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CAUCHY–KUBOTA–TYPE INTEGRAL FORMULAE FOR THE GENERALIZED COSINE TRANSFORMS

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The paper considers the action of Radon transforms on Grassmann manifolds for some special functions. These functions are positive powers of the volume of certain parallelepipeds. As a consequence some integral geometric formulas of Cauchy–Kubota–type for the generalized cosine transforms on Grasmannians are proved. Some applications of these results in convex and stochastic geometry, stereology and geometric tomography are discussed.

§1. INTRODUCTION

This section contains the notation most of the definitions and the main results. Their proofs and corollaries are given in sections 4 – 6. Various applications of the generalized cosine transforms in geometry are touched upon in section 2. The history of the problem is discussed in section 3.

1.1. Preliminaries. Let $G(k,d)$ be the Grassmann manifold of all linear k–dimensional subspaces of $\mathbb{R}^d$, $d \geq 3$. The *Radon transform on Grassmannians* and its *dual* we introduce following the paper of Grinberg [10]: for $1 \leq i < j \leq d-1$

$$(R_{ij}f)(\xi) = \int\limits_{\eta \in G(i,d):\eta\subset\xi} f(\eta)\,\sigma(d\eta), \quad (R_{ji}g)(\eta) = \int\limits_{\xi\in G(j,d):\xi\supset\eta} g(\xi)\,\sigma(d\xi),$$

where $f \in L^1(G(i,d))$, $g \in L^1(G(j,d))$, $\xi \in G(j,d)$, $\eta \in G(i,d)$, $\sigma(\cdot)$ is the unique rotation invariant measure on the appropriate integration space with total mass 1. Such Radon transforms find numerous applications in convex geometry, see [3] – [5], [8], [9].

We make use of the following notation:

$L\{a_1,\ldots,a_k\}$ = the k-flat spanned by the vectors $a_1,\ldots,a_k$,

$Vol^{(k)}(a_1,\ldots,a_k)$ = the non-oriented volume of the parallelepiped spanned by $a_1,\ldots,a_k$ (we shall often omit the dimension of the volume),

$Vol^{(k)}(\xi)$ = the non-oriented volume of the parallelepiped spanned by the orthonormal basis vectors of the k-flat ξ,

$e_1,\ldots,e_d$ = the Cartesian unit basis vectors in $\mathbb{R}^d$,

$\omega_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of the $(d-1)$-dimensional sphere $\mathcal{S}^{d-1}$,

$d\eta$ = the rotation invariant measure on a Grassmannian with total mass 1,

$b(\xi)$ = a set of the orthonormal basis vectors $a_1,\ldots,a_k$, such that $\xi = L\{a_1,\ldots,a_k\}$ for $\xi \in G(k,d)$.

1.2. Main results. In Section 4, we study the action of R_{ij} and R_{ji} on the functions of the type $f(\eta) = [\eta,\zeta_0]^\alpha$ ($f(\eta) = [\eta^\perp,\zeta_0]^\alpha$ in the case of the dual Radon transform) for η from $G(i,d)$ (or $G(j,d)$, respectively) and $\alpha > 0$, where

$$[a,b] = Vol^{(2d-n-m)}(a_{n+1},\ldots,a_d,b_{m+1},\ldots,b_d)$$

and $\zeta_0 = L\{e_1,\ldots,e_k\}$. Here $a_{n+1},\ldots,a_d$ and $b_{m+1},\ldots,b_d$ are some orthonormal bases in the orthogonal complements $a^\perp$ and $b^\perp$ for $a \in G(n,d)$, $b \in G(m,d)$.

Theorem 1.1. Let $\xi \in G(j,d)$, $\zeta_0 = L\{e_1,\ldots,e_k\}$, $i < j$, $i+k \geq d$, $d \geq 3$, $\alpha > 0$. Then

$$(R_{ij}[\cdot,\zeta_0]^\alpha)(\xi) = \int\limits_{\eta\in G(i,d):\eta\subset\xi} [\eta,\zeta_0]^\alpha\,\sigma(d\eta) = c(\alpha)[\xi,\zeta_0]^\alpha, \tag{1.1}$$

where

$$c(\alpha) = \prod_{l=0}^{d-k-1} \frac{\Gamma\left(\frac{j-l}{2}\right)\Gamma\left(\frac{i-l+\alpha}{2}\right)}{\Gamma\left(\frac{i-l}{2}\right)\Gamma\left(\frac{j-l+\alpha}{2}\right)}. \tag{1.2}$$

Proposition 1.1. Let $i < j \le k < d$, $d \ge 3$. Then for $\alpha > 0$ the following relation holds:

$$(R_{ji}[\cdot^{\perp}, \zeta_0]^{\alpha})(\eta) = c^*(\alpha)[\eta^{\perp}, \zeta_0]^{\alpha}, \tag{1.3}$$

where $\eta \in G(i,d)$,

$$c^*(\alpha) = \prod_{l=0}^{d-k-1} \frac{\Gamma\left(\frac{d-i-l}{2}\right)\Gamma\left(\frac{d-j-l+\alpha}{2}\right)}{\Gamma\left(\frac{d-j-l}{2}\right)\Gamma\left(\frac{d-i-l+\alpha}{2}\right)}. \tag{1.4}$$

