_i(Y)$, $i \geqslant 1$, are i.i.d. random compact sets in $\mathbb{R}^{d-k}$ with $\Xi_i(Y) = \bigcup_{y\in Y}(\Xi_i - \pi_{d-k}(O_i^T y))$ (see also (4.2)).

We first show that $T(y_n; X_m, Y_{n-1})$ gives just the probability that the origin $\mathbf{o}$ lies in all the union sets $\Xi(y_n; X_m, \{y_j\})$, $j = 1, \ldots, n-1$. With the above-introduced notation, it is easily seen that

$$\mathsf{P}(\mathbf{o} \notin \Xi(y_n; X_m, Y)) = \exp\{-\lambda\mathsf{E}|\Xi_0(y_n; X_m, Y)|_{d-k}\} = \exp\{-E(y_n; X_m, Y)\}.$$

Taking into account the relations $\sum_{\emptyset\subseteq Y\subseteq Y_{n-1}}(-1)^{|Y|} = 0$ and $\Xi(y_n; X_m, \emptyset) = \emptyset$ combined with the second part of (4.15), we find by applying the inclusion–exclusion principle that

$$\begin{aligned} T(y_n; X_m, Y_{n-1}) &= \sum_{\emptyset\subseteq Y\subseteq Y_{n-1}} (-1)^{|Y|}\mathsf{P}(\mathbf{o} \notin \Xi(y_n; X_m, Y)) \\ &= \sum_{\emptyset\subset Y\subseteq Y_{n-1}} (-1)^{|Y|-1}\mathsf{P}\Big(\bigcup_{y\in Y}\{\mathbf{o} \in \Xi(y_n; X_m, \{y\})\}\Big) \\ &= \mathsf{P}\Big(\bigcap_{j=1}^{n-1}\{\mathbf{o} \in \Xi(y_n; X_m, \{y_j\})\}\Big) = \mathsf{E}\Big(\prod_{j=1}^{n-1}\mathbf{1}_{\Xi(y_n;X_m,\{y_j\})}(\mathbf{o})\Big), \end{aligned}$$

whence, again by Fubini's theorem, it follows that

$$\int_{(b(\mathbf{o},2\varrho))^{n-1}} T(y_n; X_m, Y_{n-1})\,\mathrm{d}Y_{n-1} = \mathsf{E}\Big(\int_{b(\mathbf{o},2\varrho)} \mathbf{1}_{\Xi(y_n;X_m,\{y\})}(\mathbf{o})\,\mathrm{d}y\Big)^{n-1}.$$

Furthermore, the subadditivity of the Dirac measure $\mathbf{1}_{(\cdot)}(\mathbf{o})$, combined with the inclusion relation $\Xi_i(y_n; X_m, \{y\}) \subseteq (\Xi_i - \pi_{d-k}(O_i^T x_m)) \cap (\Xi_i - \pi_{d-k}(O_i^T y_n)) \cap (\Xi_i - \pi_{d-k}(O_i^T y))$, shows that

$$\begin{aligned} \int_{b(\mathbf{o},2\varrho)} \mathbf{1}_{\Xi(y_n;X_m,\{y\})}(\mathbf{o})\,\mathrm{d}y &\leqslant \sum_{i\geqslant 1}\int_{b(\mathbf{o},2\varrho)} \mathbf{1}_{(\Xi_i(y_n;X_m,\{y\})+P_i)\times\mathbb{R}^k}(\mathbf{o})\,\mathrm{d}y \\ &\leqslant (4\varrho)^k\sum_{i\geqslant 1}\mathbf{1}_{(\Xi_i-\pi_{d-k}(O_i^T x_m))\cap(\Xi_i-\pi_{d-k}(O_i^T y_n))}(-P_i)|\Xi_i|_{d-k}. \end{aligned}$$

