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EFFICIENT SIMULATION OF STABLE RANDOM FIELDS AND ITS APPLICATIONS

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ABSTRACT

Two methods to approximate stable random fields are presented. The methods are based on approximating the kernel function in the integral representation of such fields. Error bounds for the approximation error are derived and the approximations are used to simulate stable random fields. The simulation methodology is applied to a portfolio of storm insurance policies in order to spatially predict insurance claims.

Keywords: insurance, simulation, stable random fields.

INTRODUCTION

In many cases, the normal distribution is a reasonable model for real phenomena. If one considers the cumulative outcome of a great amount of influence factors, the normal distribution assumption can be justified by the Central Limit Theorem which states that the sum of a large number of independent and identically distributed random variables can be approximated by a normal distribution if the variance of these variables is finite. However, many real phenomena such as stock returns and claim sizes in storm insurance exhibit rather heavy tails. Stable distributions remedy this drawback by still being the limit distribution of a sum of independent and identically distributed random variables, but allowing for an infinite variance and heavy tails.

In order to include the spatial structure of real phenomena, stable random fields may be an appropriate model. Stable random fields can be represented as a stochastic integral of a deterministic kernel as integrand and a stable random measure as integrator. The kernel basically determines the dependence structure, whereas the stable random measure inhibits the probabilistic characteristics of the random field. As already noted by Cohen *et al.* (2008), practitioners have to try a variety of kernels and stable random measures to find the model that best fits their needs.

Once the model is fixed, it is desirable to be able to perform simulations of the stable random field. There are several papers that are devoted to this problem. In Biermé and Scheffler (2008), Stoev *et al.* (2004) and Wu *et al.* (2004), the fast Fourier transform is used for the simulation of linear fractional stable processes, whereas in Dury (2001), a wavelet representation of a certain type of fractional stable processes was applied to simulate sample paths. Furthermore, Cohen *et al.* (2008) give a general framework for the simulation of fractional fields.

In this paper, we consider certain classes of stable random fields for which the kernel function is assumed to be Hölder-continuous or bounded which is a less restrictive assumption. Based on the respective assumption, we derive estimates for the approximation error when the kernel function is approximated by a step function or by a certain truncated wavelet series. The approximation allows for simulation since the integral representation of the stable field reduces to a finite sum of random variables in this case. The proofs of the theorems are given in Karcher *et al.* (2009).

In Section 2, we present the main results for the approximation error which is made when the kernel function is replaced by a step function or a truncated wavelet series. Section 3 is devoted to a brief simulation study where we apply the derived formulas for the approximation error to the simulation of two particular stable random fields. In Section 4, we present an application of the simulation methods to a portfolio of storm insurance policies.

APPROXIMATION OF STABLE RANDOM FIELDS

Let $0 < \alpha \leq 2$ and $\alpha \neq 1$. In the following, we consider random fields of the form

$$X(t) = \int_{\mathbb{R}^d} f_t(x) M(dx), \quad t \in \mathbb{R}^d, \quad d \in \mathbb{N},$$

where $f_t \in L^\alpha(\mathbb{R}^d)$ and M is an α-stable random measure with Lebesgue control measure and constant skewness intensity β. For a detailed introduction and a thorough treatment of stable distributions and stable random processes, we refer to Samorodnitsky and Taqqu (1994). Our goal is to approximate sample paths of X for a variety of kernel functions f_t, $t \in \mathbb{R}^d$.

In the following, two approaches will be presented. They are both based on approximating the kernel function appropriately and determining bounds for the error resulting from this approximation.

MEASURING THE APPROXIMATION ERROR

We consider a set of functions $\{\tilde{f}_t^{(n)}\}_{t\in\mathbb{R}^d}$ such that $\tilde{f}_t^{(n)} \in L^\alpha(\mathbb{R}^d)$ for all $t \in \mathbb{R}^d$ and $n \in \mathbb{N}$. The corresponding α-stable random field is denoted by

$$\tilde{X}^{(n)}(t) := \int_{\mathbb{R}^d} \tilde{f}_t^{(n)}(x) M(dx), \quad t \in \mathbb{R}^d.$$

We know that for $\alpha \neq 1$ and for each $t \in \mathbb{R}^d$, $\tilde{X}^{(n)}(t)$ converges to $X(t)$ in probability if and only if

$$\int_{\mathbb{R}^d} |f_t(x) - \tilde{f}_t^{(n)}(x)|^\alpha dx$$

converges to 0 as n goes to infinity, see for instance Samorodnitsky and Taqqu (1994). Therefore, we can use $\tilde{X}^{(n)}(t)$ as an approximation for $X(t)$ if $\tilde{f}_t^{(n)}$ approximates f_t sufficiently well. We choose

$$\begin{aligned} Err(X(t),\tilde{X}^{(n)}(t)) &:= \left\| f_t(x) - \tilde{f}_t^{(n)}(x) \right\|_{L^\alpha} \\ &:= \left(\int_{\mathbb{R}^d} |f_t(x) - \tilde{f}_t^{(n)}(x)|^\alpha dx \right)^{1/\alpha} \end{aligned}$$

to measure the error resulting from this approximation. This choice can be further justified as follows.

Since $X(t)$ and $\tilde{X}^{(n)}(t)$ are jointly α-stable random variables for all $t \in \mathbb{R}^d$, the difference $X(t) - \tilde{X}^{(n)}(t)$ is also an α-stable random variable. The scale parameter of $X(t) - \tilde{X}^{(n)}(t)$ is given by

$$\sigma_{X(t)-\tilde{X}^{(n)}(t)} = \left(\int_{\mathbb{R}^d} |f_t(x) - \tilde{f}_t^{(n)}(x)|^\alpha dx \right)^{1/\alpha},$$

so that

$$Err(X(t),\tilde{X}^{(n)}(t)) = \sigma_{X(t)-\tilde{X}^{(n)}(t)}.$$

Furthermore, let us consider the quantity

$$\mathbb{E}|X(t) - \tilde{X}^{(n)}(t)|^p, \quad 0 < p < \alpha,$$

that is the mean error between $X(t)$ and $\tilde{X}^{(n)}(t)$ in the L^p-sense.