In Section 5, we study some interesting connections between the generalized cosine transforms and Radon transforms. Let $\mathbf{M}(G(i,d))$ be the Banach space of all signed measures on $G(i,d)$ with finite total variation, and $C(G(i,d))$ be the space of all continuous functions on $G(i,d)$. Introduce for $i+j \ge d$ the *generalized cosine transform* $T_{ij} : \mathbf{M}(G(i,d)) \longmapsto C(G(j,d))$,

$$(T_{ij}\theta)(\xi) = \int\limits_{G(i,d)} [\xi,\eta]\,\theta(d\eta), \tag{1.5}$$

where $\theta \in \mathbf{M}(G(i,d))$, $\xi \in G(j,d)$. In particular, if $\theta(d\eta) = f(\eta)d\eta$, meaning that θ is absolute continuous with respect to $d\eta$, we write

$$(T_{ij}f)(\xi) = \int\limits_{G(i,d)} [\xi,\eta] f(\eta)\,d\eta.$$

To achieve greater generality, we imbed the generalized cosine transforms into the new families of operators $\{T_{ij}^{\alpha}\}$, $\{\tilde{T}_{ij}^{\alpha}\}$, where α is a positive parameter (see Section 2 for details):

$$T_{ij}^{\alpha}, \tilde{T}_{ij}^{\alpha} : \mathbf{M}(G(i,d)) \longmapsto C(G(j,d)),$$

$$(T_{ij}^{\alpha}\theta)(\xi) = \int\limits_{G(i,d)} [\xi,\eta]^{\alpha}\,\theta(d\eta), \quad \left(\tilde{T}_{ij}^{\alpha}\theta\right)(\xi) = \int\limits_{G(i,d)} [\xi,\eta^{\perp}]^{\alpha}\,\theta(d\eta), \quad j \ge i.$$

On integrable functions the above transforms can be introduced as before. Evidently, these families comprise generalized cosine (T_{ij}^{1}) transforms. It is also clear, that

$$\left(\tilde{T}_{ij}^{\alpha}\theta\right)(\xi) = \left(T_{d-i,j}^{\alpha}\theta^{\perp}\right)(\xi), \tag{1.6}$$

where $\theta^{\perp}(d\nu) = \theta(d\nu^{\perp})$ for $\nu \in G(d-i,d)$.

In Section 5, the first Cauchy–Kubota – type formula for operators T_{ij}^{α} is proved.

Proposition 1.2. [First Cauchy–Kubota – type formula] For any $\alpha > 0$ and dimensions i, j, k with $i+k \geq d$, $i < j$ the following integral relation is valid on the space $\mathbf{M}(G(k,d))$:

$$R_{ij} T^{\alpha}_{ki} = c(\alpha) T^{\alpha}_{kj}, \tag{1.7}$$

where the constant $c(\alpha)$ is defined in (1.2).

Two corollaries of the above proposition and their stereological meaning are discussed at the end of the section. Some interesting consequences of the double fibration relation for T^{α}_{ij} are considered in Section 6, of which the more important is the following.

Proposition 1.3 [Second Cauchy–Kubota – type formula] For all $i < j$, $i + k \geq d$, $\alpha > 0$ and all absolute integrable functions $\varphi \in L^1(G(j,d))$,

$$\left(T^{\alpha}_{jk}\varphi\right)(\zeta) = c^{-1}(\alpha) T^{\alpha}_{ik}\left(R_{ji}\varphi\right)(\zeta), \quad \zeta \in G(k,d), \tag{1.8}$$

where $c(\alpha)$ is given by (1.2).

Some upper bounds for the weighted images of Radon transforms are given at the end of §6.

§2. THE MEANING OF T^{α}_{ij} IN STOCHASTIC GEOMETRY

The transforms (1.5) generalize the well-known notion of the *spherical cosine transform:*

$$T\theta(u) = \int\limits_{\mathcal{S}^{d-1}} |\langle u, v\rangle| \, \theta(dv) = \left(T_{d-1,1}\theta\right)(u), \quad u \in \mathcal{S}^{d-1}.$$

There exists exhaustive literature on the spherical cosine transform and its use in geometry (see e.g. [1], [2], [6], [16], [17]).

The generalized cosine transforms find important applications in convex and stochastic geometry as well. Namely, the *i-th projection function* $v_i(K;\cdot)$ of a *zonoid* K is the generalized cosine transform of its *projection generating measure* $\rho_i(K,\cdot)$:

$$v_i(K;\eta) = \left(T_{i,d-i}\rho_i(K,\cdot)\right)(\eta^{\perp}), \quad \eta \in G(i,d) \tag{2.1}$$

(cf. [8]). By definition, $v_i(K;\eta)$ is the i-dimensional volume of the orthogonal projection of K onto η.

Furthermore, in stochastic geometry $(T_{ij}\theta)(\eta)$ means the *rose of intersections* of a stationary stochastic process of i-dimensional manifolds (or affine flats) Φ_i^d in $\mathbb{R}^d$ with unit intensity and directional distribution measure $\theta(\cdot)$ with an arbitrary j-dimensional flat η through the origin.

Roughly speaking, a stochastic process Φ_i^d of affine i-flats can be thought of as a random countable collection of affine i-flats in $\mathbb{R}^d$. Its stationarity means the invariance of its probabilistic properties with respect to all translations. The intensity of Φ_i^d is the mean i-volume content of its flats in a unit observation window. The directional distribution of Φ_i^d is the probability distribution of the direction $r(\xi)$ of the *typical* i-flat $\xi \in \Phi_i^d$, i.e., of a flat picked up "at random" among the others in any realization of Φ_i^d.

The *direction* $r(\xi)$ of an affine i-flat ξ is the unique i-flat through the origin parallel to ξ (cf. [20] for exact definitions). Sometimes it is necessary to restore the probabilistic characteristics of such spatial stochastic processes, if one observes the process of intersections of the original process with all affine j–flats η. Due to stationarity, it is enough to consider only $\eta \in G(j,d)$.

One of the characteristics of the latter process is the so-called *rose of intersections*, i.e. the mean i-content of intersection planes $\Phi_i^d \cap \eta$ in a unit test window within η for arbitrary $\eta \in G(j,d)$. As shown in [18] and [19], the retrieval of the directional distribution θ of Φ_i^d from its rose of intersections $T_{ij}\theta$ is possible for some particular dimensions i and j.

