Wiener sausage and sensor networks

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Overview

- Sensor networks
- Wiener sausage
  - Mean volume and surface area
  - Almost sure approximation and convergence of curvature measures
- Boolean model of Wiener sausages
  - Capacity functional and volume fraction
  - Contact distribution function and covariance function
  - Specific surface area
- Outlook

Motivation: sensor networks

- Finding an object in a medium using a totally random strategy (G. Kesidis et al. (2003), S. Shakkottai (2004)): a set of sensors with reach $r > 0$ move randomly in the medium. Target detection area of a sensor = Wiener sausage S_r

Paths of two sensors with reach $r = 20$ and their target detection areas

Motivation: sensor networks

- Sensor network: Sensors are located "at random" in space moving according to the Brownian motion.
- Target detection area of such network: the Boolean model of Wiener sausages
- Target detection probability: $p = 1 - e^{-\lambda \mathbb{E}\,|S_r|}$, where λ is the intensity of the Poisson process of initial positions of sensors and $\mathbb{E}\,|S_r|$ is the mean volume of the Wiener sausage
- Other specific intrinsic volumes? $\Longrightarrow$ find other mean intrinsic volumes of S_r
- E.g., the mean surface area $\mathbb{E}\,\mathcal{H}^{d-1}(\partial S_r)$

Polyconvex sets and sets of positive reach

- Polyconvex sets: $K = \cup_{i=1}^n K_i$, where K_i are convex bodies.
- Sets of positive reach: For a closed subset $A \subseteq \mathbb{R}^d$, define

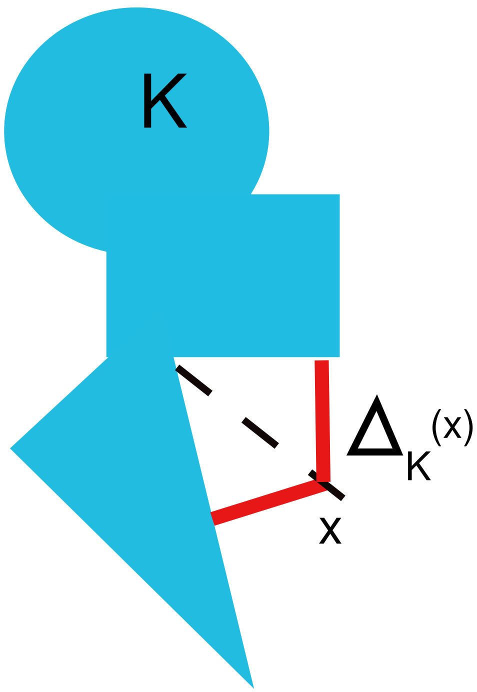
$$\mathrm{reach}\, A = \sup\{r \geq 0 : \forall x \in \mathbb{R}^d,\ \Delta_A(x) < r \implies \mathrm{card}\, \Sigma_A(x) = 1\},$$

where $\Delta_A(x)$ is the distance from $x \in \mathbb{R}^d$ to A and

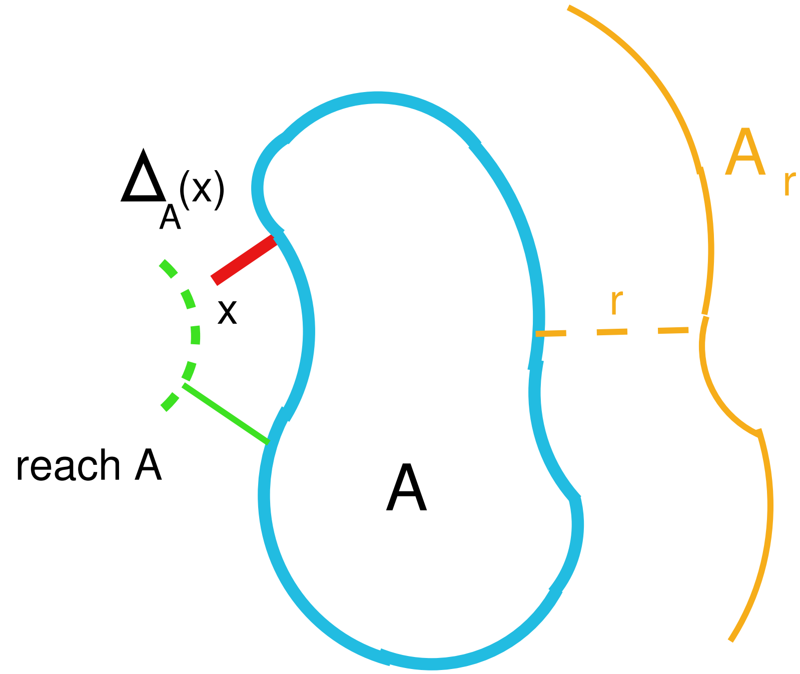
$$\Sigma_A(x) = \{a \in A : |x - a| = \Delta_A(x)\}$$

is the set of all metric projections of x on A. A is said to be of positive reach if $\mathrm{reach}\, A > 0$.

Polyconvex sets and sets of positive reach



A polyconvex set



A set of positive reach

The parallel set of A, $r > 0$:

$$A_r = A \oplus B_r(o) = \{x \in \mathbb{R}^d : \Delta_A(x) \leqslant r\}$$

Intrinsic volumes

- Steiner's formula (H. Federer (1959)): If A is a set of positive reach then

$$|A_r| = \sum_{i=0}^{d} \omega_i r^i V_{d-i}(A), \quad 0 \le r < \operatorname{reach} A,$$

where

- $V_d(\cdot) = |\cdot|$ is the Lebesgue measure in $\mathbb{R}^d$ (volume),
- ω_i is the volume of the unit i-ball
- V_i is the i-th intrinsic volume of A, $i = 0, \ldots, d$. In particular, V_0 is the Euler-Poincaré characteristic and V_{d-1} is one half of the surface area.

Curvature measures

- Local version of Steiner's formula: For any Borel subset $F \subset \mathbb{R}^d$

$$|(A_r \setminus A) \cap \xi_A^{-1}(F)| = \sum_{i=1}^{d} \omega_i r^i C_{d-i}(A; F), \quad 0 \le r < \operatorname{reach} A,$$

  where

  - $\xi_A(x)$ is the nearest point of A to x
  - the signed Radon measure $C_i(A; \cdot)$ concentrated on ∂A is the ith curvature measure of A, $0 \le i \le d-1$
  - If ∂A is compact then $C_i(A; \mathbb{R}^d) = V_i(A)$, $i = 0, \dots, d-1$.