In the last line, we have replaced the integral of $\mathbf{1}_{\Xi_i+P_i}(\pi_{d-k}(O_i^T y))$ over the ball $b(\mathbf{o}, 2\varrho)$ by the larger term $(4\varrho)^k|\Xi_i|_{d-k}$. Some elementary algebraic rearrangements and the application of the higher-order Campbell's formula (1.2), together with the reflection invariance of stationary Poisson processes, enable us to rewrite the $(n-1)$st moment of the random sum

$$Z(x_m, y_n) = \sum_{i\geqslant 1} \mathbf{1}_{(\Xi_i - \pi_{d-k}(O_i^T x_m))\cap(\Xi_i - \pi_{d-k}(O_i^T y_n))}(-P_i)|\Xi_i|_{d-k}$$

in the following way:

$$\sum_{j=1}^{n-1} \sum_{n_1+\cdots+n_j=n-1}^{>} \frac{(n-1)!}{j!n_1!\cdots n_j!} \mathsf{E} \sum_{i_1,\ldots,i_j\geqslant 1}^{*} \prod_{q=1}^{j} \bigl(\mathbf{1}_{(\Xi_{i_q} - \pi_{d-k}(O_{i_q}^T x_m))\cap(\Xi_{i_q} - \pi_{d-k}(O_{i_q}^T y_n))}(P_{i_q})|\Xi_{i_q}|_{d-k}^{n_q}\bigr)$$
$$= \sum_{j=1}^{n-1} \sum_{n_1+\cdots+n_j=n-1}^{>} \frac{\lambda^j (n-1)!}{j!n_1!\cdots n_j!} \prod_{q=1}^{j} \mathsf{E}\bigl(\bigl|(\Xi_0 - \pi_{d-k}(O_0^T x_m)) \cap (\Xi_0 - \pi_{d-k}(O_0^T y_n))\bigr|_{d-k} |\Xi_0|_{d-k}^{n_q}\bigr).$$

Together with

$$\int_{b(\mathbf{o},2\varrho)} \mathsf{E}\bigl(\bigl|(\Xi_0 - \pi_{d-k}(O_0^T x_m)) \cap (\Xi_0 - \pi_{d-k}(O_0^T y_n))\bigr|_{d-k} |\Xi_0|_{d-k}^{n_1}\bigr)\,\mathrm{d}y_n \leqslant (4\varrho)^k \mathsf{E}|\Xi_0|_{d-k}^{n_1+2},$$

we arrive at

$$\int_{(b(\mathbf{o},2\varrho))^n} T(y_n; X_m, Y_{n-1})\,\mathrm{d}Y_n \leqslant (4\varrho)^{k(n-1)} \int_{(b(\mathbf{o},2\varrho))} \mathsf{E}\bigl(Z(x_m, y_n)\bigr)^{n-1}\,\mathrm{d}y_n \leqslant B_n \varrho^{kn}$$

with B_n as given in (4.13). Hence, the second estimate in (4.10) is proved.

From (4.11) and (4.12) we obtain the first estimate of (4.10) with a recursive relation for the constants A_n with $A_1 = 4^k \lambda \mathrm{e}^{\lambda M_1} M_2$ and $A_0 = 1$. More precisely,

$$\int_{(b(\mathbf{o},2\varrho))^n} \bigl|S(X_m, Y_n)\bigr|\,\mathrm{d}Y_n \leqslant A_1 \varrho^k A_{n-1} \varrho^{k(n-1)} + \mathrm{e}^{2\lambda M_1} \varrho^{kn} \sum_{j=1}^{n-1} \binom{n-1}{j} B_{j+1} A_{n-j-1} = A_n \varrho^{kn},$$

which gives (4.14). Thus, the proof of Lemma 3 is completed. □

We are now in a position to prove the estimate

$$\sup_{X_m} \int_{(b(\mathbf{o},2\varrho))^n} \bigl|c(X_m, Y_n)\bigr|\,\mathrm{d}Y_n \leqslant C_{m,n} \varrho^{kn} \quad \text{for any } m, n \geqslant 1, \tag{4.16}$$