Since $X(t) - \tilde{X}^{(n)}(t)$ is an α-stable random variable, we have

$$\mathbb{E}|X(t) - \tilde{X}^{(n)}(t)|^p < \infty, \quad 0 < p < \alpha$$

and

$$\mathbb{E}|X(t) - \tilde{X}^{(n)}(t)|^p = \infty, \quad p \geq \alpha.$$

For $0 < p < \alpha$, $0 < \alpha < 2$ and $\alpha \neq 1$, this quantity can be written as

$$\left(\mathbb{E}|X(t) - \tilde{X}^{(n)}(t)|^p \right)^{1/p} = c_{\alpha,\beta_t}(p) \cdot \sigma_{X(t)-\tilde{X}^{(n)}(t)},$$

where

$$\begin{aligned} \left(c_{\alpha,\beta_t}(p)\right)^p = \frac{2^{p-1}\Gamma(1-\frac{p}{\alpha})}{p\int_0^\infty u^{-p-1}\sin^2 u\, du} \\ \cdot \left(1+\beta_t^2 \tan^2\frac{\alpha\pi}{2}\right)^{p/2\alpha} \\ \cdot \cos\left(\frac{p}{\alpha}\arctan\left(\beta_t \tan\frac{\alpha\pi}{2}\right)\right) \end{aligned}$$

and

$$\beta_t = \frac{\int_{\mathbb{R}^d} |f_t(x) - \tilde{f}_t^{(n)}(x)|^\alpha \mathrm{sign}(f_t(x) - \tilde{f}_t^{(n)}(x))\beta dx}{\int_{\mathbb{R}^d} |f_t(x) - \tilde{f}_t^{(n)}(x)|^\alpha dx}.$$

We remind that β is the constant skewness intensity of the α-stable random measure. The above implies that for $0 < p < \alpha$, we have

$$Err(X(t),\tilde{X}^{(n)}(t)) = \frac{1}{c_{\alpha,\beta_t}(p)} \left(\mathbb{E}|X(t) - \tilde{X}^{(n)}(t)|^p \right)^{\frac{1}{p}}.$$

The goal is now to find a set of functions $\{\tilde{f}_t^{(n)}\}_{t\in\mathbb{R}^d}$ such that $Err(X(t),\tilde{X}^{(n)}(t))$ is less than a predetermined critical value.

We see that the problem of approximating the α-stable random field X reduces to an approximation problem of the corresponding kernel functions. Two approaches to approximate the kernel functions are presented in the following.

STEP FUNCTION APPROXIMATION

First, we restrict our setting to the observation window $[-T,T]^d$ with $T>0$ and consider

$$X(t)=\int_{\mathbb{R}^d} f_t(x)M(dx),\quad t\in[-T,T]^d.$$

We denote by $supp(f_t)$ the support of f_t for each $t\in[-T,T]^d$ and assume that

$$\bigcup_{t\in[-T,T]^d} supp(f_t)\subset[-A,A]^d$$

for an $A>0$. Then $X(\cdot)$ can be written as

$$X(t)=\int_{[-A,A]^d} f_t(x)M(dx),\quad t\in[-T,T]^d.$$

For any natural number $n\geq 1$ and $k=(k_1,...,k_d)\in\mathbb{Z}^d$ with $-n\leq k_1,...,k_d<n$, let

$$\xi_k=\left(k_1\frac{A}{n},\cdots,k_d\frac{A}{n}\right),$$
$$\Delta_k=\left[k_1\frac{A}{n},(k_1+1)\frac{A}{n}\right)\times\cdots\times\left[k_d\frac{A}{n},(k_d+1)\frac{A}{n}\right).$$

We define the step function

$$\tilde{f}_t^{(n)}(x):=\sum_{|k|\leq n} f_t(\xi_k)\,\mathbb{1}_{\Delta_k}(x)$$

to approximate f_t, where $|k|\leq n$ is meant to be componentwise, i. e. $-n\leq k_i<n$ for $i=1,...,d$. Then we have

$$\tilde{X}^{(n)}(t)=\int_{[-A,A]^d}\tilde{f}_t^{(n)}(x)M(dx)=\sum_{|k|\leq n} f_t(\xi_k)M(\Delta_k).$$

In the last sum, $M(\Delta_k)$, $|k|\leq n$, are independent α-stable random variables which can be simulated as presented in Chambers *et al.* (1976).

The following theorem provides error bounds for $Err(X(t),\tilde{X}^{(n)}(t))$ for Hölder-continuous functions f_t.

Theorem 1
Assume that $0<\alpha\leq 2$, $\alpha\neq 1$ and the functions f_t are Hölder-continuous for all $t\in[-T,T]^d$, i. e. for $x,y\in[-A,A]^d$, it holds

$$|f_t(x)-f_t(y)|\leq C_t\cdot||x-y||_2^{\gamma_t},\quad t\in[-T,T]^d,$$

for some $0<\gamma_t\leq 1$ and $C_t>0$, where $\|\cdot\|_2$ denotes the Euclidean norm. Then for any $t\in[-T,T]^d$ we have for all $n\geq 1$ that

$$Err(X(t),\tilde{X}^{(n)}(t))\leq C_t\cdot 2^{d/\alpha}\left(\frac{d}{1+\alpha\gamma_t}\right)^{1/\alpha}A^{\gamma_t+d/\alpha}\left(\frac{1}{n}\right)^{\gamma_t}$$

for $0<\alpha<1$ and

$$Err(X(t),\tilde{X}^{(n)}(t))\leq C_t\cdot 2^{d}\left(\frac{d}{1+\alpha\gamma_t}\right)^{1/\alpha}A^{\gamma_t+d/\alpha}\left(\frac{1}{n}\right)^{\gamma_t+d(1/\alpha-1)}$$

for $1<\alpha\leq 2$.

