On the Expected Surface Area of the Wiener Sausage

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For parallel neighborhoods of the paths of the d–dimensional Brownian motion, so–called *Wiener sausages*, formulae for the expected surface area are given for any dimension $d \geq 2$. It is shown by means of geometric arguments that the expected surface area is equal to the first derivative of the mean volume of the Wiener sausage with respect to its radius.

1 Introduction

The trace of a moving spherical particle involved in a d–dimensional Brownian motion observed within a certain time interval $[0, T]$, where $0 < T < \infty$, is often called a *Wiener sausage*. It is used in physics, chemistry, biology and technology to model various phenomena; see [31] and the references therein. Further recent applications are wireless sensor networks; see e.g. [11] and [22].

There exists an extensive literature on Wiener sausages; see e.g. [25] and the references therein. Nevertheless, relatively little is known so far about the geometry of Wiener sausages due to the complex nature of their realizations. One possible description of the geometric structure of (sufficiently regular) subsets of $\mathbb{R}^d$ is given by their $d+1$ *intrinsic volumes* or *Minkowski functionals* including the usual volume, surface area and other curvature measures as well as the Euler–Poincaré characteristic; see e.g. [5]. As far as it is known to the authors, the only Minkowski functional of Wiener sausages studied in the literature is their expected d–dimensional Lebesgue measure. Thus, explicit formulae for the expected volume of Wiener sausages and its asymptotic behavior for $T \to 0$ or $T \to \infty$ can be found in [9], [13], [23]; see also [2]. The results of corresponding simulation studies in three dimensions are discussed in [31]. Further limit theorems and deviation results for the volume of Wiener sausages are proved in [15] and [27]. Asymptotic long–time behavior of its moment generating function is given in [4], [26], [28]; see also [25], pp. 201 and 315.

In the present paper, the mean surface area of Wiener sausages is determined for all dimensions greater than one. However, there remains an interesting unsolved problem to obtain corresponding formulae for the other $d-1$ expected intrinsic volumes which would then give a more complete description of the geometric properties of Wiener sausages. These intrinsic volumes are well defined; see Corollary 4.4 of this paper.

The paper is organized as follows. In Section 2, some preliminary properties of Wiener sausages are discussed and the main results are stated. In particular, in Theorem 2.2, a representation formula for the expected surface area of Wiener sausages is given. The proof of this theorem is postponed to Section 4. It makes use of some auxiliary geometric results on the differentiability of the volume of parallel neighborhoods of compact sets given in Section 3.

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Fig. 1 A realization of S_r for $r = 10$ (left) and $r = 40$ (right)

2 Wiener Sausage

2.1 Random Closed Sets

Let $\{W(t) : t \geq 0\}$ be the *Wiener process* with variance σ^2 initiated at $x \in \mathbb{R}$ defined on the probability space $(\Omega, \mathfrak{F}, P)$. Let $\{W_1(t)\}, \ldots, \{W_d(t)\}$ be d independent Wiener processes starting at $x_1, \ldots, x_d \in \mathbb{R}$, respectively. Then, the random function $\{X(t), t \geq 0\}$ with $X(t) = \big(W_1(t), \ldots, W_d(t)\big)$ is called a d–dimensional *Brownian motion* initiated at $(x_1, \ldots, x_d) \in \mathbb{R}^d$; see e.g. [3].

The *Minkowski sum* of two sets A and B in $\mathbb{R}^d$ is given by $A \oplus B = \{x + y : x \in A, y \in B\}$. If B is the ball $B_r(o)$ of radius $r \geq 0$ in $\mathbb{R}^d$ centered at the origin, the set $A_r = A \oplus B_r(o)$ is often referred to as r*–parallel neighborhood of* A. The operation $A \mapsto A_r$ is known as *dilation*. Let $\mathcal{F}$ ($\mathcal{C}$) be the family of all closed (compact) subsets in $\mathbb{R}^d$, respectively. Denote by $\sigma_{\mathcal{F}}$ the σ–algebra generated by the sets $\{F \in \mathcal{F} : F \cap C \neq \emptyset\}$, $C \in \mathcal{C}$. A *random closed set* (RACS) Ξ is a random variable with values in $\mathcal{F}$, i.e., $\Xi : \Omega \to \mathcal{F}$ is a $(\mathfrak{F}, \sigma_{\mathcal{F}})$–measurable mapping. For the general theory of random sets, see e.g. [16], [17], [21].

Let $T > 0$ be a fixed a time instant and let $S(T) = \{X(t) : 0 \leq t \leq T\} \subseteq \mathbb{R}^d$ denote the Brownian path in $\mathbb{R}^d$. We shall often write only S instead of $S(T)$ unless the time T is changed. The set $S_r = S(T) \oplus B_r(o)$, $r \geq 0$ is called a *Wiener sausage*; see e.g. [25], p. 64.

In Figure 1, a simulated realization of S_r for $\sigma = 1$, $T = 25000$ and two dilation radii $r = 10$ and $r = 40$ is given.

Lemma 2.1 *For any $r > 0$, the Wiener sausage S_r is a compact RACS.*

Proof. Notice that for each $\omega \in \Omega$ the set $S(\omega)$ is compact as a continuous image of the compact interval $[0, T]$. To prove the measurability of S required in the definition of a random closed set, it is sufficient to show that the indicator function $\mathbf{1}(S \cap C = \emptyset)$ is a random variable for all $C \in \mathcal{C}$. Indeed, it holds

$$\mathbf{1}(S \cap C = \emptyset) = \mathbf{1}(X(t) \notin C,\ t \in [0, T]) = \mathbf{1}(\tau_C^x > T)\,,$$

where $\tau_C^x = \inf\{t \geq 0 : X(t) \in C\}$ is the first hitting time of the set C for the Wiener process X started at x. It is well known that τ_C^x is a random variable; cf. e.g. [30, § 6.1]. Hence, $\mathbf{1}(\tau_C^x > T)$ is measurable and S is a compact random closed set. Since the dilation preserves this property (see [21], p. 23), the Wiener sausage S_r is a RACS as well. □

2.2 Main Results

Let V_d denote the Lebesgue measure and $\mathcal{H}^s$ the s–dimensional Hausdorff measure in $\mathbb{R}^d$; see e.g. [20]. Then $\mathcal{H}^{d-1}(\partial S_r)$ is a random variable, cf. [1] and references therein. The main result of the present paper is the following representation formula of the expected surface area of the Wiener sausage S_r.

