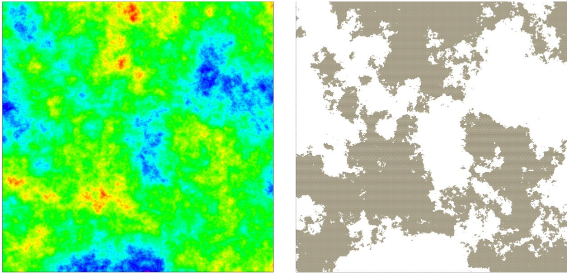

A CLT for excursion sets of dependent random fields

Joint work with
Alexander Bulinski and Florian Timmermann

Evgeny Spodarev | 8.07.2010
IWAP 2010
Madrid

Overview

- Motivation
- The central limit theorem
  - One-dimensional CLT:
    - General case
    - Shot noise case
    - Gaussian case
  - Multivariate CLT
  - Statistical version of the CLT
- A statistical hypothesis test
- Is the paper surface Gaussian?
- Other CLTs for excursion sets of some classes of random fields

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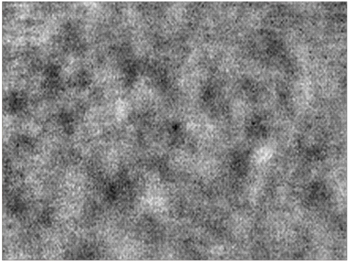
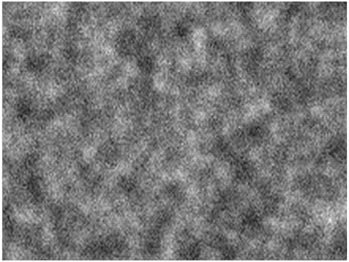


Paper surface

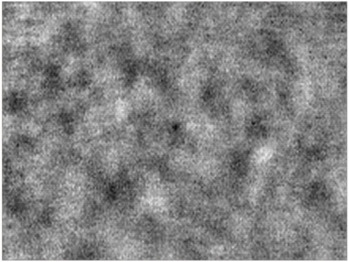


Simulated Gaussian random field

$$\mathrm{E}X(t) = 126$$

$$r(t) = 491 \exp\left(-\frac{\|t\|_2}{56}\right)$$

- Is the paper surface Gaussian?



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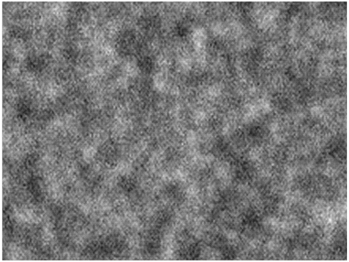

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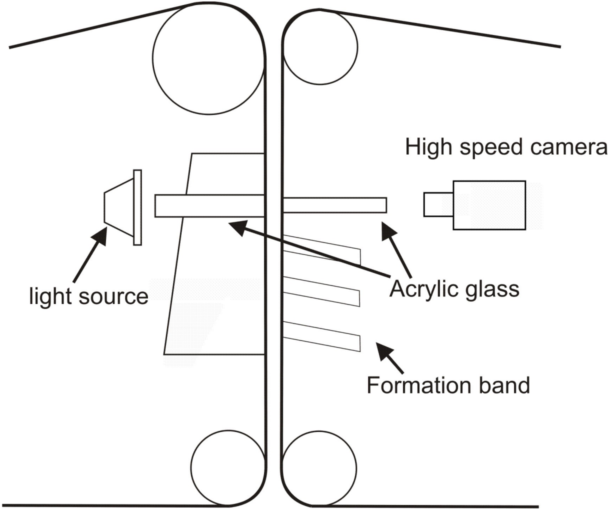
How to gain the paper data