The transforms of the parametric family $\{T_{ij}^\alpha\}$ can also get an interpretation in terms of stochastic geometry:

$$(T_{ij}^\alpha\theta)(\xi) = \int\limits_{G(i,d)} [\xi,\eta]^\alpha \, \theta(d\eta) = E_\theta\left([\xi,\eta]^\alpha\right)$$

is the moment of order α of the quantity $[\xi,\eta]$ = generalized "sine of the angle" between the direction η of the affine planes of the process Φ_i^d and a fixed *test* direction $\xi \in G(j,d)$ with respect to the directional distribution θ. All the results of this paper are valid for arbitrary real $\alpha > 0$. As far as it is known to the author, the present work is the first attempt to treat those moments.

One can also find another interpretation of T_{ij}^α at least in $\mathbb{R}^2$ and $\mathbb{R}^3$, which can be extended analogously to arbitrary dimensions d. It can be shown (see [20], pp. 286 – 303), that

$$\mathcal{A} \longmapsto \frac{\int_{\mathcal{A}} [\xi, \eta]\, \theta(d\eta)}{\int_{G(i,d)} [\xi, \eta]\, \theta(d\eta)}, \quad \mathcal{A} \text{ measurable}$$

is the distribution of "sine" of the typical intersection angle between the test hyperplane (or line) ξ and Φ_i^d ($d = 2$ or $d = 3$). If we rewrite T_{ij}^{α} in the form

$$c \cdot \frac{\int_{G(i,d)} [\xi, \eta]^{\alpha - 1} [\xi, \eta]\, \theta(d\eta)}{(T_{ij}\theta)(\xi)} = c \cdot E\left([\xi, \eta]^{\alpha-1} | \eta \in \Phi_i^d,\ \eta \cap \xi \neq \emptyset\right),$$

where $c = (T_{ij}\theta)(\xi)$, we get that $(T_{ij}^{\alpha}\theta)(\xi)$ is proportional to the moment of order $\alpha - 1$ of the absolute value of the "sine" of the typical intersection angle of Φ_i^d with a fixed test flat (line) ξ.

The meaning of $\left(\tilde{T}_{ij}^{1}\theta\right)(\xi)$ also becomes transparent: it is the rose of intersections of a stationary $(d-i)$-flat process with directional distribution $\theta^{\perp}(\cdot)$ and intensity 1 with a j-flat ξ (cf. relation (1.6)).

It would be of some interest to illustrate the use of Proposition 1.3 in stochastic geometry. Namely, for the values $\alpha = 1$, $d = 3$, $i = 1$, $j = 2$ and $k = 2$, by (1.8),

$$(T_{22}\varphi)(\zeta) = \frac{\pi}{2} T_{12}(R_{21}\varphi)(\zeta), \quad \zeta \in G(k, d). \tag{2.2}$$

The transform $(T_{22}\varphi)(\zeta)$ is the rose of intersections of the stationary process of planes Φ_2^3 in three dimensions with a test plane ζ. The process Φ_2^3 has the unit intensity and the directional distribution with density φ. By (2.2), this rose of intersections is equal to the rose of intersections T_{12} of the process Φ_1^3 of lines with the same test plane ζ, where this new process Φ_1^3 has the unit intensity and directional distribution density $R_{21}\varphi$ obtained from φ by integration.

§3. SOME HISTORICAL REMARKS

A special case of Proposition 1.1 for the dual Radon transform with $\alpha = 1$ and dimension $k = j$ can be found in lemma 4.1 of [7]. The argument there uses the connection between volumes $[\cdot,\cdot]$, mixed volumes and projection functions. Then the following Cauchy–Kubota formula (equation (2.3) of [7]) is applied:

$$R_{ij}\left(v_i(K;\cdot)\right)(\xi) = \frac{i!k_i(j-i)!k_{j-i}}{j!k_j} V_i\left(Pr_{\xi}\{K\}\right) \tag{3.1}$$

for a convex body K and all $\xi \in G(j,d)$, where $i < j < d$, $k_d = \dfrac{2\pi^{d/2}}{d\Gamma(d/2)}$ is the volume of the d-dimensional unit ball, $Pr_\xi\{a\}$ denotes the orthogonal projection of the vector (plane, convex body) a onto the plane ξ, $v_i(K;\eta)$ is the i-th projection function of K, and $V_i(L)$ is the i*-th intrinsic volume* (cf. [15]) of a convex body L.

Our approach differs substantially from that of [7]. It allows us to gain more generality in dimensions of involved linear subspaces and positive powers of the volume. Corollary 5.1 generalizes a well-known relation for the spherical Radon and usual cosine transforms (cf. relation (5.12) of [18] and references therein):

$$R_{1r}T_{d-1,1} = \frac{2k_{r-1}}{\omega_r}T_{d-1,r}. \tag{3.2}$$

The name "Cauchy–Kubota – type" for the formulas stated in Propositions 1.2, 1.3 and 6.2 is due to the resemblance between the left-hand side of (1.7) and relation (3.1). Indeed, suppose $\alpha = 1$, $i < j$, and let the convex body K be a zonoid. By (2.1) and the duality relation (2.3) of [5], the left-hand side of (3.1) rewrites

$$\left(R_{d-i,d-j}\left(T^1_{i,d-i}\rho_i(K,\cdot)\right)\right)(\xi^\perp),$$

or, equivalently,

$$\left(R_{ij}\left(\tilde{T}^1_{d-i,i}\rho_i^\perp(K,\cdot)\right)\right)(\xi).$$

In spite of this similarity, the Cauchy–Kubota formula does not follow from our results; nor can they be deduced as a direct corollary of (3.1).

The classical Cauchy–Kubota formulas and their versions can be found in Ch. 13, §1, 2 of [14], [15], p. 295, and [12], p. 126.