Brownian motion

- Wiener process $\{W(t) : t \geq 0\}$: a random process with continuous paths defined on $(\Omega, \mathfrak{F}, P)$ such that
  - $W(0) = x \in \mathbb{R}$ a.s.,
  - W has independent increments,
  - $W(t) - W(s) \sim N(0, \sigma^2(t-s))$, $0 \leq s < t$.
- Brownian motion in $\mathbb{R}^d$ initiated at $x = (x_1, \ldots, x_d)$:

$$X(t) = (W_1(t), \ldots, W_d(t)), \quad t \geq 0,$$

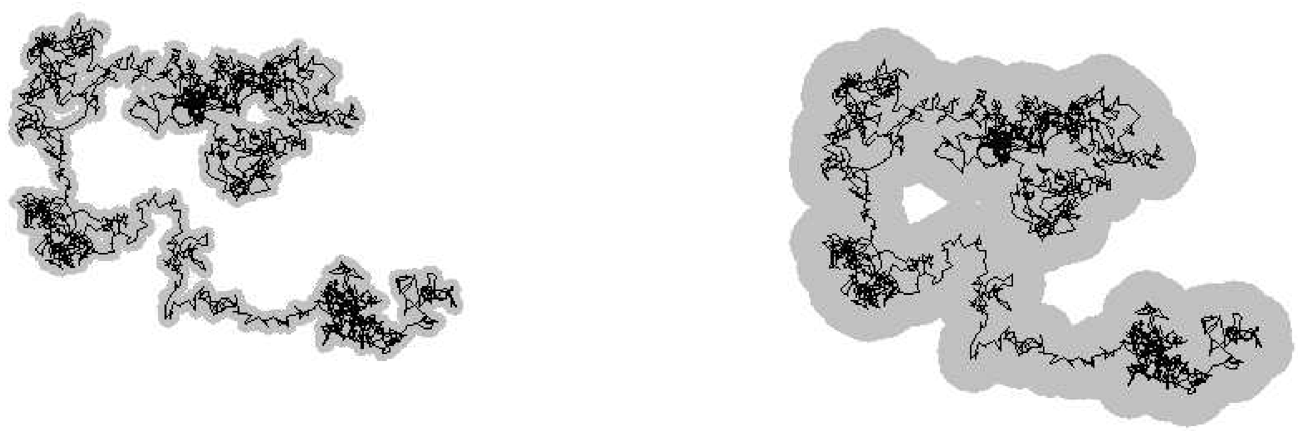
where $W_1, \ldots, W_d$ are independent Wiener processes starting at $x_1, \ldots, x_d \in \mathbb{R}$.

Wiener sausage

Let $S(T) = \{X(t) : t \in [0,T]\}$ be the path of X up to time $T > 0$.

- Wiener sausage S_r of radius $r > 0$:

$$S_r = S(T) \oplus B_r(o) = \left\{x \in \mathbb{R}^d : \Delta_{S(T)}(x) \leqslant r\right\}.$$



$r = 10$ $r = 40$

A realization of S_r

Intrinsic volumes of the Wiener sausage

- Intrinsic volumes $V_0(S_r), \ldots, V_d(S_r)$ are well–defined a.s. for $d \leqslant 3$, $r > 0$;
  - $V_d(S_r) = V_S(r)$
  - $2V_{d-1}(S_r) = \mathcal{H}^{d-1}(\partial S_r)$
  - $V_i(S_r) = (-1)^{d-i-1} V_i\left(\overline{\mathbb{R}^d \setminus S_r}\right)$, $i = 0, \ldots, d-2$, where ∂S_r is a Lipschitz manifold with $\mathrm{reach}\left(\overline{\mathbb{R}^d \setminus S_r}\right) > 0$ a.s.
- Compute $\mathbb{E}\, V_i(S_r)$, $i = 0, \ldots, d$.
  It is proved that $\mathbb{E}\, V_i(S_r) < \infty$, $i = d, d-1$ for all $d \geqslant 2$ and $\mathbb{E}\, V_0(S_r) < \infty$ for $d = 2$ (RSS 09, RSM 09).

Mean volume of the Wiener sausage

- Explicit formulae
  - $d = 2$: A. Kolmogoroff and M. Leontowitsch (1933)
  - $d = 3$: F. Spitzer (1964)
  - $d \geqslant 4$: A. Berezhkovskii *et al.* (1989)
- Asymptotics of the volume
  - R. Getoor (1965)
  - M. Donsker und S. Varadhan (1975)
  - J.-F. Le Gall (1988): CLT for shrinking Wiener sausage ($T \to \infty$ or $r \to 0$)
  - M. van den Berg and E. Bolthausen (1994)
  - $\ldots$

Other mean curvature measures

- Mean surface area: RSS (2009)
- Support measures and mean curvature functions: G. Last (2006)
- Other mean intrinsic volumes $\mathbb{E}\, V_i(S_r)$, $i = 0, \ldots, d-2$: an open problem. Approximations can be obtained numerically (RSM (2009))

Mean volume of the Wiener sausage

A. Berezhkovskii, Yu. Makhnovskii, R. Suris (1989) : for $d \geqslant 2$

$$\mathbb{E}\,|S_r| = \omega_d r^d + \frac{d(d-2)}{2}\omega_d\,\sigma^2 r^{d-2} T + \frac{4d\,\omega_d\,r^d}{\pi^2}\int\limits_0^\infty \frac{1-e^{-\frac{\sigma^2 y^2 T}{2r^2}}}{y^3\left(J_\nu^2(y)+Y_\nu^2(y)\right)}\,dy\,,$$

where $J_\nu(y)$ and $Y_\nu(y)$ are Bessel functions of the first and second kind of order $\nu = (d-2)/2$ and $\omega_d = \pi^{d/2}/\Gamma(1+d/2)$ is the volume of the unit d-ball.

Mean volume of the Wiener sausage

Bessel functions $J_\nu(y)$ and $Y_\nu(y)$ are linearly independent solutions of the Bessel diff. equation $y^2 f''(y) + y f'(y) + (y^2 - \nu^2) f(y) = 0$, and

$$J_\nu(y) = \sum_{k=0}^{\infty} \frac{(-1)^k (y/2)^{2k+\nu}}{k!\Gamma(\nu + k + 1)},$$

$$Y_\nu(y) = \frac{J_\nu(y)\cos(\nu\pi) - J_{-\nu}(y)}{\sin(\nu\pi)}.$$

In three dimensions, the above formula for the mean volume of S_r simplifies to

$$\mathbb{E}\,|S_r| = \frac{4}{3}\pi r^3 + 4\sigma r^2 \sqrt{2\pi T} + 2\pi\sigma^2 r T\,.$$