Theorem 2.2 *The expected surface area $\mathbb{E}\mathcal{H}^{d-1}(\partial S_r)$ of S_r is finite and given by*

$$\mathbb{E}\mathcal{H}^{d-1}(\partial S_r) = \frac{\mathrm{d}\,\mathbb{E}\,V_d(S_r)}{\mathrm{d}\,r} \tag{1}$$

for any $r > 0$ in dimensions two and three. For dimensions $d \geq 4$, formula (1) holds at least for almost all $r > 0$.

The *proof* of Theorem 2.2 will be postponed to Section 4.

Notice that as a consequence of (1), further formulae can be obtained for the expected surface area of S_r. In connection with this, the Bessel functions $J_\nu(y)$ and $Y_\nu(y)$ of the first and second kind of order $\nu = (d-2)/2$ are considered; see e.g. [29].

Corollary 2.3 *Let $d \geq 2$ and $r > 0$. Then, it holds that*

$$\mathbb{E}\mathcal{H}^{d-1}(\partial S_r) = d\omega_d r^{d-1}\left(1 + \frac{\sigma^2 T(d-2)^2}{2r^2} + \frac{4d}{\pi^2}\int\limits_0^\infty \frac{1-\left(1+\sigma^2 T y^2/(dr^2)\right)e^{-\frac{\sigma^2 y^2 T}{2r^2}}}{y^3\left(J_\nu^2(y)+Y_\nu^2(y)\right)}\,\mathrm{d}y\right) \tag{2}$$

for almost all radii $r > 0$, where $\omega_d = \pi^{d/2}/\Gamma(1+d/2)$ is the volume of the unit d–dimensional ball $B_1(o)$. For $d = 2, 3$, formula (2) *holds for all $r > 0$. Moreover, in the case $d = 3$, it simplifies to*

$$\mathbb{E}\mathcal{H}^2(\partial S_r) = 4\pi r^2 + 8r\sigma\sqrt{2\pi T} + 2\pi\sigma^2 T\,. \tag{3}$$

Proof. It is well–known (see e.g. [2]) that the expected volume $\mathbb{E}\,V_d(S_r)$ of the Wiener sausage S_r can be given by

$$\mathbb{E}\,V_d(S_r) = \omega_d r^d + \frac{d(d-2)}{2}\omega_d\,\sigma^2 r^{d-2} T + \frac{4d\,\omega_d\,r^d}{\pi^2}\int\limits_0^\infty \frac{1-e^{-\frac{\sigma^2 y^2 T}{2r^2}}}{y^3\left(J_\nu^2(y)+Y_\nu^2(y)\right)}\,dy\,. \tag{4}$$

Furthermore, in three dimensions, the latter formula simplifies to

$$\mathbb{E}\,V_3(S_r) = \frac{4}{3}\pi r^3 + 4\sigma r^2\sqrt{2\pi T} + 2\pi\sigma^2 r T\,,$$

see also [11] and [23]. By Theorem 2.2, differentiating the above expressions with respect to r yields the formulae (2) and (3). To see this we still have to show that it is possible to differentiate with respect to r under the integral sign in (4). Introduce the notation

$$f(y,r) = \frac{1-e^{-\frac{\sigma^2 y^2 T}{2r^2}}}{y^3(J_\nu^2(y)+Y_\nu^2(y))}$$

and notice that

$$\frac{\mathrm{d}}{\mathrm{d}r}f(y,r) = \frac{\varphi(\frac{\sigma^2 y^2 T}{2r^2})}{y^3(J_\nu^2(y)+Y_\nu^2(y))},$$

where $\varphi(z) = 1 - e^{-z} - \frac{2}{d}ze^{-z}$. It is easy to verify that φ is increases on $(0,\infty)$ from 0 to 1, and that $\varphi(z) \leq z$ for all $z > 0$. Thus we have an upper bound

$$\frac{\mathrm{d}}{\mathrm{d}r}f(y,r) \leq \frac{\max\{1, \frac{\sigma^2 y^2 T}{2r_0^2}\}}{y^3(J_\nu^2(y)+Y_\nu^2(y))}$$

valid for all $y > 0$ and $r > r_0$ with a given $r_0 > 0$. The well-known asymptotic properties of the Bessel functions can be used now to verify that the latter bound is integrable on $(0,\infty)$, which justifies the interchange of integration and differentiation, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}r}\int_0^\infty f(y,r) = \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}r}f(y,r)\,dy, \quad r > r_0. \tag{5}$$

Since $r_0 > 0$ was arbitrary, (5) holds for all $r > 0$. □

Notice that in the three–dimensional case, the mean surface area of the Wiener sausage is given by the simple analytical expression given in (3), whereas in other dimensions the formula (2) can be assessed only numerically. This main difference comes from the theory of parabolic partial differential equations due to the close connections between the heat equation and the Brownian motion as a diffusion process; see e.g. [13] and [23].

Remark The following asymptotic behaviour of $\mathbb{E}\mathcal{H}^{d-1}(\partial S_r)$ holds as the radius r of the Wiener sausage tends to zero. Namely, using (2) and asymptotic properties of Bessel functions one can show that

$$\mathbb{E}\mathcal{H}^{d-1}(\partial S_r) \sim \begin{cases} \pi\sigma^2 T r^{-1}\log^{-2} r & \text{if } d=2, \\ 2\pi\sigma^2 T & \text{if } d=3, \\ d\omega_d\sigma^2 T\frac{(d-2)^2}{2} r^{d-3} & \text{if } d\geq 4 \end{cases}$$

as $r \to 0$, where $f(r) \sim g(r)$ means that $\lim_{r\to 0} f(r)/g(r) = 1$.