§4. RADON TRANSFORMS OF THE POWER OF THE VOLUME

Proof of Theorem 1.1. We fix an orthonormal basis $\xi_1,\ldots,\xi_j$ of $\xi \in G(j,d)$, so that $\xi_1,\ldots,\xi_i$ is an orthonormal basis of $\eta \in G(i,d)$, $i < j$. Let

$$(\eta)^\perp_\xi = L\{\xi_{i+1},\ldots,\xi_j\} \tag{4.1}$$

be the orthogonal complement of η in ξ. We denote by $\xi_{j+1}, \ldots, \xi_d$ a certain orthonormal basis of $\xi^\perp$. Then the vectors $\xi_{i+1}, \ldots, \xi_d$ form an orthonormal basis in $\eta^\perp$. If $[\xi, \zeta_0] = 0$, i.e. $dim\,(\xi^\perp \cap \zeta_0^\perp) > 0$, then $[\eta, \zeta_0] = 0$ for all $\eta \subset \xi$, since $\xi^\perp \subset \eta^\perp$ and $dim\,(\eta^\perp \cap \zeta_0^\perp) > 0$. Hence, formula (1.1) holds automatically. It means that in the following it suffices to prove (1.1) for the case $[\xi, \zeta_0] \neq 0$, i.e.

$$\xi^\perp \cap L\{e_{k+1}, \ldots, e_d\} = \{0\}. \tag{4.2}$$

The following relation holds:

$$[\eta, \zeta_0] \equiv Vol(\xi_{i+1}, \ldots, \xi_d, e_{k+1}, \ldots, e_d) =$$

$$= Vol(\xi_{j+1}, \ldots, \xi_d, e_{k+1}, \ldots, e_d) \cdot Vol\left(Pr_{L^\perp\{\xi_{j+1}, \ldots, \xi_d, e_{k+1}, \ldots, e_d\}}\{\xi_{i+1}, \ldots, \xi_j\}\right),$$

or, briefly,

$$[\eta, \zeta_0] = [\xi, \zeta_0] Q(\xi, \eta), \tag{4.3}$$

where $Q(\xi, \eta)$ denotes the $(j-i)$-dimensional volume of the parallelepiped spanned by projections of $\xi_{i+1}, \ldots, \xi_j$ (cf. (4.1)) onto the plane

$$L^\perp\{\xi_{j+1}, \ldots, \xi_d, e_{k+1}, \ldots, e_d\} = L^\perp(\xi^\perp, \zeta_0^\perp). \tag{4.4}$$

Thus, by (4.3) we have

$$\int\limits_{\eta \in G(i,d): \eta \subset \xi} [\eta, \zeta_0]^\alpha \sigma(d\eta) = c_\alpha(\xi) [\xi, \zeta_0]^\alpha,$$

$$c_\alpha(\xi) = \int\limits_{\eta \in G(i,d): \eta \subset \xi} Q(\xi, \eta)^\alpha \sigma(d\eta). \tag{4.5}$$

We write $Q(\xi, \eta)$ in a different form. First we give another representation to the plane (4.4):

$$L^\perp(\xi^\perp, \zeta_0^\perp) = \xi \cap \zeta_0. \tag{4.6}$$

Indeed, if $\tau \in L^\perp(\xi^\perp, \zeta_0^\perp)$, then $\tau \perp \{\xi_{j+1}, \ldots, \xi_d\}$ and $\tau \perp \{e_{k+1}, \ldots, e_d\}$. Hence, $\tau \in L^\perp\{\xi_{j+1}, \ldots, \xi_d\} = \xi$, $\tau \in L^\perp\{e_{k+1}, \ldots, e_d\} = \zeta_0$, or $\tau \in \xi \cap \zeta_0$. Thus, $L^\perp(\xi^\perp, \zeta_0^\perp) \subseteq \xi \cap \zeta_0$. Since $dim\left(L^\perp(\xi^\perp, \zeta_0^\perp)\right) = dim(\xi \cap \zeta_0)$ as we use the obvious formula

$$dim(a \cap b) \geq dim(a) + dim(b) - d. \tag{4.7}$$

The relation (4.6) is proved. Now show that

$$Q(\xi,\eta) = Vol^{(d-k+j-i)}\left(\xi_{j+1},\ldots,\xi_d, b\Big((\xi\cap\zeta_0)^{\perp}_{\xi}\Big)\right) \stackrel{def}{=} [\xi\cap\zeta_0,\eta]_{\xi}. \tag{4.8}$$

By definition of $Q(\xi,\eta)$, owing to (4.6),

$$Q(\xi,\eta) = Vol^{(j-i)}\left(Pr_{\xi\cap\zeta_0}(\eta)^{\perp}_{\xi}\right). \tag{4.9}$$

The following formula holds for any flats a and c in arbitrary ambient space:

$$[a^{\perp},c] \stackrel{def}{=} Vol\left(b(a),b(c^{\perp})\right) = Vol(Pr_c(a)) = Vol\left(Pr_{a^{\perp}}\left(c^{\perp}\right)\right).$$

Hence, equality (4.9) yields

$$Q(\xi,\eta) = Vol^{(j-i)}\left(Pr_{\xi\cap\zeta_0}(\eta)^{\perp}_{\xi}\right) = Vol\left(Pr_{\eta}(\xi\cap\zeta_0)^{\perp}_{\xi}\right) = [\xi\cap\zeta_0,\eta]_{\xi}$$

(here the ambient space is ξ). Thus, relation (4.8) is proved, and one obtains by (4.5)

$$c_{\alpha}(\xi) = \int_{\eta\in G(i,d):\eta\subset\xi} [\xi\cap\zeta_0,\eta]^{\alpha}_{\xi}\,\sigma(d\eta).$$