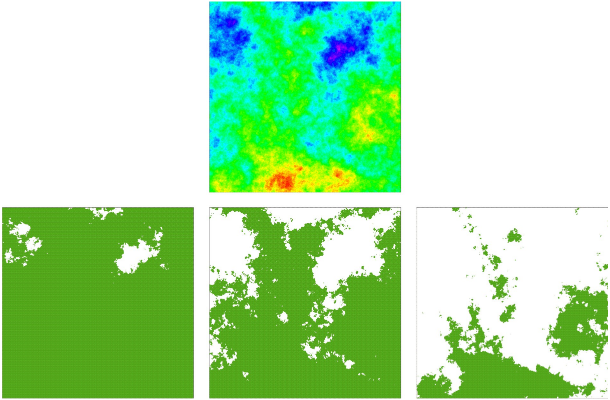
Definition

Let X be a measurable real-valued function on some measurable space, and let T be a measurable subset of that space. Then for $u \in \mathbb{R}$

$$A_u(X, T) := \{t \in T : X(t) \geq u\}$$

is called the excursion set of X in T over the level u, and $|A_u(X, T)|$ denotes its volume.

First works by Yu. K. Belyaev from 1967 - 1975
R. J. Adler from 1980 - now
A. V. Ivanov and N. N. Leonenko (1989)



Centered Gaussian random field on $[0, 1]^2$,

$r(t) = \exp(-\|t\|_2 / 0.3)$,

Levels: $u = -1.0, 0.0, 1.0$

Definition

A sequence of compact Borel sets $(W_n)_{n\in\mathbb{N}}$ is called a Van Hove sequence (VH) if $W_n \uparrow \mathbb{R}^d$ with

$$\lim_{n\to\infty} |W_n| = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{|\partial W_n \oplus B_r(0)|}{|W_n|} = 0, \quad r > 0.$$

Considered random fields

Let $X = \{X(t), t \in \mathbb{R}^d\}$ have the following properties:

- square-integrable
- strictly stationary
- has a continuous covariance function $r(t) = \mathrm{Cov}(X(o), X(t)), t \in \mathbb{R}^d$
- $|r(t)| = \mathcal{O}\left(\|t\|_2^{-\alpha}\right)$ for some $\alpha > 3d$ as $\|t\|_2 \to \infty$
- $X(0)$ has a bounded density
- quasi-associated

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Definition

A random field $X = \{X(t), t \in T\}$ with finite second moments is called quasi-associated if

$$|\mathrm{cov}\left(f\left(X_I\right), g\left(X_J\right)\right)| \leq \sum_{i \in I} \sum_{j \in J} \mathrm{Lip}_i\left(f\right) \mathrm{Lip}_j\left(g\right) |\mathrm{cov}\left(X\left(i\right), X\left(j\right)\right)|$$

for all finite disjoint subsets $I, J \subset T$, and for any Lipschitz functions $f : \mathbb{R}^{|I|} \to \mathbb{R}$, $g : \mathbb{R}^{|J|} \to \mathbb{R}$.

Theorem (One-dimensional CLT - general case)

Let X be the above random field and $u \in \mathbb{R}$. Then, for any sequence of VH-growing sets $W_n \subset \mathbb{R}^d$, one has

$$\frac{|A_u(X, W_n)| - \mathrm{P}(X(0) \geq u) \cdot |W_n|}{\sqrt{|W_n|}} \xrightarrow{\mathrm{d}} \mathcal{N}\left(0, \sigma^2(u)\right)$$

as $n \to \infty$. Here

$$\sigma^2(u) = \int_{\mathbb{R}^d} \operatorname{cov}\left(\mathbb{1}\{X(0) \geq u\}, \mathbb{1}\{X(t) \geq u\}\right) dt.$$

Special case - Shot noise random fields

The above CLT holds for a stationary shot noise random field $X = \{X(t), t \in \mathbb{R}^d\}$ given by $X(t) = \sum_{i \in \mathbb{N}} \xi_i \varphi(t - x_i)$ where