where $C_{m,n}$ depends on m, n, λ, and $M_1, \ldots, M_{n+1}$. From (4.9) we already know that (4.16) is true for $n = 1$ and any $m \geqslant 1$. Inserting the first estimate of (4.10) with constants (4.14) on the rhs of (4.7), we get

$$\int_{(b(\mathbf{o},2\varrho))^n} \bigl|c(X_m, Y_n)\bigr|\,\mathrm{d}Y_n \leqslant C_{m-1,n} \varrho^{kn} + A_n \varrho^{kn} + \sum_{j=1}^{n-1} \binom{n}{j} A_j \varrho^{kj} C_{m-1+j,n-j} \varrho^{k(n-j)},$$

which immediately implies estimate (4.16) and the double-index recursion formula

$$C_{m,n} = A_n + \sum_{j=1}^{n} \binom{n}{j} A_{n-j} C_{m-1+n-j,j} \quad \text{with } C_{0,n} = 0 \text{ for } m, n \geqslant 1. \tag{4.17}$$

This equation allows us to determine successively all constants $C_{m,n}$ starting with $C_{m,2}$ depending on A_1 and A_2 for all $m \geqslant 1$ and afterwards $C_{m,3}$ depending on A_1, A_2, and A_3 for all $m \geqslant 1$, etc. For example, we have $C_{m,2} = A_2 + 2A_1C_{m,1} + C_{m-1,2}$ leading to $C_{m,2} = mA_2 + m(m+1)A_1C_{1,1}$ for $m \geqslant 1$.

Having in mind the identity $c(\{\mathbf{o}\} \cup Y_n) = c(\{\mathbf{o}\}, Y_n)$, we deduce from (4.3) and (4.16) that

$$\big|\,\mathsf{Cum}_n\big(|\Xi \cap \varrho W|_d\big)\big| \leqslant |W|_d C_{1,n-1}\varrho^{d+k(n-1)}, \tag{4.18}$$

where $C_{1,n-1}$ depends on λ and $M_1, \dots, M_n$. In the final step, we determine the growth of the constants $C_{1,n-1}$ in dependence on $n \geqslant 2$ under assumption (1.5). In this case, we have $M_n \leqslant n!a^{-n}m_a$ for $n \in \mathbb{N}$, so that formula (4.13) yields

$$B_n \leqslant 4^{kn}(n-1)!\sum_{j=1}^{n-1}\frac{\lambda^j}{j!}\frac{m_a^j}{a^{j+1}}\sum_{n_1+\cdots+n_j=n-1}^{>}(n_1+2)\prod_{i=1}^{j}\frac{(n_i+1)}{a^{n_i}}.$$

Since $n+1 \leqslant 2^n$ for $n \in \mathbb{N}$, we have

$$\sum_{n_1+\cdots+n_j=n-1}^{>}(n_1+2)\prod_{i=1}^{j}\frac{(n_i+1)}{a^{n_i}} \leqslant \frac{n}{a^{n-1}}\sum_{n_1+\cdots+n_j=n-1}^{>}2^{n_1+1}\prod_{i=2}^{j}2^{n_i} = \frac{n2^n}{a^{n-1}}\binom{n-2}{j-1},$$

which, in turn, gives

$$B_n \leqslant \frac{2^n4^{kn}}{a^n}n!\sum_{j=1}^{n-1}\frac{\lambda^j m_a^j}{a^j}\binom{n-2}{j-1} = \frac{\lambda m_a}{a}\left(\frac{2\cdot 4^k}{a}\right)^n\left(1+\frac{\lambda m_a}{a}\right)^{n-2}n! \quad \text{for } n \geqslant 2.$$

In summary, using the abbreviations

$$A = \frac{2^{2k+1}}{a}\big(1+\exp\{\lambda\mathsf{E}|\Xi_0|_{d-k}\}\big) \quad \text{and} \quad B = \frac{\lambda\mathsf{E}\exp\{a|\Xi_0|_{d-k}\}}{a},$$

the positive constants A_n and B_n in (4.10) satisfy the estimates $A_1 \leqslant AB$ and

$$A_n \leqslant A^nB(1+B)^{n-1}n! \quad \text{and} \quad B_n \leqslant B\left(\frac{2^{2k+1}}{a}\right)^n(1+B)^{n-2}n! \quad \text{for } n \geqslant 2. \tag{4.19}$$