We prove that $c_{\alpha}(\xi)$ does not depend on ξ. According to (4.2), it is sufficient to consider only the case

$$\xi^{\perp}\cap\zeta_0^{\perp} = \{0\}. \tag{4.10}$$

For any $\xi\in G(j,d)$ there exists a rotation $\gamma\in SO(d)$, such that $\xi=\gamma\xi_0$, $\xi_0 = L\{e_1,\ldots,e_j\}$. Then

$$c_{\alpha}(\xi) = \int_{\eta\in G(i,d):\eta\subset\xi_0} [\gamma\xi_0\cap\zeta_0,\gamma\eta]^{\alpha}_{\gamma\xi_0}\,\sigma(d\eta) = \int_{\eta\in G(i,d):\eta\subset\xi_0} [\xi_0\cap\gamma^{-1}\zeta_0,\eta]^{\alpha}_{\xi_0}\,\sigma(d\eta),$$

and (4.10) has the form

$$(\gamma\xi_0)^{\perp}\cap\zeta_0^{\perp} = \gamma\xi_0^{\perp}\cap\zeta_0^{\perp} = \xi_0^{\perp}\cap\gamma^{-1}\zeta_0^{\perp} = \{0\}.$$

Without loss of generality, one can substitute γ for γ^{-1}. Thus, we have to show that

$$\tilde{c}_{\alpha}(\gamma) \stackrel{def}{=} \int_{\eta\in G(i,d):\eta\subset\xi_0} [\xi_0\cap\gamma\zeta_0,\eta]^{\alpha}_{\xi_0}\,\sigma(d\eta) \tag{4.11}$$

is constant on the set

$$G_{jk} = \left\{ \gamma \in SO(d) : \xi_0^{\perp} \cap \gamma\zeta_0^{\perp} = \{0\} \right\}. \tag{4.12}$$

First, let us first prove that

$$dim(\xi_0 \cap \gamma\zeta_0) = j + k - d, \quad \gamma \in G_{jk}.$$

By (4.7), the dimension of $\xi_0 \cap \gamma\zeta_0$ can not be less than $j+k-d$. Let us prove that it also can not be greater than $j+k-d$. Suppose, ex adverso, that $dim(\xi_0 \cap \gamma\zeta_0) = m > j+k-d$. Let $\tau_1, \dots, \tau_m$ be the basis in $\xi_0 \cap \gamma\zeta_0$. Amplify it to the bases in ξ_0 and $\gamma\zeta_0$:

$$\xi_0 = L\{\tau_1, \dots, \tau_m, \tau_{m+1}, \dots, \tau_j\}, \quad \gamma\zeta_0 = L\{\tau_1, \dots, \tau_m, \tilde{\tau}_{m+1}, \dots, \tilde{\tau}_k\}.$$

The number of distinct unit vectors in ξ_0, $\gamma\zeta_0$ is equal to $j+k-m$. As $m > j+k-d$, that number is less than d. So there exists at least one unit vector $x \in \mathbb{R}^d$, that does not belong to the linear hull of the bases in ξ_0 and $\gamma\zeta_0$. Then $x \in \xi_0^{\perp} \cap (\gamma\zeta_0)^{\perp}$. We arrived at the contradiction with (4.12).

Thus, we proved that any transform $\gamma \in G_{jk}$ preserves the dimension of $\beta \stackrel{def}{=} \xi_0 \cap \gamma\zeta_0 \subset \xi_0$. Identifying ξ_0 with $\mathbb{R}^j$, we can rewrite the relation (4.11) as follows:

$$\bar{c}_\alpha(\beta) = \int\limits_{\eta \in G(i,d):\eta \subset \mathbb{R}^j} [\beta, \eta]^\alpha_{\mathbb{R}^j}\, \sigma(d\eta) = \int\limits_{G(i,j)} [\beta, \eta]^\alpha_{\mathbb{R}^j}\, d\eta,$$

$\beta \subset \xi_0$, $dim(\beta) = j+k-d$. Now prove that $\bar{c}_\alpha(\beta)$ does not depend on $\beta \in G(j+k-d, j)$. Indeed, by rotation invariance (since $dim(\beta)$ does not depend on $\gamma \subset G_{jk}$),

$$\bar{c}_\alpha(\beta) = \bar{c}_\alpha(\beta_0), \quad \beta_0 = L\{e_1, \dots, e_{j+k-d}\}.$$

Thus, we have proved that

$$c(\alpha) = \int\limits_{G(i,j)} [\beta_0, \eta]^\alpha_{\mathbb{R}^j}\, d\eta$$

is a constant, $\beta_0 = L\{e_1, \dots, e_{j+k-d}\} \in G(j+k-d, j)$. Now our aim is to calculate $c(\alpha)$. For $n+r \geq m$, $\beta_0 = L\{e_1, \dots, e_r\}$ we put

$$b_\alpha(n, m, r) = \int\limits_{G(n,m)} [\beta_0, \eta]^\alpha_{\mathbb{R}^m}\, d\eta.$$

Then $c(\alpha) = b_\alpha(i,j,j+k-d)$. We calculate $b_\alpha(n,m,r)$ for all n,m,r, such that $n+r \geq m \geq 2$. At the first step, we prove that

$$b_\alpha(k,d,i) = b_\alpha(i,j,i+j-d) \cdot b_\alpha(k,d,j) \tag{4.13}$$

for $i+k \geq d$, $i < j$, $i \geq d/2$, $d \geq 2$. Integrate the equality

$$\int\limits_{\eta \in G(i,d):\eta \subset \xi} [\eta,\zeta]^\alpha \, \sigma(d\eta) = c(\alpha)[\xi,\zeta]^\alpha \tag{4.14}$$

with respect to ζ, where $\zeta \in G(k,d)$ and $\xi \in G(j,d)$ ((4.14) follows from (1.1) by rotation invariance). By Fubini's theorem,

$$\int\limits_{\eta \in G(i,d):\eta \subset \xi} \left(\int\limits_{G(k,d)} [\eta,\zeta]^\alpha \, d\zeta \right) \sigma(d\eta) = c(\alpha) \int\limits_{G(k,d)} [\xi,\zeta]^\alpha \, d\zeta.$$

By rotation invariance, the integrand in parentheses in the left-hand side does not depend on η, and is equal to $b_\alpha(k,d,i)$, so we can write

$$b_\alpha(k,d,i) \int\limits_{\eta \in G(i,d):\eta \subset \xi} \sigma(d\eta) = c(\alpha) b_\alpha(k,d,j).$$

As the total mass of the measure σ is one, the above relation completes the proof of (4.13).