Mean surface area of the Wiener sausage

Theorem 1. *Let S_r be the Wiener sausage in $\mathbb{R}^d$, $d \geqslant 2$. Then, it holds*

$$\begin{aligned}\mathbb{E}\mathcal{H}^{d-1}(\partial S_r) &= d\omega_d r^{d-1} + \frac{4d^2\,\omega_d\,r^{d-1}}{\pi^2}\int_0^\infty \frac{1-e^{-\frac{\sigma^2 y^2 T}{2r^2}}}{y^3(J_\nu^2(y)+Y_\nu^2(y))}\,dy \\ &+ d\omega_d\sigma^2 r^{d-3}T\left(\frac{(d-2)^2}{2} - \frac{4}{\pi^2}\int_0^\infty \frac{e^{-\frac{\sigma^2 y^2 T}{2r^2}}}{y(J_\nu^2(y)+Y_\nu^2(y))}\,dy\right)\end{aligned}$$

for almost all radii $r > 0$. For $d = 2, 3$, this formula holds for all $r > 0$. In the case $d = 3$, it simplifies to

$$\mathbb{E}\mathcal{H}^2(\partial S_r) = 4\pi r^2 + 8r\sigma\sqrt{2\pi T} + 2\pi\sigma^2 T\,.$$

Mean surface area of the Wiener sausage

Asymptotic behaviour (RSS 09)

$$\mathbb{E}\mathcal{H}^{d-1}(\partial S_r) \sim \begin{cases} \pi\sigma^2 T r^{-1} \log^{-2} r & \text{if } d = 2, \\ 2\pi\sigma^2 T & \text{if } d = 3, \\ d\omega_d \sigma^2 T \frac{(d-2)^2}{2} r^{d-3} & \text{if } d \geqslant 4 \end{cases} \quad \text{as } r \to 0.$$

Idea of the proof of Theorem 1

Theorem 2. *For $d \leq 3$ and any $r > 0$, it holds*

$$E\,\mathcal{H}^{d-1}(\partial S_r) = \frac{\mathrm{d}\,E\,V_S(r)}{\mathrm{d}\,r},$$

where $V_S(r) = |S_r|$. For dimensions $d \geqslant 4$, this relation holds for almost all $r > 0$.

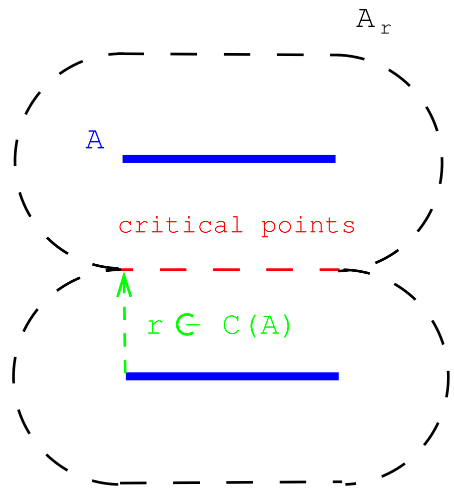
follows from the dominated convergence theorem and

Lemma 1. *It holds*

$$\mathcal{H}^{d-1}(\partial S_r) \overset{a.s.}{=} V_S'(r)$$

for all $r > 0$ if $d = 2, 3$ and almost all $r > 0$ if $d \geqslant 4$.

Idea of the proof of Theorem 1



The point $x \in \mathbb{R}^d$ is called critical for A if

$$x \in \text{conv}\{a \in A : |x - a| = d(A, x)\}.$$

The radius $r > 0$ is critical for A if $\exists$ a critical point x for A with $d(A, x) = r$. Let $C(A)$ be the set of critical dilation radii r for A.

Idea of the proof of Theorem 1

Lemma 1 follows from

Theorem 3. *Let $A \subset \mathbb{R}^d$ be any compact set. If $r \in (0, \infty) \setminus C(A)$ then $V_A'(r)$ exists and equals $\mathcal{H}^{d-1}(\partial A_r)$. Here $A_r = \{x \in \mathbb{R}^d : d(A, x) \leqslant r\}$ is the parallel neighborhood of A of radius $r > 0$ and $C(A)$ is the set of critical dilation radii for A.*

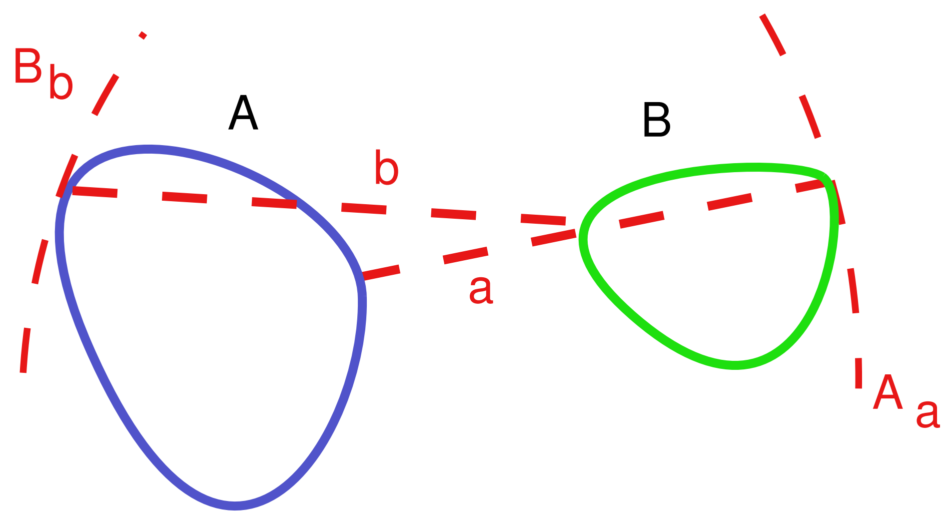
and

Theorem 4. *Let $d \leq 3$. Then for any $r > 0$, $r \notin C\big(S(T)\big)$ a. s. For $d \geqslant 4$, $r \notin C\big(S(T)\big)$ a.s. for almost all radii $r > 0$. Here $S(T)$ is the path of the Brownian motion up to time instant $T > 0$.*

Other mean intrinsic volumes

- Compute $\mathbb{E}\, V_i(S_r)$, $i = 0, \ldots, d-2$
- Idea: Approximate the Wiener sausage S_r a.s. by random polyconvex sets S_r^n with $\mathbb{E}\, V_i(S_r^n) \to \mathbb{E}\, V_i(S_r)$ as $n \to \infty$, $i = 0, \ldots, d$
- Method: Use any a.s. piecewise linear approximation S^n of the Brownian path S w.r.t. the Hausdorff metric to approximate the Wiener sausage $S_r = S \oplus B_r(o)$ by $S_r^n = S^n \oplus B_r(o)$, e.g. the Haar-Schauder approximation
- Goal: Obtain $\mathbb{E}\, V_i(S_r)$ by computing $\mathbb{E}\, V_i(S_r^n)$ and using $\mathbb{E}\, V_i(S_r^n) \to \mathbb{E}\, V_i(S_r)$, $n \to \infty$, $i = 0, \ldots, d-2$

Approximation of the Wiener sausage



Hausdorff metric: For two nonempty compacts $A, B \subset \mathbb{R}^d$

$$d_H(A,B) = \min\{a > 0 : A \subseteq B \oplus B_a(o), B \subseteq A \oplus B_a(o)\}.$$

Convergence of curvature measures

Theorem 5.