The first relation follows from (4.14) by induction on n. In fact, by $M_2 \leqslant 2m_a/a^2$, we have

$$A_1 = 4^k\lambda e^{\lambda M_1}M_2 \leqslant 4^k\lambda e^{\lambda M_1}\frac{2m_a}{a^2} = \frac{2^{2k+1}}{a}e^{\lambda M_1}B \leqslant AB,$$

and, for $n \geqslant 2$, we combine the recursive relation (4.14) with $A_0 = 1$ and the second (already proved) estimate in (4.19):

$$\begin{aligned} A_n &= A_1A_{n-1} + e^{2\lambda M_1}\sum_{j=0}^{n-2}\binom{n-1}{j}A_jB_{n-j} \\ &\leqslant A_1A_{n-1} + e^{2\lambda M_1}B\sum_{j=0}^{n-2}\binom{n-1}{j}A_j\left(\frac{2^{2k+1}}{a}\right)^{n-j}(1+B)^{n-j-2}(n-j)!. \end{aligned}$$

Replacing A_j by $A^j B(1+B)^{j-1} j!$ for $j = 1, \ldots, n-1$, we find after some elementary calculations the asserted first estimate in (4.19).

In the same way, the recursive relation (4.17) suggests an inductive proof of the estimate

$$C_{m,n} \leqslant 2^{m-1} 4^{n-1} A^n B(1+B)^{n-1} n! \quad \text{for } n, m \geqslant 1,$$

whence with (4.18) the desired estimate (2.3) follows, completing the proof of Theorem 1.

Now, we apply the general lemma on large deviations including an optimal Berry–Esseen bound proved by Statulevičius in [18] (see also Lemma 2.3 in [15]). This result is formulated for a single random variable ξ satisfying $\mathsf{E}\xi = 0$, $\mathsf{Var}(\xi) = 1$ and $|\,\mathsf{Cum}_n(\xi)| \leqslant n! H/\Delta^{n-2}$ for $n \geqslant 2$ and some $H \geqslant 1/2$ and $\Delta > 0$. In our specific situation, ξ is chosen to be the standardized d-volume $V_\varrho^{(d,k)}$, i.e.,

$$\xi = \frac{V_\varrho^{(d,k)} - \mathsf{E} V_\varrho^{(d,k)}}{\sqrt{\mathsf{Var}(V_\varrho^{(d,k)})}} = \frac{V_\varrho^{(d,k)} - \varrho^d |W|_d (1 - \exp\{-\lambda \mathsf{E}|\Xi_0|_{d-k}\})}{\sigma_\varrho \varrho^{(d+k)/2}}$$

with distribution function $F_\varrho(x) = \mathsf{P}(\xi \leqslant x)$. Using (2.3) and the notation introduced in Section 2, we obtain that

$$|\,\mathsf{Cum}_n(\xi)| \leqslant (n-1)! \frac{H_a \Delta_a^{n-2} \varrho^{d+k(n-1)}}{(\mathsf{Var}(V_\varrho^{(d,k)}))^{n/2}} \leqslant n! H_{a,\varrho}/\Delta_{a,\varrho}^{n-2},$$

where $H_{a,\varrho} = H_a/2\sigma_\varrho^2$ ($\geqslant 1/2$ by (2.3) for $n = 2$) and $\Delta_{a,\varrho} = \varrho^{(d-k)/2}\sigma_\varrho/\Delta_a$.