By lemma 2.2 (a) of [13] (where the operators A and A^* are applied to a constant function),

$$b_\alpha(d-1,d,d-k) = b_\alpha(d-k,d,d-1) = \frac{\omega_{d-k}\omega_k}{\omega_d} \int\limits_0^1 (1-t^2)^{(k-2)/2} t^{d-k-1+\alpha} dt. \tag{4.15}$$

The integral in the right-hand side of (4.15) is equal to

$$\frac{1}{2} \int\limits_0^1 (1-u)^{k/2-1} u^{\frac{d-k+\alpha}{2}-1} du = \frac{1}{2} \frac{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{d+\alpha-k}{2}\right)}{\Gamma\left(\frac{d+\alpha}{2}\right)}.$$

By (4.15),

$$b_\alpha(d-1,d,d-k) = \frac{1}{2} \frac{\omega_{d-k}\omega_k}{\omega_d} \frac{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{d+\alpha-k}{2}\right)}{\Gamma\left(\frac{d+\alpha}{2}\right)} = \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d+\alpha-k}{2}\right)}{\Gamma\left(\frac{d-k}{2}\right)\Gamma\left(\frac{d+\alpha}{2}\right)}.$$

Thus, we have proved that

$$b_\alpha(d-1,d,d-k) = b_\alpha(d-k,d,d-1) = \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d+\alpha-k}{2}\right)}{\Gamma\left(\frac{d-k}{2}\right)\Gamma\left(\frac{d+\alpha}{2}\right)}. \tag{4.16}$$

Now we show that for all k, $b_\alpha(d-k,d,k) = b_\alpha(k,d,d-k)$. By definition, for some $\eta \in G(k,d)$,

$$b_\alpha(d-k,d,k) = \int\limits_{G(d-k,d)} [\beta_0,\eta]^\alpha \, d\eta = \int\limits_{G(d-k,d)} [\beta_0^\perp,\eta^\perp]^\alpha \, d\eta =$$

$$= \int\limits_{G(k,d)} [\beta_0^\perp,\eta^\perp]^\alpha \, d\eta^\perp = \int\limits_{G(k,d)} [\beta_0^\perp,\nu]^\alpha \, d\nu = b_\alpha(k,d,d-k).$$

The above is true since $[\beta,\eta] = [\beta^\perp,\eta^\perp]$ for all $\beta \in G(d-k,d)$, $\eta \in G(k,d)$. Furthermore, by (4.16) we have for all $1 \le r \le d-1$,

$$b_\alpha(d-1,d,r) = b_\alpha(r,d,d-1) = \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{r+\alpha}{2}\right)}{\Gamma\left(\frac{r}{2}\right)\Gamma\left(\frac{d+\alpha}{2}\right)}. \tag{4.17}$$

By (4.15) and (4.17), the following equality is true:

$$b_\alpha(k,d,d-2) = b_\alpha(d-2,d-1,k-1) b_\alpha(k,d,d-1) = \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{k+\alpha}{2}\right)}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{d+\alpha}{2}\right)} \cdot \frac{\Gamma\left(\frac{d-1}{2}\right)\Gamma\left(\frac{k-1+\alpha}{2}\right)}{\Gamma\left(\frac{k-1}{2}\right)\Gamma\left(\frac{d-1+\alpha}{2}\right)}.$$

In the same way, by induction on r, for $\alpha > 0$

$$b_\alpha(k,d,d-r) = \prod_{l=0}^{r-1} \frac{\Gamma\left(\frac{d-l}{2}\right)\Gamma\left(\frac{k-l+\alpha}{2}\right)}{\Gamma\left(\frac{k-l}{2}\right)\Gamma\left(\frac{d-l+\alpha}{2}\right)},$$

or, more generally,

$$b_\alpha(k,d,r) = \prod_{l=0}^{d-r-1} \frac{\Gamma\left(\frac{d-l}{2}\right)\Gamma\left(\frac{k-l+\alpha}{2}\right)}{\Gamma\left(\frac{k-l}{2}\right)\Gamma\left(\frac{d-l+\alpha}{2}\right)}, \quad k+r \ge d. \tag{4.18}$$

Then as $c(\alpha) = b_\alpha(i,j,j+k-d)$ and by (4.18) we get (1.2). Theorem 1.1 is proved.

Proof of Proposition 1.1. Let $\eta \subset \xi \in G(j,d)$. Then $\xi^\perp \subset \eta^\perp$, and using the duality relation (2.3) of [5] for the Radon transform, one can write

$$(R_{ji}[\cdot^\perp,\zeta_0]^\alpha)(\eta) = \left(R_{d-j,d-i}[\cdot,\zeta_0]^\alpha\right)(\eta^\perp).$$

Above $d-j < d-i$, $\eta^\perp \in G(d-i,d)$, so we can use the result of Theorem 1.1:

$$\left(R_{d-j,d-i}[\cdot,\zeta_0]^\alpha\right)(\eta^\perp) = c^*(\alpha)[\eta^\perp,\zeta_0]^\alpha,$$

where $c^*(\alpha) = b_\alpha(d-j,d-i,d-i+k-d) = b_\alpha(d-j,d-i,k-i)$ (cf. proof of Theorem 1.1). By (4.18), we get

$$c^*(\alpha) = \prod_{l=0}^{d-k-1} \frac{\Gamma\left(\frac{d-i-l}{2}\right)\Gamma\left(\frac{d-j-l+\alpha}{2}\right)}{\Gamma\left(\frac{d-j-l}{2}\right)\Gamma\left(\frac{d-i-l+\alpha}{2}\right)},$$

and the assertion is proved.