As a consequence, for $0<\alpha<1$ the error bound converges to zero as n goes to infinity. For $1<\alpha\leq 2$, the condition $d<\frac{\alpha\gamma_t}{\alpha-1}$ has to be fullfilled to get such a convergence.

Remark 2
Suppose that the conditions of Theorem 1 hold true. If the support of f_t is not compact, we first need to estimate

$$X(t)=\int_{R^d} f_t(x)M(dx)$$

by

$$X_K(t)=\int_{[-K,K]^d} f_t(x)M(dx)$$

For $K>0$ large enough, the approximation error is small since

$$Err(X(t),X_K(t))=\left(\int_{\mathbb{R}^d\setminus[-K,K]^d}|f_t(x)|^\alpha dx\right)^{1/\alpha}$$

tends to 0 if K goes to infinity. Let $\varepsilon>0$. If $1<\alpha\leq 2$, choose $K>0$ such that $Err(X(t),X_K(t))\leq\varepsilon/2$. We can apply Theorem 1 to $X_K(\cdot)$ such that

$$Err(X_K(t),\tilde{X}_K^{(n)}(t))\leq\varepsilon/2$$

for $n\in\mathbb{N}$ large enough. Then

$$\begin{aligned}Err(X(t),\tilde{X}_K^{(n)}(t)) &\leq Err(X(t),X_K(t))\\ &\quad +Err(X_K(t),\tilde{X}_K^{(n)}(t))\\ &\leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

If $0<\alpha<1$, choose $K>0$ such that $Err(X(t),X_K(t))\leq\varepsilon^\alpha/2$. Again, we can apply Theorem 1 to $X_K(\cdot)$ such that

$$Err(X_K(t),\tilde{X}_K^{(n)}(t))\leq\varepsilon^\alpha/2$$

for $n \in \mathbb{N}$ large enough. Then

$$\begin{aligned} Err(X(t),\tilde{X}_K^{(n)}(t)) &\leq \Big(Err(X(t),X_K(t)) \\ &\quad + Err(X_K(t),\tilde{X}_K^{(n)}(t))\Big)^{1/\alpha} \\ &\leq \left(\frac{\varepsilon^\alpha}{2}+\frac{\varepsilon^\alpha}{2}\right)^{1/\alpha} = \varepsilon. \end{aligned}$$

Remark 3
More generally, we could consider random fields of the form

$$X(t) = \int_{[-A,A]^q} f_t(x)M(dx), \quad t \in [-T,T]^d,$$

where $q \in \mathbb{N}$ and $d \in \mathbb{N}$. In this case, under the assumptions of Theorem 1, the error bounds hold true with d replaced by q.

APPROXIMATION BY WAVELET SERIES

Series representation of kernel functions

Let $f_t \in L^\alpha(\mathbb{R}^d)$, $t \in \mathbb{R}^d$, $0 < \alpha \leq 2$, $\alpha \neq 1$, and let $\{\xi_i\}_{i\in I}$ be a basis for $L^\alpha(\mathbb{R}^d)$, where I is an index set. Then f_t can be represented as

$$f_t = \sum_{i\in I} a_i \cdot \xi_i \tag{1}$$

for certain constants $a_i \in \mathbb{R}$. In order to approximate f_t, one can truncate (1) such that it consists only of a finite number of summands. In Biermé and Scheffler (2008), the trigonometric system is used to approximate the kernel function of certain stable random fields. In this paper, we will go another way and analyse whether a wavelet system may also be appropriate for the simulation of α-stable random fields.

The Haar system

In this section, we consider a specific wavelet basis, the so-called *Haar basis*. For a detailed treatment of wavelets, we refer to DeVore (1998), DeVore and Lucier (1992) and Urban (2008) and proceed with the definition of the Haar basis.

Definition 4
Consider the function

$$\varphi^{Haar}(x) := \frac{1}{(2A)^{1/2}}, \quad x \in [-A,A],$$

and the corresponding mother wavelet defined by

$$\Psi^{Haar} := \varphi^{Haar}(2x) - \varphi^{Haar}(2x-1).$$

Then, translation by j and dilation by 2^k of the mother wavelet Ψ^{Haar} yields $\Psi^{Haar}_{j,k} := 2^{k/2}\Psi^{Haar}(2^k \cdot -j)$, $2^k \leq j \leq 2^{k+1}-1$, $k \in \mathbb{N}_0$, that form together with φ^{Haar} an (orthonormal) basis of $L^2([-A,A])$ which is called Haar basis of $L^2([-A,A])$.

The Haar basis can be extended to d dimensions as follows.

Let $\Psi^0 := \varphi^{\text{Haar}}$, $\Psi^1 := \Psi^{\text{Haar}}$ and E be the set of nonzero vertices of the unit cube $[0,1]^d$. Consider the multivariate functions Ψ^e, $e = (e_1,...,e_d) \in E$, defined by

$$\Psi^e(x_1,...,x_d) := \Psi^{e_1}(x_1)\cdots\Psi^{e_d}(x_d), \quad x \in [-A,A]^d.$$