3 Parallel Neighbourhoods

In the first part of this section, some preliminary facts from the geometry of sets with positive reach and Lipschitz manifolds are given. Later on, in Section 3.2, it will be shown that the first derivative of the volume of a dilated set with respect to the dilation radius equals its surface area for "non–critical" values of dilation radii. Notice that in dimensions two and three, the set of "critical" radii is of Lebesgue measure zero for any compact set, cf. Theorem 3.1.

3.1 Preliminaries

3.1.1 Distance Function

Let A be a nonempty compact subset of $\mathbb{R}^d$. The *distance function* $\Delta_A : \mathbb{R}^d \to [0,\infty)$ of A is defined by

$$\Delta_A(x) = \min\{|x-a| : a \in A\}, \qquad x \in \mathbb{R}^d .$$

Clearly, Δ_A is a Lipschitzian function with Lipschitz constant 1. For any $r \geq 0$, the r-parallel neighbourhood to A rewrites in terms of the distance function as $A_r = \{x \in \mathbb{R}^d : \Delta_A(x) \leq r\}$. Given two nonempty compact subsets A, B of $\mathbb{R}^d$, their *Hausdorff distance* is defined as

$$\Delta_H(A,B) = \max\bigl\{\max_{a\in A} \Delta_B(a), \max_{b\in B} \Delta_A(b)\bigr\} .$$

It is well known that Δ_H is a metric. For any $x \in \mathbb{R}^d$, we denote by

$$\Sigma_A(x) = \{a \in A : |x-a| = \Delta_A(x)\}$$

the set of all points in A which are nearest to x. The set $\Sigma_A(x)$ is always nonempty, by the compactness of A. Following Ferry [7], we say that $x \in \mathbb{R}^d$ is a *critical point* of Δ_A if x lies in the closed convex hull of $\Sigma_A(x)$. Notice that this property is equivalent to that the subgradient of Δ_A contains the origin which is used more commonly as definition of a critical point; see [8].

A point x is called *regular* if it is not critical. A number $r > 0$ is a *critical value* of Δ_A is there exists a critical point x of Δ_A with $\Delta_A(x) = r$. We shall denote by $C(A) \subseteq (0,\infty)$ the set of all critical values of Δ_A. In Figure 2, the set A consists of two parallel segments located at the distance $b > 0$. It is clear that the dilation radius $r = b/2$ is a critical value of Δ_A. The dashed line in the middle of Figure 2 is the set of critical points of Δ_A.

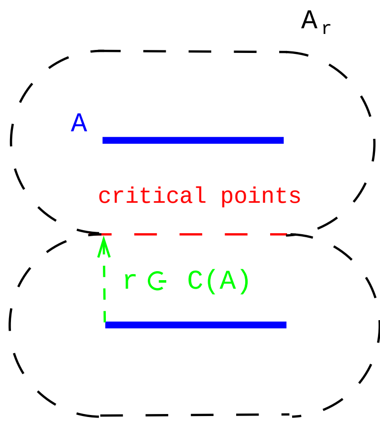

Fig. 2 The set of critical points for $r \in C(A)$

3.1.2 Sets with Positive Reach

The *reach* of a closed subset $A \subseteq \mathbb{R}^d$ is given by

$$\operatorname{reach} A = \sup\{r \geq 0 : \forall x \in \mathbb{R}^d,\, \Delta_A(x) < r \implies \operatorname{card} \Sigma_A(x) = 1\}\,,$$

see [5]. As examples of sets with positive reach, consider convex closed sets (with infinite reach) or sets with compact and C^2–smooth boundaries. Note that the union of two intersecting convex bodies typically does not have positive reach. The tangent cone $\operatorname{Tan}(A, a)$ is always a closed convex cone if $a \in \partial A$ and $\operatorname{reach} A > 0$. The normal cone is defined as the dual cone to the tangent cone

$$\operatorname{Nor}(A, a) = \{u : \; u \cdot v \leq 0 \text{ for all } v \in \operatorname{Tan}(A, a)\}$$

and it is a closed convex cone as well.

For a compact set $A \subseteq \mathbb{R}^d$ with finite positive reach, the Steiner formula holds only for sufficiently small radii. More exactly, we have

$$V_d\,(A_r) = \sum_{i=0}^{d} \omega_i r^i V_{d-i}(A)\,, \qquad 0 \leq r < \operatorname{reach} A\,, \tag{6}$$

where V_d is the Lebesgue measure in $\mathbb{R}^d$ (volume), ω_i is the volume of the unit i-ball and V_i is the i-th intrinsic volume of A, $i = 0, \ldots, d$ (see [5]). The functionals V_i are motion invariant and additive. In particular, V_0 is the Euler–Poincaré characteristic and V_{d-1} is one half of the surface area.

A local version of the Steiner formula holds for (not necessarily compact) sets with positive reach. Let $\xi_A(x)$ be the nearest point of A to x whenever $\Delta_A(x) < \operatorname{reach} A$. Then we have for any Borel subset F of $\mathbb{R}^d$

$$V_d\,\left((A_r \setminus A) \cap \xi_A^{-1}(F)\right) = \sum_{i=1}^{d} \omega_i r^i C_{d-i}(A; F)\,, \quad 0 \leq r < \operatorname{reach} A\,, \tag{7}$$

where $C_i(A; \cdot)$ is the ith *curvature measure* of A; it is a signed Radon measure concentrated on ∂A for $0 \leq i \leq d-1$ and $C_{d-1}(A; \cdot)$ is the restriction of the $(d-1)$–dimensional Hausdorff measure to ∂A provided that A is d–dimensional. If ∂A is compact then $C_i(A; \mathbb{R}^d) = V_i(A)$, $i = 0, \ldots, d-1$.

3.1.3 Lipschitz Manifolds

A set $A \subseteq \mathbb{R}^d$ is a ($(d-1)$–dimensional) *Lipschitz manifold* if A is locally representable as the graph of a Lipschitzian function, i.e., for any $a \in A$ there exists a neighbourhood U of a (in $\mathbb{R}^d$), a unit vector $u \in \mathbb{R}^d$ and a Lipschitzian function $\phi : u^\perp \to \mathbb{R}$ such that $A \cap U = \operatorname{graph} \phi \cap U$ ($u^\perp$ denotes the $(d-1)$–dimensional subspace of $\mathbb{R}^d$ perpendicular to u). Of course, any Lipschitz manifold is a topological manifold and any smooth manifold is a Lipschitz manifold, but not vice versa.