These estimates and the lemma in [18], p. 133, imply the asymptotic relations (2.5), (2.6) and the Berry–Esseen bound (2.8) stated in Theorem 2. It should be noted that, according to the general result in [15] or [18], relations (2.5) and (2.6) hold in a smaller interval $0 \leqslant x \leqslant \delta^* \Delta_{a,\varrho}$ for $\delta^* < \delta(1+\delta)/2$, where $\delta \in (0,1)$ is uniquely determined by the equation $(1-\delta)^3 = 6H_{a,\varrho}\delta$ giving $\delta(1+\delta)/2 \leqslant 1/2(1+4H_{a,\varrho})$. However, a careful check of the original proof reveals that (2.5) and (2.6) remain valid for larger x-values because, in contrast to [18], the explicitly known coefficients (2.7) of the Cramér series $\mu(x) := \sum_{s \geqslant 0} \mu_s^{(\varrho)} (x/\sigma_\varrho \varrho^{(d+k)/2})^s$ can be estimated directly by means of (2.3). For this, we use (2.3) and $\binom{s+j+1}{j} \leqslant 2^{s+j}$ and get that, for any $s \geqslant 0$,

$$\begin{aligned}
|\mu_s^{(\varrho)}| &\leqslant \frac{1}{(s+2)(s+3)} \sum_{j=1}^{s+1} 2^{s+j} \sum_{s_1+\cdots+s_j=s+1}^{>} \prod_{i=1}^{j} \left(\frac{\Delta_a^{s_i} \varrho^{k s_i} H_a}{\sigma_\varrho^2} \right) \\
&= \frac{2^s \Delta_a^{s+1} \varrho^{k(s+1)}}{(s+2)(s+3)} \sum_{j=1}^{s+1} \binom{s}{j-1} \left(\frac{2H_a}{\sigma_\varrho^2} \right)^j \\
&= \frac{4H_{a,\varrho}\Delta_a \varrho^k}{(s+2)(s+3)} \bigl(2\Delta_a \varrho^k (1+4H_{a,\varrho})\bigr)^s.
\end{aligned}$$

Thus, $\mu(x)$ converges absolutely for $|x| \leqslant \Delta_{a,\varrho}/2(1+4H_{a,\varrho})$ such that $|\mu(x)| \leqslant 2H_{a,\varrho}\Delta_a\varrho^k$, proving the validity of (2.5) and (2.6) in the desired interval $0 \leqslant x \leqslant \Delta_{a,\varrho}/2(1+4H_{a,\varrho})$, which completes the proof of Theorem 2.

5 APPENDIX

In the previous sections, we were dealing with the volume fraction of the random set $\Xi_{\mathrm{cyl}}^{(d,k)}(\lambda, Q)$ in various Borel sets. For this purpose, it suffices to consider the stationary 0–1-random field $\{\mathbf{1}_\Xi(x),\ x \in \mathbb{R}^d\}$ the finite-dimensional distributions of which are given by the family of n-point probabilities $p_\Xi^{(n)}(x_1, \ldots, x_n) =$

$\mathsf{P}(x_1 \in \varXi,\ \ldots,\ x_n \in \varXi)$ or by $p^{(n)}_{\varXi^c}(x_1,\ldots,x_n) = \mathsf{P}(x_1 \notin \varXi,\ \ldots,\ x_n \notin \varXi)$ for $x_1,\ldots,x_n \in \mathbb{R}^d$, $n \in \mathbb{N}$. In general, these distributions do not describe the properties of a random set $\varXi$ completely (see [13]). In case of the random set (1.4), one can choose the canonical probability space $[\Omega, \mathfrak{F}, \mathsf{P}]$ on which the marked Poisson point process $\varPi_{\lambda,Q}$ (introduced in Section 1) is defined so that the mapping $\mathbb{R}^d \times \Omega \ni (x,\omega) \mapsto \mathbf{1}_{\varXi(\omega)}(x)$ turns out to be $\mathfrak{B}(\mathbb{R}^d) \otimes \mathfrak{F}$-measurable. This follows by repeating the arguments (with obvious changes) proving Proposition 1 in [4]. Hence, Fubini's theorem and the probability generating functional (1.1) can be applied to calculate the n-point probabilities of the complement $\varXi^c$,