- $\{x_i\}$ is a homogeneous Poisson point process in $\mathbb{R}^d$ with intensity $\lambda \in (0, \infty)$
- $\{\xi_i\}$ is a family of i.i.d. non–negative random variables with $\mathrm{E}\,\xi_i^2 < \infty$ and the characteristic function φ_ξ
- $\{\xi_i\}$, $\{x_i\}$ are independent
- $\varphi : \mathbb{R}^d \to \mathbb{R}_+$ is a bounded and uniformly continuous Borel function with

$$\varphi(t) \leq \varphi_0(\|t\|_2) = O\left(\|t\|_2^{-\alpha}\right) \quad \text{as} \quad \|t\|_2 \to \infty$$

for a function $\varphi_0 : \mathbb{R}_+ \to \mathbb{R}_+$, $\alpha > 3d$, and

$$\int_{\mathbb{R}^d} \left| \exp\left\{ \lambda \int_{\mathbb{R}^d} (\varphi_\xi(s\varphi(t)) - 1)\, dt \right\} \right| ds < \infty.$$

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Special case - Gaussian random fields

Consider a stationary Gaussian random field $X = \{X(t), t \in \mathbb{R}^d\}$ with the following properties:

- $X(0) \sim \mathcal{N}(a, \tau^2)$
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Theorem (One-dimensional CLT - Gaussian case)

Let X be the above Gaussian random field and $u \in \mathbb{R}$. Then, for any sequence of VH-growing sets $W_n \subset \mathbb{R}^d$, one has

$$\frac{|A_u(X, W_n)| - \Psi((u-a)/\tau)\,|W_n|}{\sqrt{|W_n|}} \xrightarrow{\text{d}} \mathcal{N}\left(0, \sigma^2(u)\right)$$

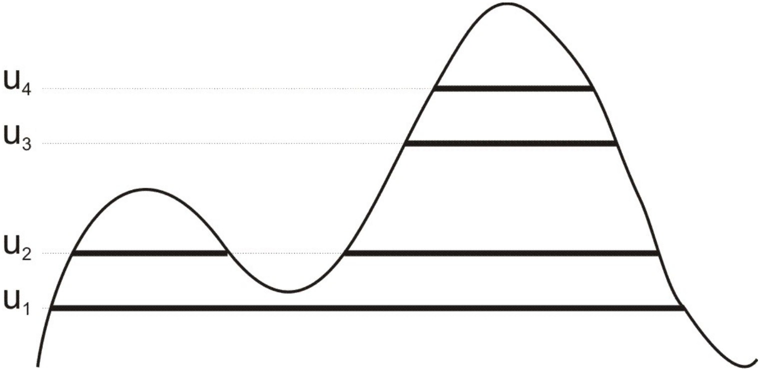
as $n \to \infty$. Here

$$\sigma^2(u) = \frac{1}{2\pi} \int_{\mathbb{R}^d} \int_0^{\rho(t)} \frac{1}{\sqrt{1-r^2}} e^{-\frac{(u-a)^2}{\tau^2(1+r)}}\, dr\, dt,$$

where $\rho(t) = \operatorname{corr}(X(0), X(t))$. In particular, for $u = a$

$$\sigma^2(a) = \frac{1}{2\pi} \int_{\mathbb{R}^d} \arcsin(\rho(t))\, dt.$$

Multi-dimensional CLT



$$S_{\vec{u}}(W_n) = \left(|A_{u_1}(X, W_n)|, \ldots, |A_{u_r}(X, W_n)|\right)^\top$$

$$\Psi(\vec{u}) = \left(\Psi((u_1 - a)/\tau), \ldots, \Psi((u_r - a)/\tau)\right)^\top$$

Theorem (Multi-dimensional CLT)

Let X be the above Gaussian random field and $u_k \in \mathbb{R}$, $k = 1, \ldots, r$. Then, for any sequence of VH-growing sets $W_n \subset \mathbb{R}^d$, one has

$$|W_n|^{-1/2} \left(S_{\vec{u}}(W_n) - \Psi(\vec{u}) \, |W_n| \right) \xrightarrow{d} \mathcal{N}(0, \Sigma(\vec{u}))$$

as $n \to \infty$. Here, $\Sigma(\vec{u}) = (\sigma_{lm}(\vec{u}))_{l,m=1}^{r}$ with

$$\sigma_{lm}(\vec{u}) = \frac{1}{2\pi} \int_{\mathbb{R}^d} \int_0^{\rho(t)} \frac{1}{\sqrt{1-r^2}} \exp\left\{ -\frac{(u_l-a)^2 - 2r(u_l-a)(u_m-a) + (u_m-a)^2}{2\tau^2(1-r^2)} \right\} dr \, dt.$$