A d–dimenional Lipschitz manifold in $\mathbb{R}^d$ with boundary is a subset $A \subseteq \mathbb{R}^d$ which is locally representable as the subgraph of a Lipschitzian function (consequently, its topological boundary is a $(d-1)$–dimensional Lipschitz manifold). For such sets, conditions were found in [19] under which its curvature measures can be defined so that the Gauss–Bonnet and principal kinematic formulae hold. These conditions are satisfied in particular if the set A itself or the closure of its complement $\overline{\mathbb{R}^d \setminus A}$ has positive reach. If, moreover, the boundary of A is compact then the total curvature measures are denoted by $V_i(A)$ and called intrinsic volumes as in the case of sets with positive reach and we always have

$$V_i\left(\overline{\mathbb{R}^d \setminus A}\right) = (-1)^{d-i-1} V_i(A)\,, \qquad i = 0, \ldots, d-1\,. \tag{8}$$

Theorem 3.1 *For any nonempty compact set $A \subseteq \mathbb{R}^d$, the following properties hold:*

(i) *if $r \in (0,\infty) \setminus C(A)$ then ∂A_r is a $(d-1)$-dimensional Lipschitz manifold and* $\operatorname{reach}\left(\overline{\mathbb{R}^d \setminus A_r}\right) > 0$*;*

(ii) *The set $C(A)$ is compact and $\mathcal{H}^{\frac{d-1}{2}}\big(C(A)\big) = 0$.*

A *proof* of Theorem 3.1 can be found in [8].

Corollary 3.2 *For any nonempty compact set A in $\mathbb{R}^d$, the curvature measures $C_i(A_r;\cdot)$ and intrinsic volumes $V_i(A_r)$ of the parallel set A_r to A are well defined for $i = 0, \ldots, d-1$ whenever $r \notin C(A)$.*

Note that the assertion about the measure of $C(A)$ in (ii) of Theorem 3.1 is nonempty only if $d \leq 3$. Examples can be found in [7] and [8] showing that $C(A)$ can contain a nondegenerate interval if $d \geq 4$.

3.2 Volume of the Parallel Neighbourhood

Let $A \subseteq \mathbb{R}^d$ be a nonempty and compact subset of $\mathbb{R}^d$. For $r \geq 0$, denote by $V_A(r)$ the volume $V_d\,(A_r)$ of the parallel neighbourhood to A. The function V_A is obviously increasing on $[0,\infty)$. Applying the co–area formula to the distance function, we obtain (see [6, Section 3.2.34])

$$V_A(r) = V_A(0) + \int_0^r \mathcal{H}^{d-1}\big(\Delta_A^{-1}\{s\}\big)\,\mathrm{d}s\,, \qquad r > 0\,. \tag{9}$$

Consequently, the function V_A is absolutely continuous and its derivative exists and equals $\mathcal{H}^{d-1}\big(\Delta_A^{-1}\{r\}\big)$ for almost all r. Note that the boundary ∂A_r is a subset of the level set $\Delta_A^{-1}\{r\}$, but the equality does not hold in general (as a counterexample, the set A in Figure 2 can be considered).

Stachó [24] derived some further properties of V_A; in particular, he showed that the one-sided derivatives $(V_A)'_-(r)$ and $(V_A)'_+(r)$ exist for any $r > 0$, but only the inequality $(V_A)'_-(r) \geq (V_A)'_+(r)$ holds in general. Stachó also proved that the arithmetic mean of the left and right derivatives of V_A always equals the Minkowski content of ∂A_r (see [24, Theorem 2]).

We shall need later the following result.

Theorem 3.3 *If $r \in (0,\infty) \setminus C(A)$ then $V_A'(r)$ exists and equals $\mathcal{H}^{d-1}(\partial A_r)$.*

In the *proof* of Theorem 3.3, which is postponed to Section 3.3, we shall need some further properties of sets with positive reach. Let $\mathrm{reach}\, A > 0$. The unit normal bundle of A is the subset of $\mathbb{R}^d \times S^{d-1}$

$$\mathrm{nor}\, A = \{(a, u) \in \partial A \times S^{d-1} :\ u \in \mathrm{Nor}\,(A, a)\}.$$

The generalized principal curvatures $k_i(x, u)$, $i = 1, \ldots, d-1$, are defined $\mathcal{H}^{d-1}$-almost everywhere on $\mathrm{nor}\, A$ and take values from $[-\mathrm{reach}\, A, \infty]$ (see [32]). If, in particular, ∂A is C^2–smooth, then $\dim \mathrm{Nor}\,(A, a) = 1$ for any $a \in \partial A$ and $k_i(a, u)$ equal, up to sign, the classical principal curvatures of differential geometry for any $(a, u) \in \mathrm{nor}\, A$.

Lemma 3.4 *Denoting $\pi : (a, u) \mapsto a$ the first coordinate projection on $\mathbb{R}^d \times \mathbb{R}^d$, we have*

$$\mathcal{H}^{d-1}\left(\pi\{(a, u) \in \mathrm{nor}\, A :\ k_i(a, u) = \infty \text{ for some } 1 \le i \le d-1\}\right) = 0.$$

P r o o f. The $(d-1)$–dimensional Jacobian of π restricted to $\mathrm{nor}\, A$ equals

$$J_{d-1}(a, u) = \prod_{i=1}^{d-1} \frac{1}{\sqrt{1 + k_i(a, u)^2}}\,, \qquad (a, u) \in \mathrm{nor}\, A\,;$$

see [32]. The Jacobian is thus zero whenever some of the generalized curvatures are infinite. The assertion follows thus by the so–called area formula; see Section 3.2.22 of [6]. □

Let from now on A be compact and denote for brevity $C_r = \overline{\mathbb{R}^d \setminus A_r}$ (clearly $\partial C_r = \partial A_r$). By Theorem 3.1, we have $\mathrm{reach}\, C_r > 0$ for $r \notin C(A)$. We can describe the tangent and normal cones of C_r as follows. Given a set $H \subseteq \mathbb{R}^d$, denote by $\mathrm{cone}\, H$ the closed convex cone spanned by H, i.e.,

$$\mathrm{cone}\, H = \{t_1 h_1 + \cdots + t_d h_d :\ h_1, \ldots, h_d \in H,\ t_1, \ldots, t_d \ge 0\},$$

and by $\mathrm{cone}^* H$ the dual cone $\mathrm{cone}^* H = \{u :\ u \cdot h \le 0 \text{ for all } h \in H\}$.