Translation by $j = (j_1,...,j_d)$ and dilation by 2^k of the functions Ψ^e yields $\Psi^e_{j,k} := 2^{k/2}\Psi^e(2^k \cdot -j)$, $2^k \leq j_i \leq 2^{k+1}-1, i = 1,...,d$, $k \in \mathbb{N}_0$, $e \in E$ that form together with $\Psi^*(x) := \frac{1}{(2A)^{d/2}}$, $x \in [-A,A]^d$ an (orthonormal) basis of $L^2([-A,A]^d)$. Then, each function $f \in L^2([-A,A]^d)$ has the expansion

$$\begin{aligned} f = (f,\Psi^*)\Psi^* &\\ + \sum_{e\in E}\sum_{k=0}^{\infty} \sum_{\substack{2^k \leq j_i \leq 2^{k+1}-1 \\ i=1,\cdots,d}} &(f,\Psi^e_{j-2^k c,k})\Psi^e_{j-2^k c,k}, \end{aligned} \tag{2}$$

where $c := (1,...,1)^T$ is a vector in $\mathbb{R}^d$. It can be shown that any function $f \in L^{\max\{\alpha,p\}}([-A,A]^d)$ with $1 < p \leq 2$ can be represented by such a wavelet series (2), cf. DeVore and Lucier (1992). However, there exist examples of functions for which (2) does not hold in particular for $p = \alpha = 1$. Therefore, we restrict our setting to kernel functions $f_t \in L^{\max\{\alpha,p\}}([-A,A]^d)$, $1 < p \leq 2$.

As in the step function approach, we restrict our setting to the observation window $[-T,T]^d$ with $T > 0$ and consider

$$X(t) = \int_{\mathbb{R}^d} f_t(x)M(dx), \quad t \in [-T,T]^d.$$

We assume again that

$$\bigcup_{t\in[-T,T]^d} supp(f_t) \subset [-A,A]^d$$

for an $A > 0$ such that $X(\cdot)$ can be written as

$$X(t) = \int_{[-A,A]^d} f_t(x)M(dx), \quad t \in [-T,T]^d.$$

Approximation and simulation

Consider a kernel function $f_t \in L^{\max\{\alpha,p\}}([-A,A]^d)$ with corresponding Haar series

$$f_t = (f_t,\Psi^*)\Psi^* + \sum_{e\in E}\sum_{k=0}^{\infty}\sum_{\substack{2^k\le j_i\le 2^{k+1}-1\\ i=1,\cdots,d}} (f_t,\Psi^e_{j-2^kc,k})\Psi^e_{j-2^kc,k}.$$

The idea is now to cut off this series at a certain detail level $k=n$, that is to approximate the kernel function f_t by

$$\tilde f^{(n)}_{t,cut} = (f_t,\Psi^*)\Psi^* + \sum_{e\in E}\sum_{k=0}^{n}\sum_{\substack{2^k\le j_i\le 2^{k+1}-1\\ i=1,\cdots,d}} (f_t,\Psi^e_{j-2^kc,k})\Psi^e_{j-2^kc,k}.$$

The following lemma provides an upper bound for the approximation error of bounded kernel functions by applying this truncation method.

Lemma 5
Let $0<\alpha\le 2$. If $1<\alpha\le 2$, let additionally $d>\alpha$. Assume that $M_t := \sup_{x\in[-A,A]^d}|f_t(x)|<\infty$. Then for any natural number n, we have

$$\|f_t-\tilde f^{(n)}_{t,cut}\|_{L^\alpha} \le \left(\frac{2^d-1}{2^{d-\alpha}-1}\right)^{1/\alpha}\cdot d^{1/\alpha} \cdot M_t\cdot(2A)^{d/\alpha}\cdot\left(\frac{1}{2^{d/\alpha-1}}\right)^n$$

if $0<\alpha<1$ and

$$\|f_t-\tilde f^{(n)}_{t,cut}\|_{L^\alpha} \le \frac{2^d-1}{2^{d/\alpha-1}-1}\cdot d \cdot M_t\cdot(2A)^{d/\alpha}\cdot\left(\frac{1}{2^{d/\alpha-1}}\right)^n$$

if $1\le\alpha\le 2$.

Remark 6
When f_t, $t\in[-T,T]^d$ is assumed to be Hölder-continuous on $[-A,A]^d$, the rate of convergence of this estimate can be improved, see Karcher et al. (2009).

Taking a wavelet basis for $L^{\max\{\alpha,p\}}([-A,A]^d)$ with $1<p\le 2$ has advantages in particular in the representation of functions with discontinuities and sharp peaks, that is functions with a certain local behavior. By simply cutting of at a certain detail level, this advantage is not honored. In this view, it is better to approximate the kernel function f_t with a truncated Haar series $\tilde f^{(n)}_t$ that contains those n summands $(f,\Psi^e_{j-2^kc,k})\Psi^e_{j-2^kc,k}$ with the largest values $\|(f,\Psi^e_{j-2^kc,k})\Psi^e_{j-2^kc,k}\|_{L^\alpha}$ and $\|(f,\Psi^*)\Psi^*\|_{L^\alpha}$, see Karcher *et al.* (2009). In this paper, an algorithm is presented which determines an approximation $\tilde f^{(n)}$ close to this particular truncation.

We denote by I the set of the indices (e,j,k) for which the summands $(f_t,\Psi^e_{j-2^kc,k})\Psi^e_{j-2^kc,k}$ are part of the truncated wavelet series $\tilde f^{(n)}_t$. Then we can write

$$\tilde f^{(n)}_t = (f_t,\Psi^*)\Psi^* + \sum_{(e,k,j)\in I}(f_t,\Psi^e_{j-2^kc,k})\Psi^e_{j-2^kc,k}$$

if $(f_t,\Psi^*)\Psi^*$ is included in the truncated series or

$$\tilde f^{(n)}_t = \sum_{(e,k,j)\in I}(f_t,\Psi^e_{j-2^kc,k})\Psi^e_{j-2^kc,k}$$

if it is not included.

In order to approximate the random field X, we use

$$\tilde X^{(n)}(t) = (f_t,\Psi^*)\cdot\frac{M([-A,A]^d)}{(2A)^{d/2}} + \sum_{(e,k,j)\in I}(f_t,\Psi^e_{j-2^kc,k})\int_{[-A,A]^d}\Psi^e_{j-2^kc,k}M(dx),$$

or

$$\tilde X^{(n)}(t) = \sum_{(e,k,j)\in I}(f_t,\Psi^e_{j-2^kc,k})\int_{[-A,A]^d}\Psi^e_{j-2^kc,k}M(dx)$$

if, again, $(f_t,\Psi^*)\Psi^*$ is not included in the truncated series.