§5. THE PROOF OF THE CAUCHY–KUBOTA – TYPE FORMULAS FOR THE GENERALIZED COSINE TRANSFORMS

Proof of Proposition 1.2. We integrate both sides of (4.14) with respect to some measure $\theta \in \mathbf{M}(G(k,d))$ and use Fubini's theorem:

$$\int\limits_{\eta \in G(i,d):\eta\subset\xi} \int\limits_{G(k,d)} [\eta,\zeta]^{\alpha}\,\theta(d\zeta)\sigma(d\eta) = c(\alpha) \int\limits_{G(k,d)} [\xi,\zeta]^{\alpha}\,\theta(d\zeta).$$

The comparison of both sides of the above equation with the expression for T_{ij}^{α} completes the proof.

The following corollary is an easy consequence of Proposition 1.2 for $\alpha = 1$.

Corollary 5.1. For any i, j, k with $i + k \geq d$, $i < j$,

$$R_{ij}T_{ki} = \frac{\omega_{i+1-d+k}\omega_{j+1}}{\omega_{j+1-d+k}\omega_{i+1}} T_{kj}.$$

The connection between generalized cosine transforms of different orders can be used in tomography or stereology, in estimation of the statistical parameters of the shape of a geometric structure (e.g. porous media, microscopic shots of tissues, fiber collections, etc.) basing on the experimental data gained from sections or, in our terms, from intersections with flats of different dimensions. The structures in question are often modeled as k-dimensional manifold processes in $\mathbb{R}^d$, which can be seen locally as k-flat processes with directional distribution θ.

The characteristics obtained from the intersections with i-flats are often the roses of intersections $T_{ki}\theta$ (or the corresponding moments $T_{ki}^{\alpha}\theta$). Proposition 1.2 states that passage from the lower-dimensional sections ($T_{ki}^{\alpha}\theta$) to higher-dimensional sections ($T_{kj}^{\alpha}\theta$, $j > i$) can be obtained by simple integration (R_{ij}).

The proof of Theorem 1.1 yields the well-known result below (cf. formula (4.18) with $\alpha = 1$), that we would like to emphasize: it is the value of the rose of intersections $T_{kr}1$ of the stationary isotropic (i.e., with the uniform distribution of directions) Poisson k-flat process (cf. [20]) with arbitrary r-flats. It coincides with the rose of intersections of the stationary isotropic Poisson r–flat process with k–planes:

Corollary 5.2. For any k and r, such that $k+r \geq d$

$$T_{kr}1 = T_{rk}1 = \frac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{r+1}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{k+r-d+1}{2}\right)} = \frac{\omega_{d+1}\omega_{k+r-d+1}}{\omega_{k+1}\omega_{r+1}}.$$

§6. DOUBLE FIBRATION FOR $\{T_{ij}^{\alpha}\}$ AND $\{\tilde{T}_{ij}^{\alpha}\}$

The following double fibration relation (cf. [11], p. 168) for R_{ij} holds for all absolutely integrable $f \in L^1(G(i,d))$, $\varphi \in L^1(G(j,d))$:

$$\int\limits_{G(j,d)} (R_{ij}f)(\xi)\varphi(\xi)\,d\xi = \int\limits_{G(i,d)} f(\eta)(R_{ji}\varphi)(\eta)\,d\eta. \tag{6.1}$$

Here R_{ji} is dual to the transform R_{ij}. Now we investigate this relation for functions $f(\eta) = [\eta, \zeta_0]^{\alpha}$, $\eta \in G(i,d)$, $i < j$.

Proof of Proposition 1.3. By Theorem 1.1, one gets from (6.1), that

$$c(\alpha) \int\limits_{G(j,d)} [\xi, \zeta_0]^{\alpha}\varphi(\xi)\,d\xi = \int\limits_{G(i,d)} [\eta, \zeta_0]^{\alpha}(R_{ji}\varphi)(\eta)\,d\eta, \tag{6.2}$$

and rotation invariance completes the proof.

Proposition 6.1. The adjoint operator of T_{ij}^{α} on $L^1(G(i,d))$ is the operator T_{ji}^{α} on $L^1(G(j,d))$:

$$\left(T_{ij}^{\alpha}\right)^* = T_{ji}^{\alpha}.$$

Proof: The desired relation

$$\int\limits_{G(i,d)} f(\eta)\left(T_{ji}^{\alpha}\varphi\right)(\eta)\,d\eta = \int\limits_{G(j,d)} \left(T_{ij}^{\alpha}f\right)(\xi)\varphi(\xi)\,d\xi$$

for $f \in L^1(G(i,d))$ and $\varphi \in L^1(G(j,d))$ follows easily from the more general relation

$$\int\limits_{G(i,d)} \left(T_{ji}^{\alpha}\theta\right)(\eta)\,\mu(d\eta) = \int\limits_{G(j,d)} \left(T_{ij}^{\alpha}\mu\right)(\xi)\,\theta(d\xi)$$

for any $\theta \in \mathbf{M}(G(j,d))$, $\mu \in \mathbf{M}(G(i,d))$, which can be seen directly by Fubini's theorem. Proposition 6.1 is proved.

Now we state the result for the operator family $\{\tilde{T}_{ij}^{\alpha}\}$ similar to Proposition 1.3.