Lemma 3.5 *If $z \in \partial A_r$ is regular for Δ_A then $\mathrm{Nor}\,(C_r, z) = \mathrm{cone}\,(\Sigma_A(z) - z)$ and $\mathrm{Tan}\,(C_r, z) = \mathrm{cone}^*\,(\Sigma_A(z) - z)$.*

P r o o f. First, notice that since z is a regular point of Δ_A, $\mathrm{cone}\,(\Sigma_A(z) - z)$ is a proper convex cone (i.e., there is a hyperplane intersecting it only in the origin) and, hence, the dual cone $\mathrm{cone}^*\,(\Sigma_A(z) - z)$ is full-dimensional. Let v be a vector from the interior of $\mathrm{cone}^*\,(\Sigma_A(z) - z)$. Then, since there are no points of A on the hemisphere $\partial B_r(z) \cap \{x :\ (x - z) \cdot v \ge 0\}$, the shifted ball $B_r(z + \varepsilon v)$ does not intersect A for sufficiently small $\varepsilon > 0$. Therefore, $z + \varepsilon v \in C_r$ for sufficiently small ε and, hence, $v \in \mathrm{Tan}\,(C_r, z)$. By the closeness of the tangent cone, we get $\mathrm{cone}^*\,(\Sigma_A(z) - z) \subset \mathrm{Tan}\,(C_r, z)$. On the other hand, if $v \notin \mathrm{cone}^*\,(\Sigma_A(z) - z)$ then there is a point $a \in A$ such that $(a - z) \cdot v > 0$. Then there exists an $\varepsilon > 0$ such that for any vector w with $|w - v| < \varepsilon$, all points of the segment $(z, z + \varepsilon w]$ have distance smaller than r from a and, therefore, v cannot be a tangent vector to C_r at z. □

Introduce the function $J_A : x \mapsto \min\{|a - x| :\ a \in \mathrm{conv}\, \Sigma_A(x)\}$, $x \in \mathbb{R}^d$. Clearly, x is a regular point of Δ_A if and only if $J_A(x) > 0$.

Lemma 3.6 *Let $z \in \partial A_r$ be a regular point of Δ_A and let ∂A_r be representable as the graph of a Lipschitzian function with Lipschitz constant $L > 0$ in some neighbourhood of z. Then*

$$J_A(z) \ge \frac{r}{\sqrt{L^2 + 1}}\,.$$

Proof. Let ∂A_r be representable in a neighbourhood of z as a graph of an L–Lipschitzian function f defined on $u^\perp$, where u is a unit vector, and assume that u points outwards of A_r. From the Lipschitz property of f we have that the whole convex cone $z + \{v : v \cdot u \geq L|v|/\sqrt{1+L^2}\}$ lies above the graph of f and, hence, belogs to $\operatorname{Tan}(C_r, z)$. Consequently, any vector from $\operatorname{Nor}(C_r, z)$ must lie in the dual cone $\{v : v \cdot u \leq -|v|/\sqrt{1+L^2}\}$. It follows by Lemma 3.5 that $(a-z)\cdot u \leq -r(L^2+1)^{-1/2}$ for any $a \in \Sigma_A(z)$ and, hence, also for any $a \in \operatorname{conv}\Sigma_A(z)$, which implies that $|a-z| \geq r(L^2+1)^{-1/2}$ for any $a \in \operatorname{conv}\Sigma_A(z)$. □

Lemma 3.7 *Let D be a set with positive reach. Assume that the generalized principal curvatures $k_i(a,u)$, $i = 1,\dots,d-1$, exist and are finite at a point $(a,u) \in \operatorname{nor} D$ such that $(a,-u) \notin \operatorname{nor} D$. Then there exists $\varepsilon > 0$ such that $B_\varepsilon(a - \varepsilon u) \subseteq D$.*

Proof. Fix some $0 < s < \operatorname{reach} D$ and consider the parallel set D_s which has $C^{1,1}$ smooth boundary. The (classical) principal curvatures of ∂D_s at $b = a + su$ exist and equal

$$k_i^s(b) = \frac{k_i(a,u)}{1 + sk_i(a,u)}, \qquad i = 1,\dots,d-1$$

(see [32]). Denote $K = \max\{0, \max_{1\leq i\leq d-1} k_i(a,u)\}$; we have

$$k_i^s(b) \leq \frac{K}{1+sK} < \frac{1}{s}, \qquad i = 1,\dots,d-1\,.$$

It follows from the basic differential calculus that, taking $s < t < s + \frac{1}{K}$, there exists a $\tau > 0$ such that $B_t(b - tu) \cap B_\tau(b) \subseteq D_s$. Since clearly $B_s(a) \cap \partial D_s = \{b\}$, the distance of $B_s(a) \setminus B_\tau(b)$ from ∂D_s is positive and, therefore, there exists an $\varepsilon > 0$ with $B_{s+\varepsilon}(a - \varepsilon u) \subseteq D_s$. It follows that $B_\varepsilon(a - \varepsilon u) \subseteq D$, which completes the proof. □

3.3 Proof of Theorem 3.3

First, we show that $(V_A)'_+(r) = \mathcal{H}^{d-1}(\partial A_r)$. By [10, Corollary 4.6], we have that $(V_A)'_+(r) = \mathcal{H}^{d-1}(\partial^+ A_r)$ for any $r > 0$, where

$$\partial^+ A_r = \left\{a \in \partial A_r : \exists x \in \mathbb{R}^d \setminus A_r, a \in \Sigma_{A_r}(x)\right\}.$$

It remains to prove that

$$\mathcal{H}^{d-1}(\partial A_r \setminus \partial^+ A_r) = 0\,. \tag{10}$$

For $x \in \partial A_r \setminus \partial^+ A_r$, there exists a unit vector u and an index $1 \leq i \leq d-1$ such that the generalized principal curvature $\kappa_i(x,u)$ of C_r is infinite. Indeed, if all principal curvatures at (x,u) were finite then the application of Lemma 3.7 with $D = C_r$ would yield $B_\varepsilon(x - \varepsilon u) \subseteq C_r$ for some $\varepsilon > 0$, and consequently $x \in \Sigma_{A_r}(x - \varepsilon u)$. We arrived at the contradiction with $x \notin \partial^+ A_r$. Thus, relation (10) follows from Lemma 3.4.