Since the Haar wavelets $\Psi^e_{j-2^kc,k}$ are simple step functions, the integrals

$$\int_{[-A,A]^d}\Psi^e_{j-2^kc,k}M(dx)$$

can be easily computed. However, the wavelet coefficients $(f_t,\Psi^e_{j-2^kc,k})$ cause problems if no closed formula of the integral of the kernel functions f_t over cubes is known. In this case, they have to be determined numerically by using the fast wavelet transform (see Urban (2008)).

The following algorithm can be used for the simulation of the considered α-stable random fields.

Algorithm

Let $M_t := \sup_{x\in[-A,A]^d} |f_t(x)| < \infty$ and $d > \alpha$. Choose $\varepsilon > 0$ as the desired level of accuracy. Choose $\varepsilon_1, \varepsilon_2 > 0$ such that $\varepsilon = \varepsilon_1 + \varepsilon_2$ if $1 < \alpha \le 2$ and $\varepsilon = (\varepsilon_1^\alpha + \varepsilon_2^\alpha)^{1/\alpha}$ if $0 < \alpha < 1$.

1. Let

$$m_t := \left\lceil \frac{\ln(\varepsilon(2^{d-\alpha}-1)^{1/\alpha}) - \ln((2^d-1)^{1/\alpha} d^{1/\alpha} M_t (2A)^{\frac{d}{\alpha}})}{(1-d/\alpha)\ln(2)} \right\rceil$$

if $0 < \alpha < 1$ and

$$m_t := \left\lceil \frac{\ln(\varepsilon(2^{d/\alpha-1}-1)) - \ln((2^d-1) d M_t (2A)^{d/\alpha})}{(1-d/\alpha)\ln(2)} \right\rceil$$

if $1 < \alpha \le 2$, where $\lceil x \rceil$ is the integral part of x and choose a number $l \in \mathbb{N}_0$ that increases the detail level m_t.

2. Calculate the wavelet coefficients for

$$(f_t, \Psi^*)\Psi^* + \sum_{e\in E} \sum_{k=0}^{m_t+l} \sum_{\substack{2^k \le j_i \le 2^{k+1}-1 \\ i=1,\cdots,d}} (f_t, \Psi^e_{j-2^k c,k}) \Psi^e_{j-2^k c,k}$$

using the fast wavelet transform with a precision of

$$\delta = \frac{\varepsilon_2}{(2A)^{d/\alpha - d/2} 2^{(n+1)d/2} \left(1 + (2^d-1)\frac{2^{(d\alpha-d)(n+1)}-1}{2^{d\alpha-d}-1}\right)^{1/\alpha}}$$

if $0 < \alpha < 1$ and

$$\delta = \frac{\varepsilon_2}{(2A)^{d/\alpha - d/2} 2^{(n+1)d/2} \left(1 + (2^d-1)\frac{2^{(d-d/\alpha)(n+1)}-1}{2^{d-d/\alpha}-1}\right)}$$

if $1 < \alpha \le 2$.

3. Take the n largest summands from

$$C := \|\widehat{(f_t, \Psi^*)}\Psi^*\|_{L^\alpha} + \sum_{e\in E} \sum_{k=0}^{m_t+l} \sum_{\substack{2^k \le j_i \le 2^{k+1}-1 \\ i=1,\cdots,d}} \|\widehat{(f_t, \Psi^e_{j-2^k c,k})} \Psi^e_{j-2^k c,k}\|_{L^\alpha}$$

and denote them by $a_1,...,a_n$, where $\widehat{(f_t, \Psi^*)}$ and $\widehat{(f_t, \Psi^e_{j-2^k c,k})}$ denote the wavelet coefficients calculated by the fast wavelet transform. The corresponding summands from

$$\widehat{(f_t, \Psi^*)}\Psi^* + \sum_{e\in E} \sum_{k=0}^{m_t+l} \sum_{\substack{2^k \le j_i \le 2^{k+1}-1 \\ i=1,\cdots,d}} \widehat{(f_t, \Psi^e_{j-2^k c,k})} \Psi^e_{j-2^k c,k}$$

are denoted by $b_1,...,b_n$. Choose the number n such that

$$C - \sum_{i=1}^{n} a_i \le \varepsilon_1 - \varepsilon_t^*,$$

where

$$\varepsilon_t^* := \left(\frac{2^d-1}{2^{d-\alpha}-1}\right)^{\frac{1}{\alpha}} d^{1/\alpha} M_t (2A)^{d/\alpha} \left(\frac{1}{2^{d/\alpha-1}}\right)^{m_t+l}$$

if $0 < \alpha < 1$ and

$$\varepsilon_t^* := \frac{2^d-1}{2^{d/\alpha-1}-1} \cdot d \cdot M_t \cdot (2A)^{d/\alpha} \cdot \left(\frac{1}{2^{d/\alpha-1}}\right)^{m_t+l}$$

if $1 < \alpha \le 2$.