$$p^{(n)}_{\varXi^c}(x_1,\ldots,x_n) = \mathsf{P}(\varXi \cap \{x_1,\ldots,x_n\} = \emptyset) = \mathsf{E}\Big(\prod_{i\geqslant 1} \mathbf{1}\{O_i((\varXi_i + P_i)\times \mathbb{R}^k) \cap \{x_1,\ldots,x_n\} = \emptyset\}\Big),$$

which immediately shows the validity of (2.1), provided that $\mathsf{E}|\varXi_0|_{d-k} < \infty$. It is a matter of fact that even the boundedness of the random $(d-k)$-volume $|\varXi_0|_{d-k}$ does not imply the closedness of the random set (1.4). In the theory of random closed sets (see [12] or [13]), the distribution of $\varXi$ is uniquely determined by its capacity functional $T_\varXi(C) = \mathsf{P}(\varXi \cap C \neq \emptyset)$ defined on the family of nonempty compact sets $C \in \mathcal{K}_d$. The union set $\varXi = \varXi^{(d,k)}_{\mathrm{cyl}}(\lambda, Q)$ is (P-a.s.) closed if any ball in $\mathbb{R}^d$ hits at most finitely many cylinders $O_i((\varXi_i + P_i)\times\mathbb{R}^k)$ with probability 1, which, in turn, is equivalent to $\mathsf{E}|\varXi_0 \oplus \pi_{d-k}(b(\mathbf{o},\varepsilon))|_{d-k} < \infty$ for some $\varepsilon > 0$ (see [3]) for general germ-grain models. Under the latter condition, the explicit form of $T_\varXi(C)$ can be calculated for (1.4). Applying again the probability generating functional (1.1) with $v(x,O,K) = \mathbf{1}\{O((K+x)\times\mathbb{R}^k) \cap C = \emptyset\}$, we have

$$\begin{aligned} 1 - T_\varXi(C) &= \mathsf{P}(\varXi \cap C = \emptyset) \\ &= \mathsf{E}\Big(\prod_{i\geqslant 1} \mathbf{1}\{O_i((\varXi_i + P_i)\times\mathbb{R}^k) \cap C = \emptyset\}\Big) \\ &= \exp\Big\{-\lambda \mathsf{E}\int_{\mathbb{R}^{d-k}} \mathbf{1}\{O_0((\varXi_0 + x)\times\mathbb{R}^k) \cap C \neq \emptyset\}\,\mathrm{d}x\Big\}. \end{aligned}$$

Since $O_0((\varXi_0 + x)\times\mathbb{R}^k) \cap C \neq \emptyset$ iff $(\varXi_0 + x) \cap \pi_{d-k}(O_0^T C) \neq \emptyset$ and the latter is equivalent to $x \in \varXi_0 \oplus (-\pi_{d-k}(O_0^T C))$, we arrive at

$$T_\varXi(C) = 1 - \exp\{-\lambda \mathsf{E}|\varXi_0 \oplus (-\pi_{d-k}(O_0^T C))|_{d-k}\};$$

see, e.g., [17].

The following lemma, which we formulate without proof, extends an analogous statement for Boolean models in [4] to unions of PCPs (1.4). This result implies that, under the assumption $\mathsf{E}|\varXi_0|_{d-k} < \infty$, the additional condition $\mathsf{E}|\varXi_0 \oplus \pi_{d-k}(b(\mathbf{o},\varepsilon))|_{d-k} < \infty$ for some $\varepsilon > 0$ is not only sufficient but even necessary for the closedness of the stationary random union set (1.4).

Lemma 4. *Let $\varXi_0$ be a compact typical cylinder base of* (1.3) *satisfying* $\mathsf{E}|\varXi_0|_{d-k} < \infty$ *and* $\mathsf{E}|\varXi_0 \oplus \pi_{d-k}(b(\mathbf{o},\varepsilon))|_{d-k} = \infty$ *for any $\varepsilon > 0$. Then* $\mathsf{P}(\varXi^{(d,k)}_{\mathrm{cyl}}(\lambda, Q)$ is closed in $\mathbb{R}^d) = 0$.