Proposition 6.2. [Third Cauchy–Kubota – type formula] For all $i < j \le k < d$, $\alpha > 0$, and all absolute integrable functions $g \in L^1(G(i,d))$,

$$\left(\tilde{T}^{\alpha}_{ik} g\right)(\zeta) = \left(c^*(\alpha)\right)^{-1} \tilde{T}^{\alpha}_{jk} \left(R_{ij} g\right)(\zeta), \quad \zeta \in G(k,d), \tag{6.3}$$

where $c^*(\alpha)$ is given by (1.4).

Proof: First, it is worth mentioning that by rotation invariance, Proposition 1.1 remains true for any $\zeta \in G(k,d)$ instead of ζ_0. One can write by (1.3) and (6.1), that

$$c^*(\alpha) \int\limits_{G(i,d)} [\eta^{\perp}, \zeta]^{\alpha} g(\eta)\, d\eta = \int\limits_{G(j,d)} [\xi^{\perp}, \zeta]^{\alpha} (R_{ij} g)(\xi)\, d\xi, \tag{6.4}$$

which together with the definition of $\tilde{T}^{\alpha}_{ij}$ completes the proof.

For $\alpha > 0$ we consider the following functionals on the space of absolute integrable functions g on $G(i,d)$ and φ on $G(j,d)$:

$$\|g\|_{(\alpha)} \stackrel{def}{=} \left| \int\limits_{G(i,d)} g(\eta) [\eta, \zeta_0]^{\alpha}\, d\eta \right|,$$

$$\|\varphi\|^{\perp}_{(\alpha)} \stackrel{def}{=} \left| \int\limits_{G(j,d)} \varphi(\xi) [\xi^{\perp}, \zeta_0]^{\alpha}\, d\xi \right|.$$

Let $\|\cdot\|_p$ denote the usual norm in L^p-spaces. The rest of the section is devoted to inequalities between the weighted images of Radon transforms and their duals as above.

Proposition 6.3. Choose the numbers $p, q > 1$, such that $1/p + 1/q = 1$.

1) Let $i < j$, $i + k \ge d$, $\alpha > 0$, and $\varphi \in L^p(G(j,d))$. Then

$$\|R_{ji}\varphi\|_{(\alpha)} \le d(\alpha, q) \|\varphi\|_p, \tag{6.5}$$

where

$$d(\alpha, q) = c(\alpha) \left(\prod_{l=0}^{d-k-1} \frac{\Gamma\left(\frac{d-l}{2}\right) \Gamma\left(\frac{j-l+\alpha+q}{2}\right)}{\Gamma\left(\frac{j-l}{2}\right) \Gamma\left(\frac{d-l+\alpha+q}{2}\right)} \right)^{1/q}, \tag{6.6}$$

and $c(\alpha)$ is defined by (1.2).

2) Let $i < j \le k < d$, $\alpha > 0$, and $g \in L^p(G(i,d))$. Then

$$\|R_{ij} g\|^{\perp}_{(\alpha)} \le d^*(\alpha, q) \|g\|_p, \tag{6.7}$$

where

$$d^*(\alpha,q) = c^*(\alpha)\left(\prod_{l=0}^{d-k-1}\frac{\Gamma\left(\frac{d-l}{2}\right)\Gamma\left(\frac{d-i-l+\alpha+q}{2}\right)}{\Gamma\left(\frac{d-i-l}{2}\right)\Gamma\left(\frac{d-l+\alpha+q}{2}\right)}\right)^{1/q}, \tag{6.8}$$

and $c^*(\alpha)$ is defined by (1.4).

Proof: First, let us prove the upper bound for the image of the dual Radon transform. By (6.2),

$$\|R_{ji}\varphi\|_{(\alpha)} \le c(\alpha)\cdot |||\varphi|||_{(\alpha)}.$$

Applying Hölder's inequality one gets

$$|||\varphi|||_{(\alpha)} \le \left(\int\limits_{G(j,d)} |\varphi(\xi)|^p\, d\xi\right)^{1/p}\left(\int\limits_{G(j,d)} [\xi,\zeta_0]^{\alpha+q}\, d\xi\right)^{1/q} = \|\varphi\|_p b^{1/q}_{\alpha+q}(j,d,k),$$

while by (4.18)

$$b^{1/q}_{\alpha+q}(j,d,k) = \left(\prod_{l=0}^{d-k-1}\frac{\Gamma\left(\frac{d-l}{2}\right)\Gamma\left(\frac{j-l+\alpha+q}{2}\right)}{\Gamma\left(\frac{j-l}{2}\right)\Gamma\left(\frac{d-l+\alpha+q}{2}\right)}\right)^{1/q}.$$

The proof of the second statement of Proposition is conducted analogously: by (6.4),

$$\|R_{ij}g\|^{\perp}_{(\alpha)} \le c^*(\alpha)\cdot |||g|||^{\perp}_{(\alpha)}.$$

By Hölder's inequality, we get

$$|||g|||^{\perp}_{(\alpha)} \le \left(\int\limits_{G(i,d)} |g(\eta)|^p\, d\eta\right)^{1/p}\left(\int\limits_{G(i,d)} [\eta^{\perp},\zeta_0]^{\alpha+q}\, d\eta\right)^{1/q} =$$

$$= \|g\|_p\left(\int\limits_{G(d-i,d)} [\nu,\zeta_0]^{\alpha+q}\, d\nu\right)^{1/q} = \|g\|_p b^{1/q}_{\alpha+q}(d-i,d,k),$$

where by (4.18)

$$b^{1/q}_{\alpha+q}(d-i,d,k) = \left(\prod_{l=0}^{d-k-1}\frac{\Gamma\left(\frac{d-l}{2}\right)\Gamma\left(\frac{d-i-l+\alpha+q}{2}\right)}{\Gamma\left(\frac{d-i-l}{2}\right)\Gamma\left(\frac{d-l+\alpha+q}{2}\right)}\right)^{1/q}.$$

Proposition 6.3 is proved.

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