If $d_H(S^n, S) \to 0$ as $n \to \infty$ almost surely, then it holds

(i) $V_{S^n}(r) \to V_S(r)$ *as* $n \to \infty$ *almost surely for any* $r > 0$*;*

(ii) $\mathcal{H}^{d-1}(\partial S_r^n) \to \mathcal{H}^{d-1}(\partial S_r)$ *as* $n \to \infty$ *almost surely for any* $r > 0$ *if* $d \leqslant 3$ *and for almost all* $r > 0$ *if* $d \geqslant 4$*;*

(iii) $C_i(S_r^n; \cdot) \to C_i(S_r;)$ *weakly as* $n \to \infty$ *almost surely for any* $r > 0$ *and* $i = 0, \ldots, d-1$ *if* $d \leqslant 3$.

Convergence of curvature measures

Corollary 1. *It holds*

(i) $\mathbb{E}\, V_{S^n}(r) \longrightarrow \mathbb{E}\, V_S(r)$ *as* $n \to \infty$ *for any* $r > 0$*;*

(ii) $\mathbb{E}\, \mathcal{H}^{d-1}\left(\partial S_r^n\right) \longrightarrow \mathbb{E}\, \mathcal{H}^{d-1}\left(\partial S_r\right)$ *as* $n \to \infty$ *for any* $r > 0$ *if* $d \leqslant 3$ *and for almost all* $r > 0$ *if* $d \geqslant 4$.

Conjecture: $\mathbb{E}\, V_i(S_r^n) \longrightarrow \mathbb{E}\, V_i(S_r)$ as $n \to \infty$ for $i = 0, \dots, d-2$ if $d \leqslant 3$.

Open problem: Find a uniform integrable upper bound for $V_i(S_r^n)$, $V_i(S_r)$, $n \in \mathbb{N}$, $i = 0, \dots, d-2$

Approximation of the Wiener sausage

Example

- For simplicity, consider $T = 1$.
- Set time instants $t_i = i/k_n$, $i = 1, \ldots, k_n$ for $k_n \to \infty$, $n \to \infty$.
- Vertices: $X^n(t_i) = X(t_i)$, $i = 1, \ldots, k_n$
- The path $S^n = \{X^n(t) : 0 \leq t \leq 1\}$ is a piecewise linear curve with $2^n + 1$ nodes lying on $S = \{X(t) : 0 \leq t \leq 1\}$.
- Set $S_r^n = S^n \oplus B_r(o)$, $n \in \mathbb{N}$.

Theorem 6. *It holds* $d_H(S^n, S) \leqslant \max_{t\in[0,1]} |X^n(t) - X(t)| \to 0$ *as* $n \to \infty$ *a.s.*

Approximation of the Wiener sausage

Special case: Haar-Schauder approximation

For the standard Brownian motion W ($\sigma^2 = T = 1$) holds

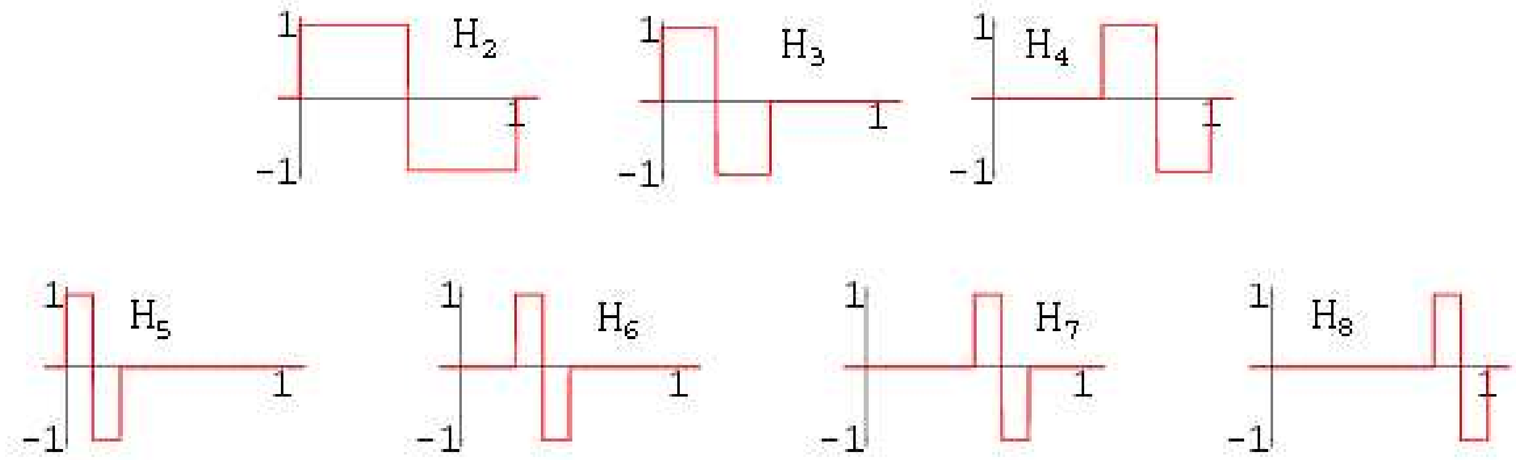
$$W(t) = \sum_{k=1}^{\infty} Y_k L_k(t), \qquad t \in [0,1]\,,$$

where

- $\{Y_n\}_{n\in\mathbb{N}}$ is the sequence of i.i.d. N$(0,1)$–distributed random variables;
- $L_k(t) = \int_0^t H_k(s)\,\mathrm{d}s$ are the Schauder functions and H_k are the Haar functions, $k \in \mathbb{N}$.

This series converges (a.s.) absolutely and uniformly on $[0,1]$.