We now verify that $(V_A)'_-(r) = \mathcal{H}^{d-1}(\partial A_r)$ as well. Since $\operatorname{reach} C_r > 0$ by Theorem 3.1, we have

$$\lim_{\varepsilon\to 0} \frac{V_d\left((C_r)_\varepsilon \setminus C_r\right)}{\varepsilon} = \mathcal{H}^{d-1}(\partial C_r) = \mathcal{H}^{d-1}(\partial A_r)$$

by Steiner formula (6). As $(C_r)_\varepsilon \setminus C_r$ is a subset of $A_r \setminus A_{r-\varepsilon}$, it is sufficient to show that

$$V_d\left(Z_{r,\varepsilon}\right) = o(\varepsilon)\,, \qquad \text{as } \varepsilon \to 0, \tag{11}$$

where $Z_{r,\varepsilon} = \left(A_r \setminus A_{r-\varepsilon}\right) \setminus \left((C_r)_\varepsilon \setminus C_r\right)$. Note that $z \in Z_{r,\varepsilon}$ if and only if $r - \varepsilon < \Delta_A(z) \leq r$ and $B_\varepsilon(z) \subseteq A_r$.

Clearly $A_{r-\varepsilon} \uparrow A_r$ as $\varepsilon \to 0$. By the compactness of A_r, $A_{r-\varepsilon}$ converges to A_r in the Hausdorff metric as $\varepsilon \to 0$ and, hence, there exists a function $\delta(\varepsilon) \to 0$ ($\varepsilon \to 0$) such that $Z_{r,\varepsilon} \subseteq A_r \setminus A_{r-\varepsilon} \subseteq (C_r)_{\delta(\varepsilon)} \setminus C_r$. Assume that ε is so small that $\delta(\varepsilon) < \operatorname{reach} C_r$.

For $x \in Z_{r,\varepsilon}$ introduce $z = \xi_{C_r}(x)$. Clearly $x - z \in \operatorname{Nor}(C_r, z)$. Take some point $a \in \Sigma_A(z)$; note that $|a - z| = r$ as $a \in \partial A$. We have $a - z \in \operatorname{Nor}(C_r, z)$ (since the interior of the ball $B_r(a)$ does not hit C_r). The vectors $x - z$ and $a - z$ are linearly independent (for otherwise, $a - z = r\frac{x-z}{|x-z|}$ would imply $\Delta_A(x) = r - |x - z| \le r - \varepsilon$, which would contradict $x \in Z_{r,\varepsilon}$), hence $\dim \operatorname{Nor}(C_r, a) \ge 2$. Denote

$$\partial^* C_r = \{z \in \partial C_r : \dim \operatorname{Nor}(C_r, z) \ge 2\}.$$

By local Steiner formula (7), we have that for $\varepsilon \to 0$

$$V_d(Z_{r,\varepsilon}) \le V_d\left(((C_r)_{\delta(\varepsilon)} \setminus C_r) \cap \xi_{C_r}^{-1}(\partial^* C_r)\right) = 2\delta(\varepsilon)C_{d-1}(C_r; \partial^* C_r) + o(\delta(\varepsilon)) = \delta(\varepsilon)\mathcal{H}^{d-1}(\partial^* C_r) + o(\delta(\varepsilon))\,.$$

Since $\mathcal{H}^{d-1}(\partial^* C_r) = 0$ by [5, Remark 4.15 (3)], we have $V_d(Z_{r,\varepsilon}) = o(\delta(\varepsilon))$. We shall prove (11) by showing that

$$\limsup_{\varepsilon \to 0} \frac{\delta(\varepsilon)}{\varepsilon} < \infty\,, \tag{12}$$

which will imply $o(\delta(\varepsilon)) = o(\varepsilon)$ for $\varepsilon \to 0$. Since ∂C_r is a compact Lipschitz manifold, there exists a constant $L > 0$ such that ∂C_r can be covered by open sets on which it can be represented as the graph of a Lipschitzian function with Lipschitz constant less then or equal to L. Thus, we have by Lemma 3.6

$$J_A(z) \ge \frac{r}{\sqrt{L^2 + 1}}\,, \qquad z \in \partial C_r\,.$$

But then for any $x \in Z_{r,\varepsilon}$ and $z = \xi_{C_r}(x)$, there must exist a point $a \in \Sigma_A(z)$ with $(a - z) \cdot (x - z) \ge \eta r |x - z|$, where $\eta = (1 + L^2)^{-1/2}$. It follows that $|x - a|^2 \le r^2 + |z - x|^2 - 2r|z - x|\eta$. We have $|x - a| > r - \varepsilon$ since $x \notin A_{r-\varepsilon}$, hence $(r - \varepsilon)^2 \le r^2 + |z - x|^2 - 2r|z - x|\eta$ and, solving the quadratic inequality, we obtain

$$|z - x| \le \eta r - \sqrt{\eta^2 r^2 - 2r\varepsilon + \varepsilon^2}\,.$$

Since this inequality holds for any $z \in Z_{r,\varepsilon}$, we have also

$$\delta(\varepsilon) \le \eta r - \sqrt{\eta^2 r^2 - 2r\varepsilon + \varepsilon^2} = \eta^{-1}\varepsilon + o(\varepsilon)$$

as $\varepsilon \to 0$, which proves (12) and completes the proof of Theorem 3.3.

4 Differentiating the Expected Volume

In this section we prove Theorem 2.2, i.e., we show that the expected surface area of the Wiener sausage S_r can be obtained by differentiating its mean volume with respect to the dilation radius r.