4. Take $\tilde{f}_t^{(n)} = \sum_{i=1}^n b_i$ as the approximation for f_t.

Remark 7
Assume that f_t is Hölder-continuous with parameters C_t and γ_t for $t \in [-T,T]^d$. Then the algorithm can be applied with m_t and ε_t^* replaced by

$$m_t := \left\lceil \frac{\ln(2\varepsilon(2^{d+\alpha\gamma_t}-1)^{\frac{1}{\alpha}}) - \ln((2^d-1)^{\frac{1}{\alpha}} d^{\frac{1}{\alpha}+\frac{\gamma_t}{2\alpha}} C_t (2A)^{\frac{d}{\alpha}+\gamma_t})}{-\ln(2^{\frac{d}{\alpha}+\gamma_t})} \right\rceil,$$

$$\varepsilon_t^* := \frac{1}{2}\left(\frac{2^d-1}{2^{d+\alpha\gamma_t}-1}\right)^{1/\alpha} \cdot d^{1/\alpha+\gamma_t/(2\alpha)} \cdot C_t \cdot (2A)^{d/\alpha+\gamma_t} \cdot \left(\frac{1}{2^{d/\alpha+\gamma_t}}\right)^{m_t+l},$$

if $0 < \alpha < 1$ and

$$m_t := \left\lceil \frac{\ln(\varepsilon(2^{d/\alpha+\gamma_t+1}-2)) - \ln((2^d-1) d^{1+\gamma_t/2} C_t (2A)^{d/\alpha+\gamma_t})}{-\ln(2^{d/\alpha+\gamma_t})} \right\rceil,$$

$$\varepsilon_t^* := \frac{2^d-1}{2^{d/\alpha+\gamma_t+1}-2} \cdot d^{1+\frac{\gamma_t}{2}} \cdot C_t \cdot (2A)^{\frac{d}{\alpha}+\gamma_t} \left(\frac{1}{2^{d/\alpha+\gamma_t}}\right)^{m_t+l},$$

if $1 < \alpha \le 2$.

We conclude this section with the main result.

Theorem 8
Assume $0 < \alpha \le 2$, $\alpha \ne 1$ and let $\varepsilon > 0$. If $\tilde{f}_t^{(n)}$ is calculated using the algorithm mentioned above, then

$$Err(X(t), \tilde{X}^{(n)}(t)) \le \varepsilon, \quad \forall t \in [-T,T]^d.$$

SIMULATION STUDY

For the simulation study, we used two different types of kernel functions for α-stable random fields of dimension $d = 2$. The first one is an Epanechnikov-type kernel function defined by

$$f_t(x) = \begin{cases} b \cdot (a^2 - \|x-t\|_2^2), & \|x-t\|_2 \leq a \\ 0, & \text{otherwise,} \end{cases} \tag{3}$$

where $a > 0$ and $b > 0$, whereas for the second one, we take

$$f_{(t_1,t_2)}(x_1,x_2) = b(a-|x_1-t_1|)(a-|x_2-t_2|) \cdot \mathbb{1}_{\{a-|x_1-t_1|\geq 0,\, a-|x_2-t_2|\geq 0\}}(x_1,x_2) \tag{4}$$

where $a > 0$ and $b > 0$. Examples of both kernel functions are plotted in Figure 1.

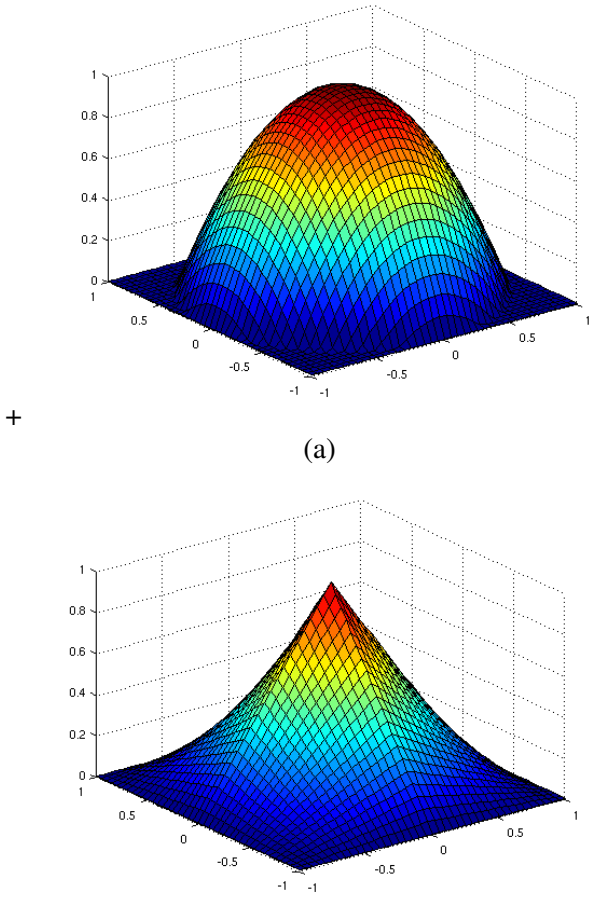

Fig. 1. *The Epanechnikov-type kernel function (3) (subfigure (a)) and the kernel function (4) (subfigure (b)).*

The main difference between these two types of kernel functions is that one can derive a simple formula for the integral of (4) over squares, but not for the integral of (3). This does not affect the step function approach since the kernel functions are only evaluated there at the points ξ_k, but it does affect the wavelet approach because the input vector consists of such integrals of (3) and (4) over squares. Therefore, we have to expect a loss in computational performance for kernel (3) with the wavelet approach in this case.

Both functions (3) and (4) are Hölder-continuous with parameters $(C_1, \gamma_1) = (2ab, 1)$ and $(C_2, \gamma_2) = (\sqrt{2}ab, 1)$, respectively. We fixed $\alpha = 1.1$, $\beta = 0$ and $[-T,T]^2 = [-1,1]^2$ for both types of kernel functions.

For the remaining parameters, we started with the following configuration: $b = 1$, $a = 1$ and $\varepsilon = 1$. Furthermore, we divided $[-1,1]^2$ into an equidistant grid of 50×50 points and chose $l = 0$ for the number of detail levels to be increased. Two realisations of the 1-stable random field X with kernels (3) and (4) are shown in Figure 2.

