Selected topics in the theory of spatial stationary flat processes

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Abstract

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Stationary k–flat processes Φ_k^d in $\mathbb{R}^d$ are considered. These are, by definition, random point processes on the phase space of all k–dimensional flats in the d–dimensional space, each realization of which is an at most countable "locally finite" collection of k–planes. Stationarity means invariance of their distributions with respect to translations in $\mathbb{R}^d$. The main characteristics of Φ_k^d are the *intensity* λ, the *directional distribution* θ, and the *rose of intersections* $T_{kr}\theta(\eta)$ with r–dimensional flats η, $k + r \geqslant d$.

The $(2k-d)$–dimensional intersections of the pairs of k-planes of Φ_k^d induce the new stationary $(2k-d)$–flat process whose intensity is called the *intersection density* of Φ_k^d. The following variational problem for a Poisson process Φ_k^d is considered: find all extremal directional distributions θ of Φ_k^d that maximize its intersection density. By means of the appropriate variational calculus, necessary conditions for a maximum are given in terms of the roses of intersections of Φ_k^d.

Suppose the intensity λ is fixed and the rose of intersections $T_{kr}\theta(\eta) = f(\eta)$ of Φ_k^d with r–flats η is given. It is known that there exists a one–to–one correspondence between f and θ for particular dimensions k and r. An important problem is to find an exact formula that would restore θ from f. The main results yield the retrieval formulae for the directional distribution θ of any stationary process Φ_{d-1}^d of hyperplanes in $\mathbb{R}^d$ from its rose of intersections $T_{d-1,r}\theta$ when the intersecting plane η has dimension r, $1 \leqslant r \leqslant d-1$. The case $d = 4, k = r = 2$ is considered separately. The whole class of directional distributions θ corresponding to the same rose of intersections f is described. The proofs involve inversions of various integral transforms and expansions in spherical harmonics.

In order to invert T_{kr}, some integral relations betweeen Radon and generalized cosine transforms on Grassmann manifolds, the so–called *Cauchy–Kubota–type* formulae, are obtained. The action of Radon

transforms R_{ij} on the functions that are the positive powers of the volumes of certain parallelepipeds is studied.

In the case $k + r < d$, the lower dimensional test flat η has, in general, no intersections with the k–flats of the process. Hence, the notion of the rose of intersections is here irrelevant. To overcome this difficulty, a counterpart to the notion of the rose of intersections, the *rose of neighborhood* is introduced and its properties are studied. It is shown that the inversion formulae that yield the directional distribution of the process Φ_k^d from its rose of intersections can be easily modified to hold for the rose of neighborhood.