Approximation of the Wiener sausage



- Haar functions: $H_1(s) = 1,\ s \in [0,1]$,

$$H_{2^m+k}(s) = \begin{cases} 2^{\frac{m}{2}}, & s \in \left[\frac{k-1}{2^m}, \frac{2k-1}{2^{m+1}}\right), \\ -2^{\frac{m}{2}}, & s \in \left[\frac{2k-1}{2^{m+1}}, \frac{k}{2^m}\right), \\ 0, & \text{otherwise} \end{cases} \qquad k = 1, 2, \ldots, 2^m, \quad m = 0, 1, 2, \ldots$$

Approximation of the Wiener sausage

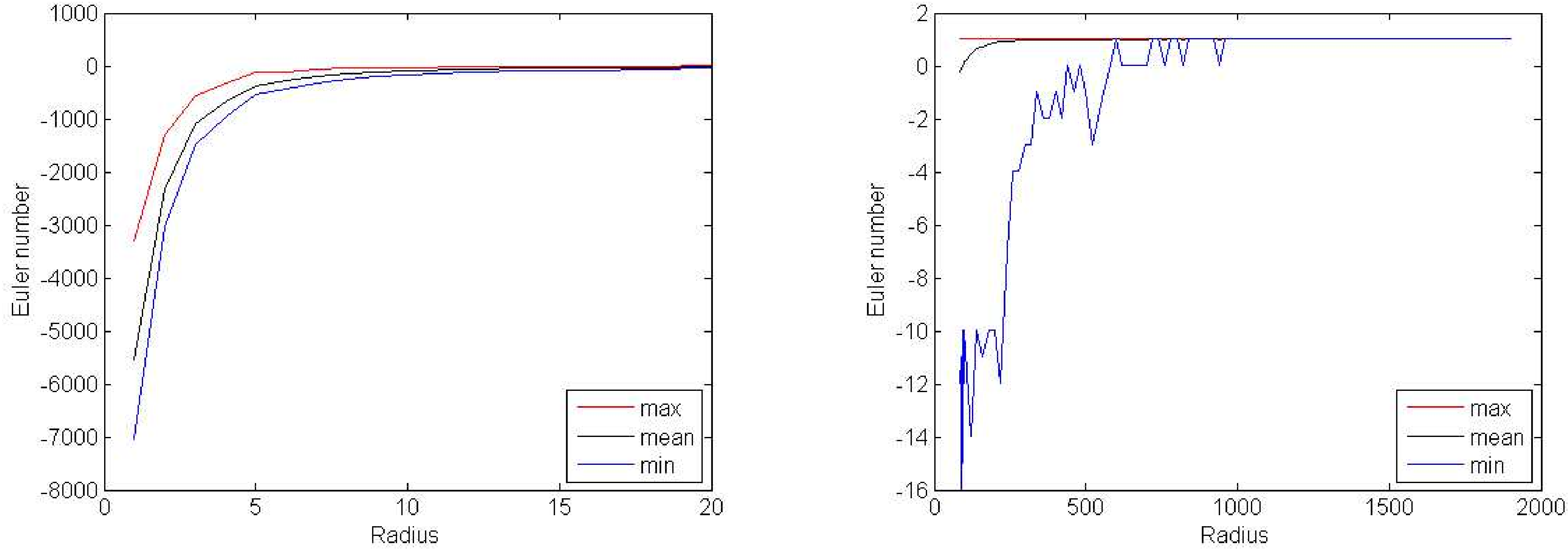
- Approximate the Brownian motion $X(t)$ by $X^n(t) = (W_1^n(t), \ldots, W_d^n(t)), \quad t \in [0,1]$, where

$$W_i^n(t) = \sum_{k=1}^{2^n} Y_{ik} L_k(t), \quad t \in [0,1],$$

and Y_{ik} are i.i.d. N$(0,1)$ random variables.

- The path $S^n = \{X^n(t) : 0 \le t \le 1\}$ is a piecewise linear curve with $2^n + 1$ nodes lying on $S = \{X(t) : 0 \le t \le 1\}$.

Approximation formula for $\mathbb{E}\, V_0(S_r)$ in 2D

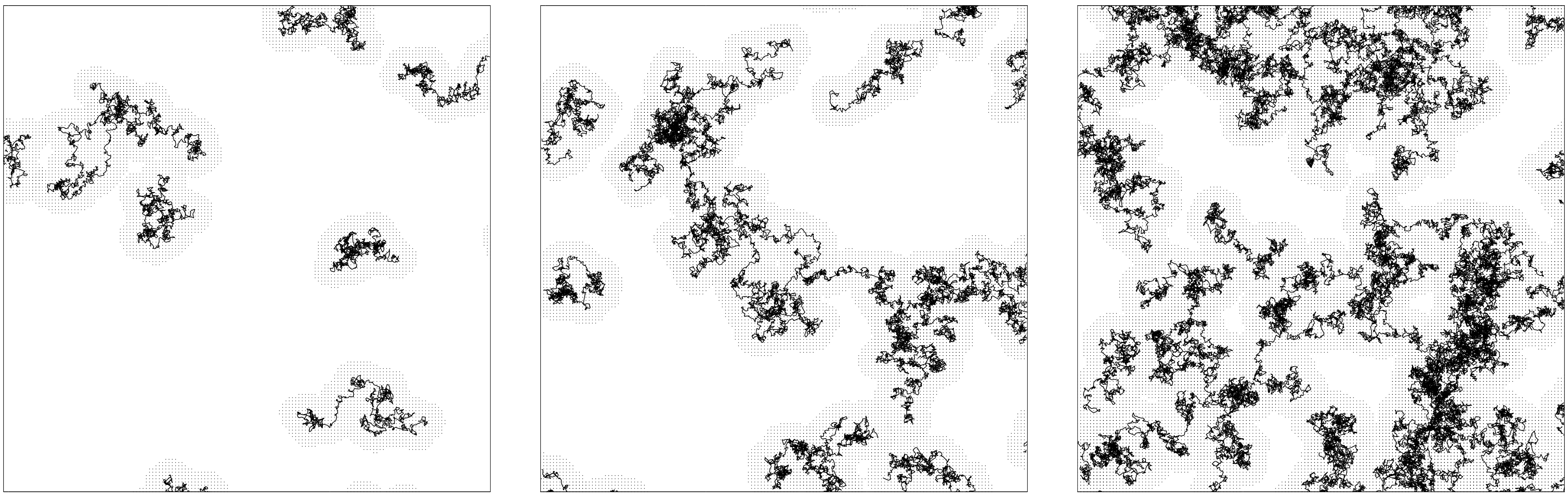


$$\mathbb{E}V_0\left(S_r\right) \approx 1 + \frac{0.0423017\left(1 - \Phi\left(\frac{r-224.899}{50.2096}\right)\right)}{\log\left(3.88182 \cdot 10^{-6}\, r^{1.88978} + 1.0\right) r^{0.153452}}\,,$$

where Φ is the c.d.f of N(0,1) – distribution.

Boolean model of Wiener sausages

$\Xi = \bigcup_{i \in \mathbb{N}} (x_i + (S_r)_i)$ where $\{x_i\}$ is a stationary Poisson point process of germs with intensity $\lambda > 0$ and grains $(S_r)_i$ are iid copies of S_r.