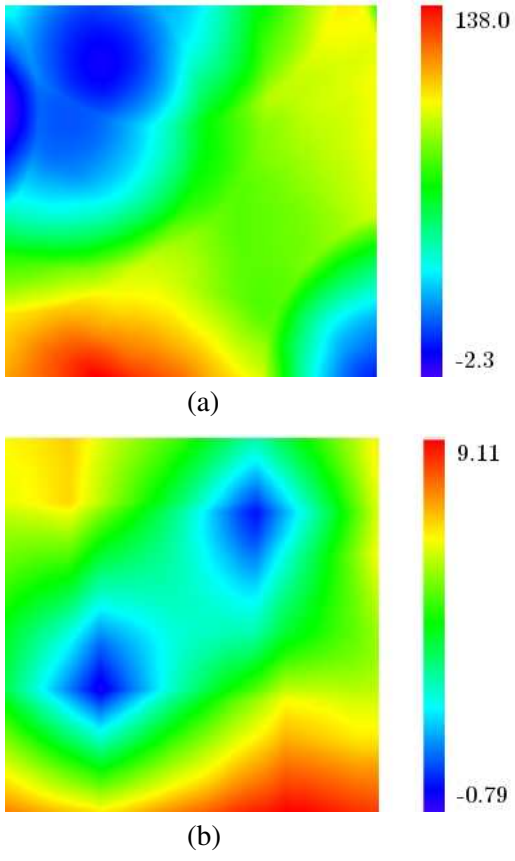

Fig. 2. *Two realisations of stable random fields with kernel (3) (subfigure (a)) and kernel (4) (subfigure (b)).*

First, we kept all parameters fixed and determined the computational time depending on the number of realisations. For the step function approach, each realisation needs the same computational time. For the wavelet approach, however, the wavelet coefficients only have to be calculated for the first realisation

and can be stored afterwards. Therefore, any further realisation needs less computational time. Table 1 shows the results for both the Epanechnikov-type kernel (3) and kernel (4). By trial-and-error, we figured out that a combination of $\varepsilon = \varepsilon_1 + \varepsilon_2 = 0.99 + 0.01$ performs quite good for the corresponding parameters in the wavelet algorithm in this case.

	Kernel (3)	Kernel (4)
Step function approach	1314.4	680.2
Wavelet approach (first realisation)	5245.0	1081.0
Wavelet approach (further realisations)	13.9	12.0

Table 1. *Computational time (in msec) for the first and further realisations.*

Second, we focused on the computational time of any further realisation except the first one and varied subsequently one of the parameters α, m (the number of pixels per row) and ε while all the other parameters were kept fixed. It turned out that the computational time increased much slower for the wavelet approach than for the step function approach when α and m were increased and ε was decreased.

Finally, we increased the parameter l successively for a field with 10×10 pixels while all other parameters were kept fixed and investigated the computational time for the wavelet approach for any further realisation except the first one. Table 2 shows the corresponding results.

l	0	1	2	3
Computational time	35.8	58.1	260.0	1050.4

l	4	5	6
Computational time	501.5	2097.5	12600.9

Table 2. *Computational time (in msec) for different values of l (kernel (4)).*

One might have expected that the computational time tends to decrease if l is increased since the wavelet series consists of less summands when keeping the same level of precision. At the same time, however, more stable random variable simulations have to be performed for the calculation of the integrals

$$\int_{[-A,A]^d} \Psi^e_{j-2^{m+l},m+l} M(dx).$$

That is why for larger values of l, the computational time tends to increase sharply.

APPLICATION TO A PORTFOLIO OF STORM INSURANCE POLICIES

The past years have demonstrated that natural disasters can cause losses in the billions so that insurance companies have to raise enormous sums to cover them (for example after the hundred year flood in August 2002 or after Hurricane Katrina in August 2005). Experts assume that both the intensity and the frequency of natural disasters continue to increase resulting in even larger and more frequent insurance claims.

In contrast to the classical stochastic risk theory dealing with one-dimensional problems, spatial analysis and modelling of risks allows for better assessment of the spatial risk situation and for regionalized premium rating. In the following, we present a method of how to fit a stable random field to insurance data in order to include the spatial structure of risks in the model. After the model has been fixed, we apply the simulation method from Section 2 to generate realisations of the stable random field.

We consider a portfolio of storm insurance policies from an Austrian insurancy company. For each zip code region in Austria, the annual average claim size over an 11-year period as well as x- and y-coordinates of the center of the zip code region are given in the data set. There are data of 2047 zip code regions in the data set.

For $i = 1, ..., 2047$ and $j = 1, ..., 11$, we denote by t_i the vector of the x- and y-coordinates of the center of the zip code region i, by $z_j(t_i)$ the average claim size of year j in zip code region i and by $\bar{z}(t_i)$ the arithmetic mean of the annual average claim sizes in zip code region i over all years. Now we consider the deviations

$$x_j(t_i) = z_j(t_i) - \bar{z}(t_i)$$

and assume that for each year, they are part of a realisation of an α-stable random field of the form

$$X(t) = \int_{\mathbb{R}^2} b \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}\|x-t\|_2^2} M(dx), \quad t \in A, \quad (5)$$

where A is the region of Austria and M has Lebesgue control measure and a constant skewness intensity β. We used all the deviations $x_j(t_i)$ in order to estimate α and β by the Maximum-Likelihood method (cf. Nolan (2001)). The estimated values are $\hat{\alpha} = 1.3562$ and $\hat{\beta} = 0.2796$. In order to estimate the parameters b and σ of the kernel function, we used an approach frequently applied in geostatistics, see for instance Wackernagel (1998).