4.1 The Case of Dimensions Two and Three

Let $S = S(T) = \{X(t) : 0 \le t \le T\}$ be the image of the interval $[0, T]$ in $\mathbb{R}^d$ under the mapping X where $X = \{X(t, \omega) : t \ge 0, \ \omega \in \Omega\}$ is the Brownian motion introduced in Section 2. Without loss of generality, assume that $X(0) = o$ almost surely. Clearly, the set $S(T) = \{S(T, \omega) : \omega \in \Omega\}$ is a random curve in $\mathbb{R}^d$ where $S(T, \omega)$ is the realization of $S(T)$ corresponding to the elementary event ω.

It follows from Theorem 3.1 that the set of critical values $C(S(T, \omega))$ has Lebesgue measure zero in dimensions two and three for any $\omega \in \Omega$. We shall combine this fact with a probabilistic argument to show that the following is true.

Theorem 4.1 *Let $d \le 3$. Then for any $r > 0$, $r \notin C(S)$ almost surely.*

The *proof* of Theorem 4.1 will be based on the following two auxiliary results.

Lemma 4.2 *Let $r > 0$ be an arbitrary fixed number. With probability one, there is no critical point $x \neq o$ of Δ_S with $o \in \Sigma_S(x)$ and $\Delta_S(x) = r$.*

Proof. Denote by $Q = \{x \in \mathbb{R}^d : x_i \geq 0 \text{ for all } i \leq d\}$ the nonnegative coordinate cone and let R be a countable dense set of rotations around the origin in $\mathbb{R}^d$. Then $\inf\{t > 0 : X(t) \in Q\}$ is a random variable, see [30, § 6.1]. Since the cone Q is recurrent for the Brownian motion $\{X(t)\}$ (see Proposition 2.13 of [18]), we have $P(\sup\{t > 0 : X(t) \in Q\} = \infty) = 1$. Furthermore, this implies that $P(\inf\{t > 0 : X(t) \in Q\} = 0) = 1$, since the process $\{\widetilde{X}(t),\, t \geq 0\}$ with

$$\widetilde{X}(t) = \begin{cases} t\,X(1/t) & \text{if } t > 0, \\ 0 & \text{if } t = 0, \end{cases}$$

has the same distribution as the Brownian motion $\{X(t)\}$. Thus, using the isotropy of the d–dimensional Brownian motion, we obtain

$$P(\inf\{t > 0 : X(t) \in \rho Q\} = 0 \text{ for all } \rho \in R) = 1\,. \tag{13}$$

Consider the set $A_r = \{\omega \in \Omega : o \in \Sigma_{S(\omega)}(x(\omega)) \text{ for some critical point } x(\omega) \text{ with } |x(\omega)| = r\}$. Then, for any $\omega \in A_r$, there is no point of $S(\omega)$ in the interior of the ball $B_r(x)$ with centre $x = x(\omega)$ and radius $r = |x|$. Using (13), this implies that $P(A_r) = 0$, since there is surely a rotation $\rho = \rho(\omega) \in R$ and $\delta = \delta(\omega) > 0$ such that $\rho Q \cap B_\delta(o)$ is contained in $B_r(x)$. □

Lemma 4.3 *If $P(r \in C(S)) > 0$ for some $r > 0$, then there exists a constant $c > 1$ such that*

$$P(s \in C(S)) > 0$$

for any $s \in [r, cr]$.

Proof. Suppose that for some $r > 0$ we have $P(B) > 0$, where $B = \{\omega \in \Omega : r \in C(S(T,\omega))\}$. Then, by Lemma 4.2, there exists a critical point $x = x(\omega)$ of $\Delta_{S(T,\omega)}$ with $\Delta_{S(T,\omega)}(x) = r$ and $|x| > r$ for almost all $\omega \in B$. Consequently, for almost all $\omega \in B$ there exists a time instant $\tau = \tau(\omega) > 0$ such that $\min_{0 \leq t \leq \tau} |X(t) - x| > r$. It follows that the event

$$M = \{\omega \in B : \exists \text{ critical point } x = x(\omega) \text{ such that } \Delta_S(x) = r,\ \min_{0 \leq t \leq t_0} |X(t) - x| > r\}$$

has positive probability for some $t_0 \in (0, T)$. Furthermore, for any $s_0 \leq t_0$, the random function

$$X^{s_0}(t) = \sqrt{\frac{T}{T - s_0}}\ \left(X\left(s_0 + \frac{T - s_0}{T}\, t \right) - X(s_0) \right)$$

is a Brownian motion due to the scaling invariance property (see, for example, [3, § IV.2]) and, hence, the random set $S^{s_0} = \{X^{s_0}(t) : 0 \leq t \leq T\}$ has the same distribution as S. Since S^{s_0} is a shift of the multiple $\sqrt{T/(T - s_0)}S'$, where $S' = \{X(t) : s_0 \leq t \leq T\}$, and $r \in C(S')$, we have that $\sqrt{T/(T - s_0)}r \in C(S^{s_0})$ for all $s_0 \leq t_0$ and $\omega \in M$. Therefore, the assertion holds with $c = \sqrt{T/(T - t_0)}$. □

Now we are able to complete the proof of Theorem 4.1. Suppose that $P(r \in C(S)) > 0$ for some $r > 0$. Then, by Fubini's theorem and by Lemma 4.3, we have

$$\mathbb{E}\,\mathcal{H}^1\{s > 0 : s \in C(S)\} = \int_0^\infty P(s \in C(S))\,\mathrm{d}s \geq \int_r^{cr} P(s \in C(S))\,\mathrm{d}s > 0\,.$$

However, by Theorem 3.1, the expectation on the left–hand side must be zero. This implies that $P(r \in C(S)) = 0$ for any $r > 0$. Thus, the statement of Theorem 4.1 is shown.

The following result immediately follows from Theorems 3.3 and 4.1.