Three realizations with volume fractions 0.25, 0.5 and 0.75 ($T = 10$, $r = 1$)

Capacity functional

- Definition and formula

$$T_\Xi(C) = P(\Xi \cap C \neq \emptyset) = 1 - e^{-\lambda \mathbb{E}\,|S_r \oplus \check{C}|}$$

for all compact C, where

$$\mathbb{E}\,|S_r \oplus \check{C}| = \int\limits_{\mathbb{R}^d} P(x \in S_r \oplus \check{C})dx = \int\limits_{\mathbb{R}^d} P\left(\tau^x_{C\oplus B_r(o)} \le T\right) dx\,.$$

$\tau^x_A = \inf\{s \geq 0 : \; X^x(s) \in A\}$ is the first hitting time of a Borel set A for the Brownian motion starting at $x \in \mathbb{R}^d$.

- Notation: $u(t,x) = P\left(\tau^x_{C\oplus B_r(o)} \le t\right)$, $x \in \mathbb{R}^d$, $t \geqslant 0$

Capacity functional

Kolmogoroff and Leontowitsch (1933), Doob (1955), Hunt (1956)

$u(t,x)$ is the unique bounded solution to the following heat conduction problem:

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\sigma^2}{2} \triangle u, && t > 0,\ x \in \mathbb{R}^d \setminus (C \oplus B_r(o)),\\
&u(0,x) = 0, && x \in \mathbb{R}^d \setminus (C \oplus B_r(o)),\\
&u(t,x) = 1, && t \geqslant 0,\ \text{for all regular } x \in \partial(C \oplus B_r(o)),
\end{aligned}$$

i.e. points x such that $P\left(\tau^x_{C\oplus B_r(o)} = 0\right) = 1$. In general, this problem has to be solved numerically.

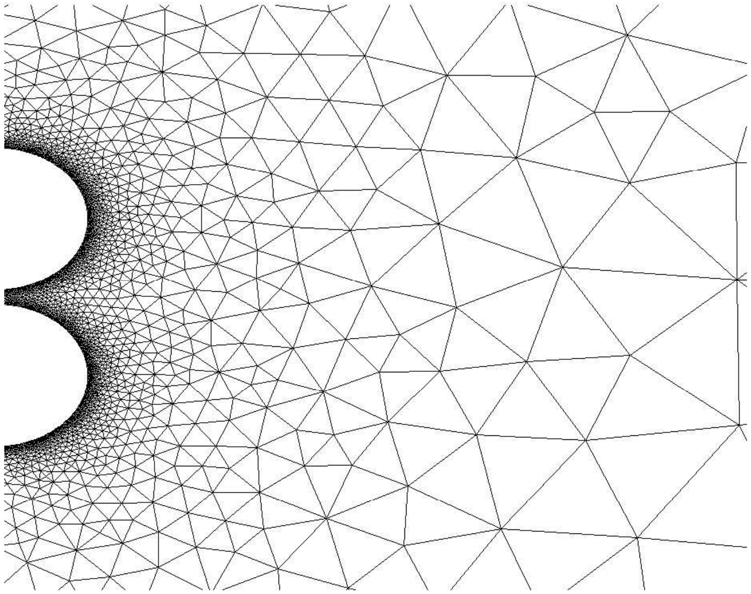
Volume fraction

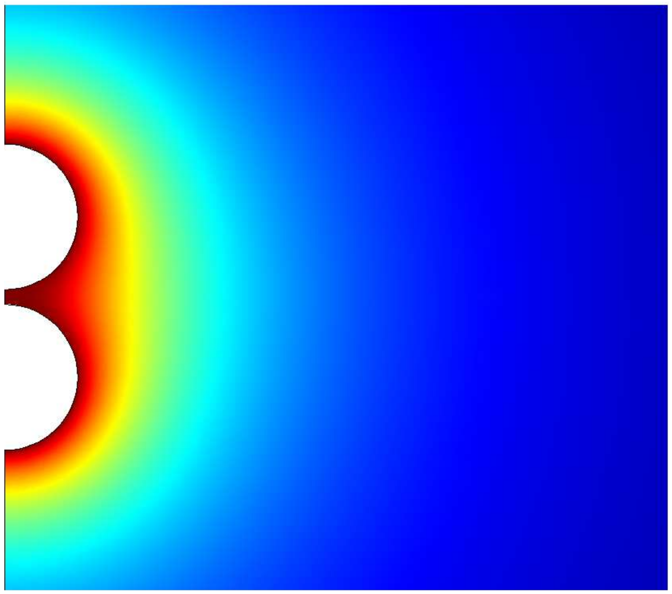
- Explicit solution: $C = \{o\}$.
  For
  $$p_\Xi = P(o \in \Xi) = \mathbb{E}\left|\Xi \cap [0,1]^d\right| = T_\Xi(\{o\})$$
  we get
  $$p_\Xi = 1 - e^{-\lambda\left(\omega_d r^d + \frac{d(d-2)}{2}\omega_d\,\sigma^2 r^{d-2} T + \frac{4d\,\omega_d\,r^d}{\pi^2}\int\limits_0^\infty \frac{1-e^{-\frac{\sigma^2 y^2 T}{2r^2}}}{y^3\left(J_\nu^2(y)+Y_\nu^2(y)\right)}\,dy\right)}.$$

Covariance function

- Numerical solution: $C = \{o, h \cdot u\}$, where $u \in \mathcal{S}^{d-1}$ and $h \geq 0$.

$$C_\Xi(h) = P(o, h \cdot u \in \Xi) = 2p_\Xi - T_\Xi(\{o, h \cdot u\})$$





Finite element method: mesh (left) and computed solution u (right) for $d = 2$, $\sigma^2 = 1$, $r = 1$, $h = 2.2$ and $t = 100$.