The proof of Lemma 4 is quite similar to that in [4] for Boolean models. The necessary changes and extensions are left to the reader.

Next, we put together some basic facts on the "method of cumulants." There exists a huge and widely scattered literature in stochastics (see, e.g., [10]) and statistical physics in connection with *cluster expansions* (see, e.g., [14]), in which cumulant techniques are employed to express the weakness of stochastic dependence between temporally (or spatially) distant parts of random processes (or fields). In statistics and probability theory, these cumulant estimates are mainly used to prove the asymptotic Gaussianity of functionals of random processes (or fields) over expanding domains. For obtaining even rates of convergence in these limit theorems and exact large-deviation probabilities based on cumulant estimates, the reader is referred to the monograph [15]. Note that, in finding optimal rates, the corresponding estimation procedures are partly rather lengthy and sophisticated (see [5] for an example).

Let us recall the definition of the *mixed cumulant* (*semi-invariant*) $\mathsf{Cum}(Y_1, \dots, Y_n)$ of n random variables $Y_1, \dots, Y_n$ (all having a finite nth moment). Following [11], we define

$$\mathsf{Cum}(Y_1, \dots, Y_n) = \mathbf{i}^{-n} \frac{\partial^n}{\partial s_1 \dots \partial s_n} \log \mathsf{E} \exp\left\{ \mathbf{i} \sum_{j=1}^{n} s_j Y_j \right\} \bigg|_{s_1 = \dots = s_n = 0}, \tag{5.1}$$

and $\mathsf{Cum}_n(Y) = \mathsf{Cum}(Y, \dots, Y)$ (by setting $Y = Y_1 = \dots = Y_n$ in (5.1)) denotes the usual nth *cumulant* of Y. From (5.1) it is easily seen that $\mathsf{Cum}(Y_1, \dots, Y_n)$ is invariant under permutation of the indices $\{1, \dots, n\}$ and $\mathsf{Cum}(\dots, aY + bZ + c, \dots) = a\,\mathsf{Cum}(\dots, Y, \dots) + b\,\mathsf{Cum}(\dots, Z, \dots)$ in each component for any $a, b, c \in \mathbb{R}^1$, $n \geqslant 2$.

Let $\{\mathbf{1}_\Xi(x),\ x \in \mathbb{R}^d\}$ be a measurable 0–1-random field, for example, when Ξ coincides with (1.4). Obviously, the mixed cumulant function

$$c_\Xi^{(n)}(x_1, \dots, x_n) = \mathsf{Cum}\big(\mathbf{1}_\Xi(x_1), \dots, \mathbf{1}_\Xi(x_n)\big)$$

equals $(-1)^n\,\mathsf{Cum}(\mathbf{1}_{\Xi^c}(x_1), \dots, \mathbf{1}_{\Xi^c}(x_n))$. By combining the identities $|\Xi \cap B_i| = \int_{B_i} \mathbf{1}_\Xi(x_i)\,\mathrm{d}x_i$ for $i = 1, \dots, n$ with the linearity of (5.1) in each component we get

$$\mathsf{Cum}\big(|\Xi \cap B_1|, \dots, |\Xi \cap B_n|\big) = (-1)^n \int\limits_{B_1} \dots \int\limits_{B_n} c_{\Xi^c}^{(n)}(x_1, \dots, x_n)\,\mathrm{d}x_1 \dots \mathrm{d}x_n \tag{5.2}$$

for any bounded $B_1, \dots, B_n \in \mathfrak{B}(\mathbb{R}^d)$. By calculating the logarithmic derivatives in (5.1) (see, e.g., [11, 15] or [5]), the cumulant function $c_{\Xi^c}^{(n)}(x_1, \dots, x_n)$ can be expressed by the k-point probabilities $p_{\Xi^c}^{(k)}(x_{i_1}, \dots, x_{i_k})$ for $1 \leqslant i_1 < \dots < i_k \leqslant n$ and $k = 1, \dots, n$.

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