Consider the so-called first-order madogram

$$\begin{aligned}\gamma(h) &= \mathbb{E}|X(t)-X(t+h)| \\ &= b\frac{1}{\sqrt{2\pi}\sigma}\mathbb{E}\left|\int_{\mathbb{R}^2}\left(e^{-\frac{1}{2\sigma^2}\|x-t\|_2^2}\right.\right. \\ &\qquad \left.\left.-e^{-\frac{1}{2\sigma^2}\|x-t-h\|_2^2}\right)M(dx)\right| \\ &= \frac{c_{\alpha,\beta_t}(1)b}{\sqrt{2\pi}\sigma}\int_{\mathbb{R}^2}\left|e^{-\frac{1}{2\sigma^2}\|x-t\|_2^2}\right. \\ &\qquad \left.-e^{-\frac{1}{2\sigma^2}\|x-t-h\|_2^2}\right|dx.\end{aligned}$$

which does only depend on the length of h, but not on the orientation. It exists since $\alpha > 1$. We note that we cannot take the variogram as described in Wackernagel (1998) since for $\alpha < 2$, the second moment of $X(t)$ does not exist for each $t \in A$.

One can think of the first-order madogram as a measure of dissimilarity between the random variables $X(t)$ and $X(t+h)$ depending on the distance $\|h\|_2$. We calculated the corresponding empirical version of the first-order madogram – the so-called experimental first-order madogram – as correspondingly defined in Wackernagel (1998) and fitted the theoretical one by the least squares method to the empirical one. Both the theoretical and experimental first-order madogram are plotted in Figure 3.

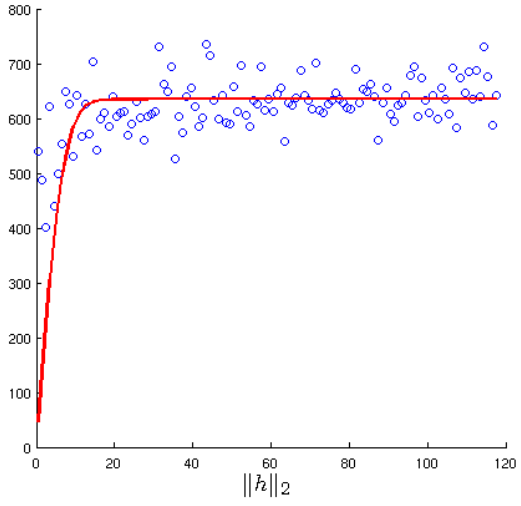

Fig. 3. *Theoretical (red curve) and experimental (blue dots) first-order madogram.*

From the least squares method, we obtained the estimates $\hat{b} = 10.7055$ and $\hat{\sigma} = 4.3472$ and simulated the α-stable random field (5). In Figure 4, a plot of a realisation of the fitted α-stable random field is shown. In order to be able to compare the picture with the real data, we used ordinary kriging (see Wackernagel (1998)) to extrapolate the deviations for the year 2000.

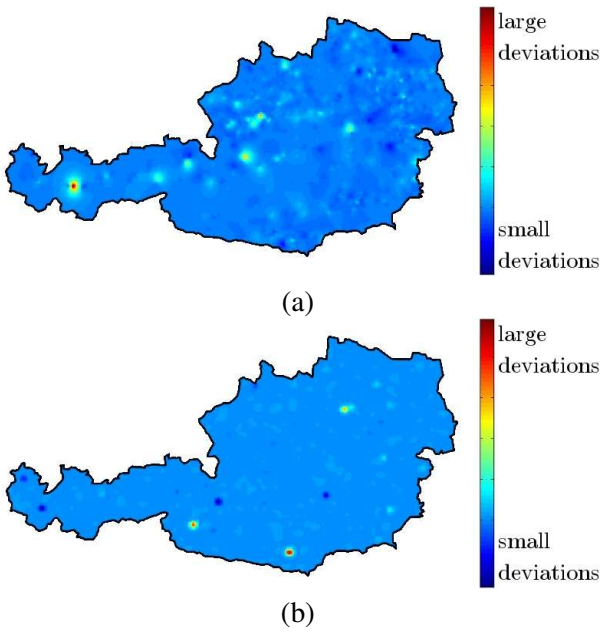

Fig. 4. *Kriged deviations for insurance year 2000 (a) and a realisaton of the fitted α-stable random field (b).*

One can see that both maps show a similar spatial structure of the deviations. There are some very large deviations which are dominant in the maps. The fact that the kriged map is smoother than the simulated one is due to the kriging technique.

In order to obtain a realisation of the actual average claim sizes in each zip code center, one can add the values in the realisation of the α-stable random field to the overall mean $\bar{z}(t_i)$ at each zip code center t_i.

SUMMARY AND OUTLOOK

We presented two approaches to simulate α-stable random fields that are based on approximating the kernel function by a step function and by a wavelet series. For both approaches, we derived estimates for the approximation error $Err(X(t),\tilde{X}^{(n)}(t))$.

In the simulation study, we saw that for the first realisation of an α-stable random field, the step function approach performs better than the wavelet approach due to the initial calculation of the wavelet coefficients. For any further realisation, however, the wavelet approach outperforms the step function approach.

We then applied the simulation methods to an portfolio of storm insurance policies by fitting an α-stable

random field to the data and generating realisations from this field.

To conclude, we want to make two remarks about the wavelet approach.

First, we have seen in the simulation study that one drawback of the wavelet approach is that the computation of the input vector for the fast wavelet transfrom may take quite a long time if no formula for the integrals

$$\int_C f_t(x)dx$$

is known, where C is a cube in $\mathbb{R}^d$. Interpolatory wavelet bases can remedy this disadvantage since for this kind of wavelets bases, the wavelet coefficients basically reduce to evaluating the kernel function at a certain point. However, the interpolatory wavelets themselves are no step functions any more such that the simulation of the integrals

$$\int\limits_{[-A,A]^d} \Psi^e_{j-2^k c,k} M(dx)$$

is much more complicated than for the Haar basis.

Second, one could use adaptive wavelet methods in order to calculate the wavelet coefficients. This might decrease the computational time for the first random field realisation. However, in the simulation study we have seen that increasing the parameter l has little advantage over the cut wavelet series ($l = 0$) since the negative effect of the increasing detail level and thus the need for more stable random variable simulations dominates the positive one of less summands in the wavelet series.

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