Covariance function

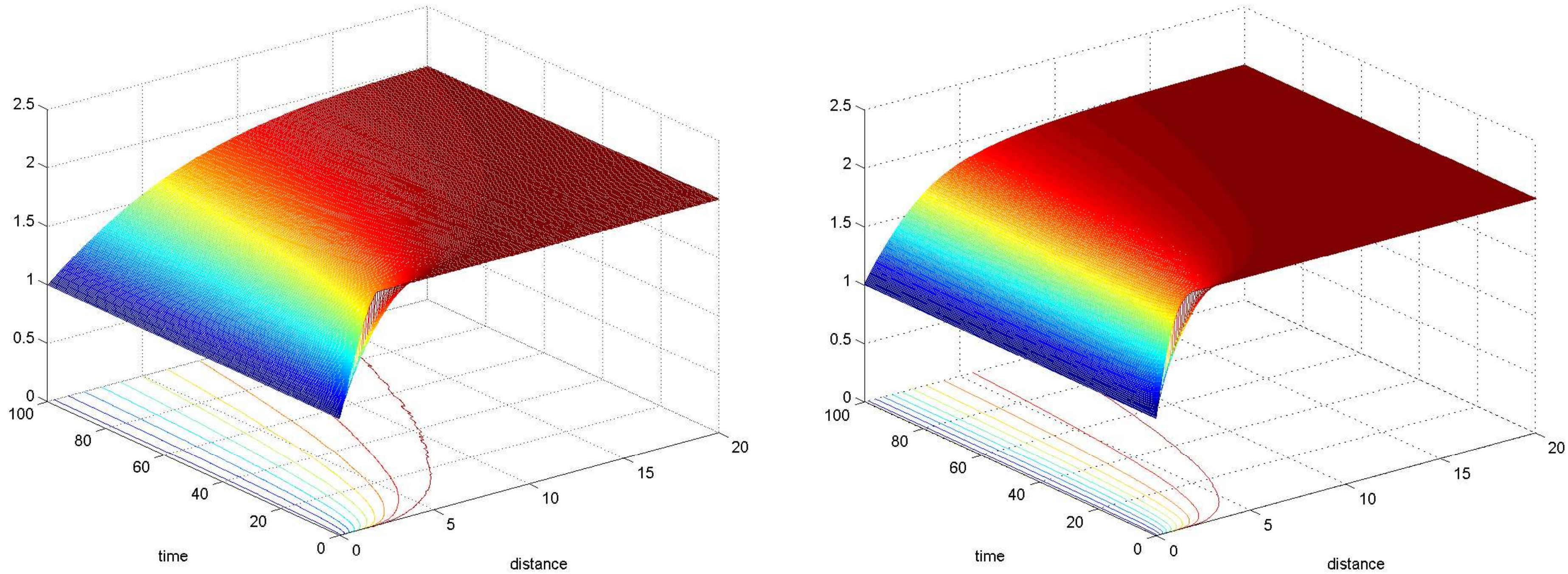
Approximation formula (CFS 08). Let $r = 1$, $\sigma^2 = 1$.

$$C_\Xi(h) \approx 2p_\Xi - 1 + (1 - p_\Xi)^{\kappa(h,t)},$$

$$\kappa(h,t) = \begin{cases} \frac{2}{\pi}\left(\pi - \arccos(\frac{h}{2r}) + \frac{h}{2r}\sqrt{1 - \frac{h}{2r}}\right), & h \le 2r,\ t = 0,\ d = 2, \\ \frac{1}{2}\left(\frac{h}{2r}\right)^3 - 2\left(\frac{h}{2r}\right)^2 + \frac{5}{2}\frac{h}{2r} + 1, & h \le 2r,\ t = 0,\ d = 3, \\ \left(\frac{h}{\nu(t)}\right)^3 - 3\left(\frac{h}{\nu(t)}\right)^2 + 3\frac{h}{\nu(t)} + 1, & h \le \nu(t),\ t > 0,\ d = 2, 3, \\ 2, \text{ otherwise}, \end{cases}$$

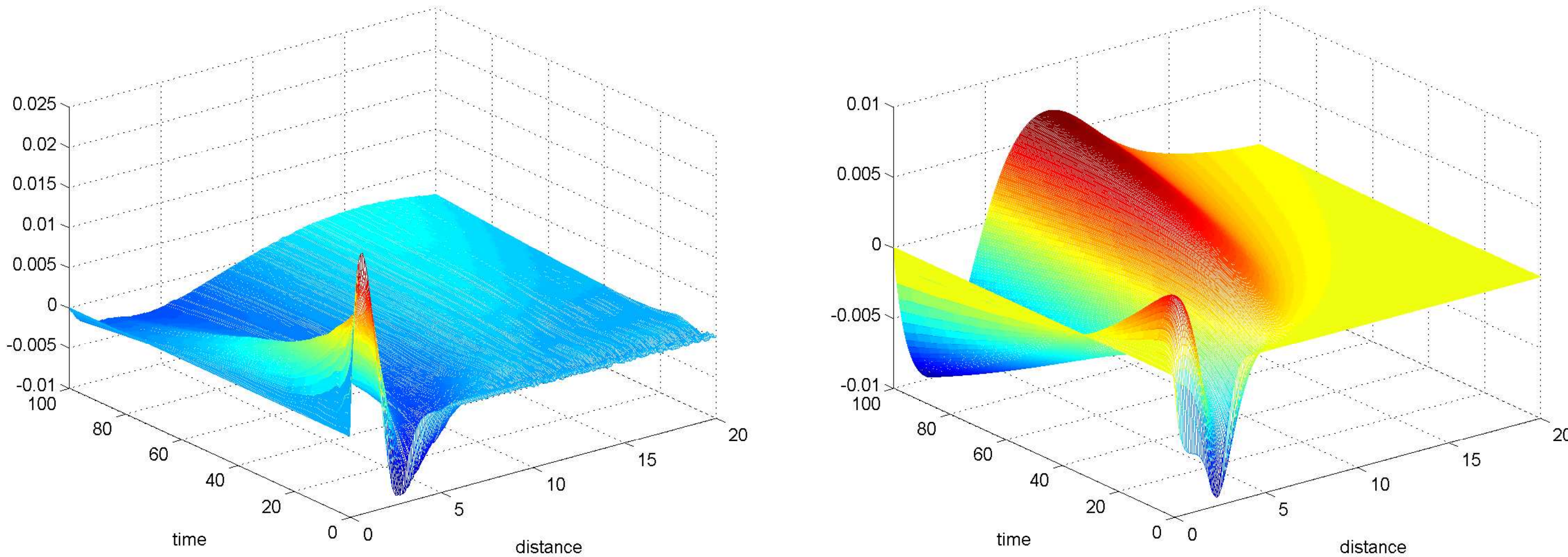
$$\nu(t) = \begin{cases} 3.124\, t^{0.3925} + 2.794,\ d = 2\,, \\ 3.744\, t^{0.2182} + 1.454,\ d = 3\,. \end{cases}$$