Theorem (Statistical version of the CLT)

Let X be the above Gaussian random field, $u_k \in \mathbb{R}$, $k = 1, \ldots, r$ and $(W_n)_{n\in\mathbb{N}}$ be a sequence of VH-growing sets. Let $\hat{C}_n = (\hat{c}_{nlm})_{l,m=1}^{r}$ be statistical estimates for the nondegenerate asymptotic covariance matrix $\Sigma(\vec{u})$, such that for any $l, m = 1, \ldots, r$

$$\hat{c}_{nlm} \xrightarrow{p} \sigma_{lm}(\vec{u}) \text{ as } n \to \infty.$$

Then

$$\hat{C}_n^{-1/2} \, |W_n|^{-1/2} \left(S_{\vec{u}}(W_n) - \Psi(\vec{u}) \, |W_n| \right) \xrightarrow{d} \mathcal{N}(0, I).$$

Hypothesis testing

$$H_0 : X \text{ Gaussian vs. } H_1 : X \text{ Non-Gaussian}$$

Test statistic:

$$T = |W_n|^{-1} \left(S_{\vec{u}}(W_n) - \Psi(\vec{u})\,|W_n|\right)^\top \hat{C}_n^{-1} \left(S_{\vec{u}}(W_n) - \Psi(\vec{u})\,|W_n|\right)$$

We know $T \xrightarrow{\text{d}} \chi^2_r$. Reject null-hypothesis if $T > \chi^2_{r,1-\nu}$.

Numerical results

Series	FTR6.3	FTR6.6	Sim. Gaussian
Resolution	218*x*138	218*x*138	218*x*138
Realizations	100	100	100
1 level			
Rejected fields ($\nu = 1\%$)	0	0	1
3 levels			
Rejected fields ($\nu = 1\%$)	5	9	3
5 levels			
Rejected fields ($\nu = 1\%$)	20	21	3
7 levels			
Rejected fields ($\nu = 1\%$)	34	31	5
9 levels			
Rejected fields ($\nu = 1\%$)	62	60	5

CLTs for excursion sets of some classes of random fields

- Functional limit theorem (invariance principle) for the volume of excursion sets of associated random fields: D. Meschenmoser, A. Shashkin (2010)
- CLT and invariance principle for the volume of excursion sets of associated infinitely divisible (e.g. α-stable) random fields: W. Karcher (2010)
- Limit theorems for the boundary length of excursion sets of Gaussian random fields: M. Kratz, J. Leon (2001); D. Meschenmoser, A. Shashkin (2010)

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References

- R. J. Adler and J. Taylor: „Random Fields and Geometry“, *Springer, New York*, 2007.
- Yu. K. Belyaev: „The general formulae for the mean number of crossings for random processes and fields, in: Bursts of Random Fields“, *Moscow Univ. Press, Moscow*, 1972
- A. V. Bulinski and A. P. Shashkin: „Limit Theorems for Associated Random Fields and Related Systems“, *World Scientific, Singapore*, 2007
- A. Bulinski, E. Spodarev, F. Timmermann: „Central limit theorems for the excursion sets volumes of weakly dependent random fields“, *submitted, 2010. http://arxiv.org/pdf/1005.0483v1*
- A. V. Ivanov and N. N. Leonenko: „Statistical Analysis of Random Fields “, *Kluwer, Dordrecht*, 1989.
- W. Karcher: „Limit theorems for the volume of the excursion sets of stationary infinitely divisible random fields “, *Preprint*, 2010.
- D. Meschenmoser and A. Shashkin: „Invariance principle for the excursion sets generated by stationary dependent random fields “, *Preprint*, 2010.
- A. Shashkin and D. Meschenmoser: „CLT for the boundary length of the excursion sets of Gaussian random fields “, *in preparation*, 2010.
- U. Pantle, V. Schmidt and E. Spodarev: „On the estimation of the integrated covariance function of stationary random fields“, *Scand, J. Stat.*, Vol. 37, 2010, pp. 47-66.

Thank you for your attention!