Corollary 4.4 *Let S_r be a Wiener sausage of radius r in $\mathbb{R}^d$, $d \le 3$. Then for any $r > 0$,*

(i) *∂S_r is a Lipschitz manifold,* $\operatorname{reach} \overline{\mathbb{R}^d \setminus S_r} > 0$ *and the curvature measures $C_i(S_r, \cdot)$ and intrinsic volumes $V_i(S_r)$ are defined for $i = 0, \dots, d-1$ almost surely;*

(ii) *$V_s(r)$ is almost surely differentiable at r, and we have $\mathcal{H}^{d-1}(\partial S_r) = V_S'(r)$ almost surely.*

4.2 The Case of Dimensions $d \ge 4$

We do not know whether the assertions of Theorem 4.1 and Corollary 4.4 hold for any dimension d. Anyway, a different method of proof must be used. However, notice that the following is true.

Lemma 4.5 *For almost all $r > 0$ the volume $V_S(r)$ is almost surely differentiable at r, and it holds*

$$P\big(\mathcal{H}^{d-1}(\partial S_r) = V_S'(r)\big) = 1$$

for $d \ge 4$ and almost all $r > 0$.

P r o o f. It follows from [10, Lemma 4.5] and Pucci's theorem in [24] that for any realization $S(\omega)$ of S the equality $\mathcal{H}^{d-1}(\partial S(\omega)_r) = V_{S(\omega)}'(r)$ holds for all $r > 0$ except for an at most countable set $K = K(\omega)$ of radii r. Thus, by Fubini's theorem we get that

$$0 = \mathbb{E}\mathcal{H}^1(K) = \int_0^\infty P(r \in K)\, dr$$

and, consequently, $P(r \in K) = 0$ for almost all $r > 0$. Hence, it holds $P\big(\mathcal{H}^{d-1}(\partial S_r) = V_S'(r)\big) = 1$ for almost all $r > 0$. $\square$

4.3 Proof of Theorem 2.2

In order to prove Theorem 2.2, we make use of the fact that we can interchange differentiation and expectation on the right–hand side of (1). For showing this, the following auxiliary result is useful.

Lemma 4.6 *Let $A \subset \mathbb{R}^d$ be an arbitrary nonempty and compact set. Then, for almost every $s, r > 0$ with $s \le r$, the derivatives $V_A'(s)$ and $V_A'(r)$ exist. Moreover, it holds that*

$$V_A'(r) \le \left(\frac{r}{s}\right)^{d-1} V_A'(s) \tag{14}$$

and

$$0 \le V_A'(r) \le \frac{d}{r}\,(R + r)^d \omega_d\,, \tag{15}$$

where R is the radius of a ball containing A, i.e. $A \subset B_R(o)$.

P r o o f. It has been shown in [12] that

$$V_A(\lambda b) - V_A(\lambda a) \le \lambda^d (V_A(b) - V_A(a))$$

for any $0 \le a \le b$ and $\lambda \ge 1$. This implies that

$$V_A'(\lambda a) = \lim_{b \to a} \frac{V_A(\lambda b) - V_A(\lambda a)}{\lambda(b-a)} \le \lambda^{d-1} \lim_{b \to a} \frac{V_A(b) - V_A(a)}{b-a} = \lambda^{d-1} V_A'(a)$$

for any $a > 0$ and $\lambda \geq 1$ such that the derivatives $V_A'(a)$ and $V_A'(\lambda a)$ exist. Recall that the derivative $V_A'(r)$ exists for almost every $r > 0$; see (9). Thus, for almost every $s, r > 0$ with $s \leq r$, we have

$$V_A'(r) \leq \left(\frac{r}{s}\right)^{d-1} V_A'(s)\,.$$

Multiplying both sides of the above inequality by $(s/r)^{d-1}$ and then integrating them with respect to s we obtain

$$V_A(r) - V_A(0) = \int_0^r V_A'(s)\,\mathrm{d}s \geq V_A'(r) \int_0^r \left(\frac{s}{r}\right)^{d-1} \mathrm{d}s = \frac{r}{d} V_A'(r)\,,$$

or, equivalently,

$$V_A'(r) \leq \frac{d}{r}\big(V_A(r) - V_A(0)\big) \leq \frac{d}{r} V_A(r) \leq \frac{d}{r}(R + r)^d \omega_d\,.$$

This completes the proof of Lemma 4.6. □

Now we can complete the proof of Theorem 2.2. It follows from Corollary 4.4 and Lemma 4.5 that

$$\mathbb{E}\,\mathcal{H}^{d-1}(\partial S_r) = \mathbb{E}\,V_S'(r) \tag{16}$$

for all $r > 0$ if $d = 2, 3$ and almost all $r > 0$ if $d \geq 4$. Furthermore, using (14), we have

$$\frac{V_S(r+h) - V_S(r)}{h} = \frac{1}{h} \int_r^{r+h} V_S'(s)\,\mathrm{d}s \leq \left(\frac{r+h}{r}\right)^{d-1} V_S'(r) \leq 2^{d-1} V_S'(r)$$

for any $h \in (0, r)$, whereas (15) implies that

$$V_S'(r) \leq \frac{d}{r}\,(\max_{t\in[0,T]} |X(t)| + r)^d \omega_d\ .$$

Notice that $\mathbb{E}(\max_{t\in[0,T]} |X(t)| + r)^d < \infty$. This follows from the inequality

$$\max_{t\in[0,T]} |X(t)| \leq \sum_{i=1}^{d} \max_{t\in[0,T]} |W_i(t)|$$

and from the well–known fact that

$$\mathbb{E}(\max_{t\in[0,T]} |W_i(t)|)^d \leq 2\,\mathbb{E}(\max_{t\in[0,T]} W_i(t))^d < \infty$$

for each $i \in \{1, \dots, d\}$, where $\{W_1(t)\}, \dots, \{W_d(t)\}$ are independent Wiener processes initiated at zero. This means that the random variables $\{(V_S(r+h) - V_S(r))/h,\ h \in (0, r)\}$ have a common integrable bound. By the dominated convergence theorem, this implies that

$$\frac{\mathrm{d}\,\mathbb{E}\,V_d(S_r)}{\mathrm{d}\,r} = \mathbb{E}\,V_S'(r)\,.$$

Thus, in view of (16), Theorem 2.2 is proved.

Note added in proof

While this paper has been refereed, Last [14, Theorem 4.5] proved the relation (1) by another method using the generalized Steiner formula from [10].

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