Covariance function



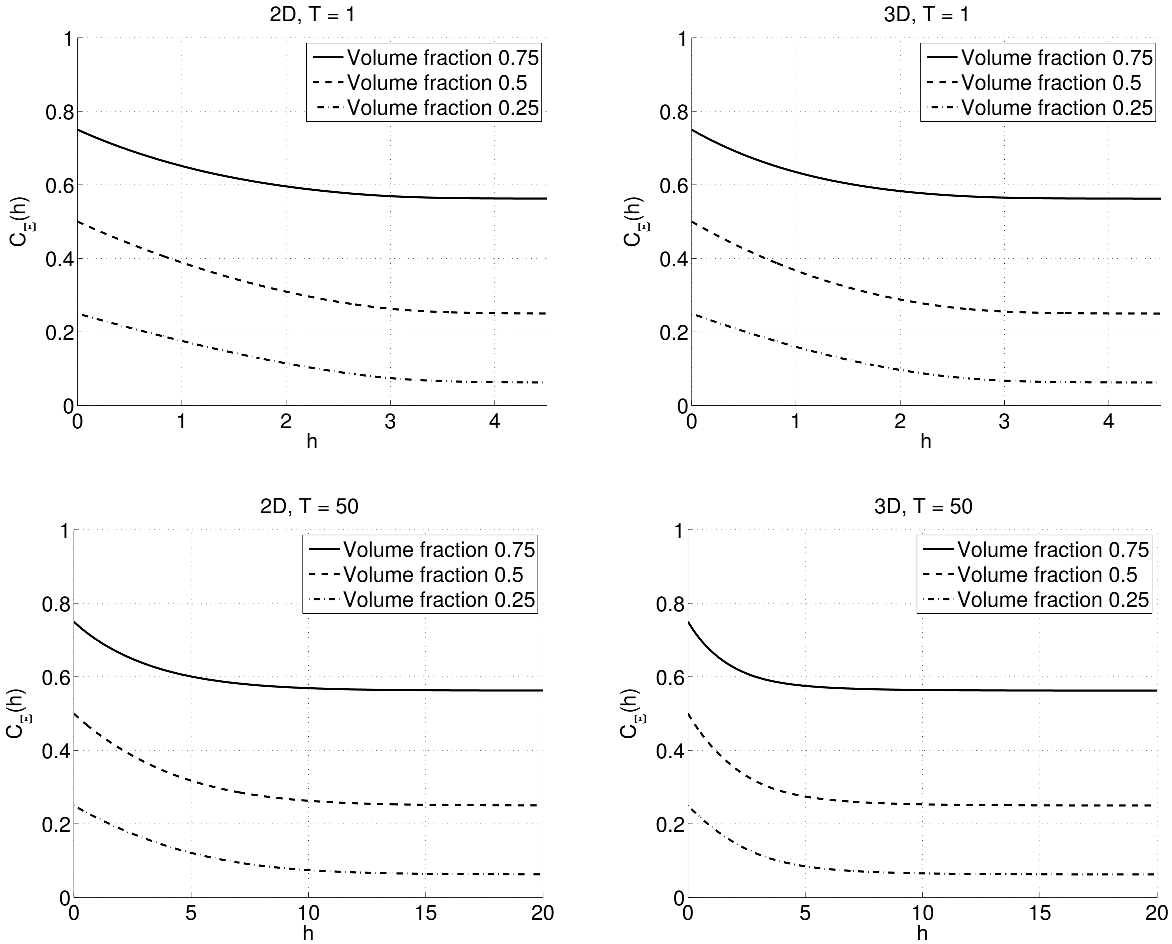
$\kappa(h,t)$ in 2D (left) resp. 3D (right) for $r=1$, $0 \le h \le 20$ and $0 \le t \le 100$

Covariance function



Relative approximation errors in 2D (left) resp. 3D (right) for $p_\Xi = 0.75$, $r = 1$, $0 \leq h \leq 20$ and $0 \leq t \leq 100$.

Covariance function



$C_{\Xi}(h)$ with $r = 1$, $\sigma^2 = 1$ and $T = 1$, $T = 50$.

Spherical contact distribution function

$$H_{B_1(o)}(\rho) = P(\Xi \cap B_\rho(o) \neq \emptyset \mid o \notin \Xi) = \frac{T_\Xi(B_\rho(o)) - T_\Xi(\{o\})}{1 - T_\Xi(\{o\})} \, .$$

It holds

$$H_{B_1(o)}(\rho) = 1 - e^{-\lambda M(d,\sigma^2,r,\rho,T)} \, ,$$

where

$$M(d,\sigma^2,r,\rho,T) = \omega_d \left((r+\rho)^d - r^d\right) + \frac{d(d-2)}{2}\omega_d \, \sigma^2 \left((r+\rho)^{d-2} - r^{d-2}\right) T$$
$$+ \frac{4d\,\omega_d}{\pi^2} \left((r+\rho)^d \int\limits_0^\infty \frac{1 - e^{-\frac{\sigma^2 y^2 T}{2(r+\rho)^2}}}{y^3 \left(J_\nu^2(y) + Y_\nu^2(y)\right)} \, dy - r^d \int\limits_0^\infty \frac{1 - e^{-\frac{\sigma^2 y^2 T}{2r^2}}}{y^3 \left(J_\nu^2(y) + Y_\nu^2(y)\right)} \, dy \right) .$$

Specific surface area

- Specific surface area of Ξ

  Constant $S_\Xi \in (0, \infty)$ defined by

$$S_\Xi(B) = \mathbb{E}\,\mathcal{H}^{d-1}(\partial\Xi \cap B) = S_\Xi \cdot |B|$$

  for all Borel sets $B \subset \mathbb{R}^d$, $d = 2, 3$.

- For $d = 2, 3$ it holds

$$S_\Xi = \lambda \mathbb{E}\,\mathcal{H}^{d-1}(\partial S_r)\, e^{-\lambda\mathbb{E}\,|S_r|}, \qquad r > 0.$$

- Example: $d = 3$

$$S_\Xi = 2\pi\lambda \left(2r^2 + 4r\sigma\sqrt{2T/\pi} + \sigma^2 T\right) e^{-2\pi\lambda r\left(2/3r^2 + 2\sigma r\sqrt{2T/\pi} + \sigma^2 T\right)}.$$

Outlook

- Formula for $E\mathcal{H}^{d-1}(\partial S_r)$ for all $r > 0$ ($d \geqslant 4$)
- Formulae for $\mathbb{E}\, V_j(S_r)$, $j \leqslant d-2$ and specific intrinsic volumes $\overline{V}_j(\Xi)$, $j \leqslant d-2$ of the Boolean model of Wiener sausages
- Limit theorems for the surface area $\mathcal{H}^{d-1}(\partial S_r)$ and other intrinsic volumes $V_j(S_r)$ of the shrinking Wiener sausage
  LT for the volume (J.-F. Le Gall (1988)): for $d = 2$, it holds

$$(\log r)^2(|S_r| + \pi/\log r)) \xrightarrow{w} c - \pi^2\gamma, \quad r \to 0,$$

  where $\sigma^2 = T = 1$ and γ is the (renormalized) Brownian local time of self–intersections. For $d \geqslant 3$: CLT.

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