Multivariate Skewness: Measures, Properties and Applications

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Introduction

- Univariate skewness;
- Third moment;
- Mardia’s skewness;
- Directional skewness.

Univariate skewness: First game's description

Five friends bet one euro each as follows:

- Each friend puts a ticket with his signature in an urn;
- A ticket is randomly chosen from the urn;
- The friend who signed that ticket wins all the euros.

Univariate skewness: First game's payoff

$$X_1 = \begin{cases} -1 & 4 \\ 0.8 & 0.2 \end{cases}$$

$$E(X_1) = 0 \quad V(X_1) = 4$$

Univariate skewness:
Second game's description

Five friends borrow an euro each, and agree to pay the money back as follows:

- Each friend puts a ticket with his signature in an urn;
- A ticket is randomly chosen from the urn;
- The friend who signed that ticket repays all the borrowed money.

Univariate skewness: Second game's payoff

$$X_2 = \begin{cases} -4 & 1 \\ 0.2 & 0.8 \end{cases}$$

$$E(X_2) = 0 \quad V(X_2) = 4$$

Univariate skewness: Definition

$$X \in \Re \quad E(X) = \mu \quad V(X) = \sigma^2$$

$$\gamma_1(X) = E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] \qquad \beta_1(X) = \gamma_1^2(X)$$

Univariate skewness: Comparing the games

$$E(X_1) = E(X_2) = 0$$

$$V(X_1) = V(X_2) = 4$$

$$\gamma_1(X_1) = 1.5 = -\gamma_1(X_2)$$

$$\beta_1(X_1) = \beta_1(X_2) = 2.25$$

Univariate skewness: Interpretation

When skewness is positive (negative), positive deviations from the mean contribute more (less) to variability than negative ones

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Values very different from the mean are often greater (smaller) than the mean itself

Univariate skewness: Symmetry

Measures γ_1 and β_1 equal zero when the underlying distribution is symmetric

$$X-\mu \equiv \mu - X \Rightarrow \gamma_1\left(X\right)=\beta_1\left(X\right)=0$$

Univariate skewness: Third game’s description

Cast a regular dice:

- Win 1 euro if the outcome is smaller than 4
- Cast it again if the outcome is greater than 3:

1. Lose 4 euros is the second outcome is smaller than 5
2. Win 5 euros in the opposite case

Univariate skewness: Third game's payoffs

$$X_3 = \begin{cases} -4 & 1 & 5 \\ 2/6 & 3/6 & 1/6 \end{cases}$$

Univariate skewness: Third game's properties

- The expected gain is zero

- P (*win*) = 2/3 > 1/3 = P (*lose*)

- Both $\gamma_1(X_3)$ and $\beta_1(X_3)$ equal zero

Problem

How does skewness generalizes
to multivariate distributions?

Kronecker product

$$A = \{a_{ij}\} \in \Re^p \times \Re^q \quad B = \{b_{ij}\} \in \Re^r \times \Re^s$$

$$A \otimes B = \{a_{ij} B\} \in \Re^{p \cdot r} \times \Re^{q \cdot s} \quad i = 1, \ldots, p \quad j = 1, \ldots, q$$

Standardized random vector

$$x \in \Re^d \quad E(x) = \mu \quad V(x) = \Sigma$$

$$\Sigma^{-1/2} = \left(\Sigma^{-1/2}\right)^T \quad \Sigma^{-1/2}\Sigma^{-1/2} = \Sigma^{-1} \quad \Sigma^{-1/2} > 0$$

$$z = \Sigma^{-1/2}\left(x - \mu\right)$$

Third moment: Definition

$$\mu_3(x) = E\left(x \otimes x^T \otimes x\right) \in \Re^{d^2} \times \Re^d$$

Third moment: Elements

Structure	Constraint	Number
$E(X_i^3)$	*none*	d
$E(X_i^2 X_j)$	$i \neq j$	$d(d-1)$
$E(X_i X_j X_h)$	$\neq i, j, h$	$d(d-1)(d-2)/6$
Total	*none*	$d(d+1)(d+2)/6$

Third moment: Example

$$x = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}, \quad \mu_{ijh} = E\left(X_i X_j X_h\right) \quad i, j, h = 1,2,3 \Rightarrow \mu_3(x) = \begin{pmatrix} \mu_{111} & \mu_{112} & \mu_{113} \\ \mu_{112} & \mu_{122} & \mu_{123} \\ \mu_{113} & \mu_{123} & \mu_{133} \\ \mu_{112} & \mu_{122} & \mu_{123} \\ \mu_{122} & \mu_{222} & \mu_{223} \\ \mu_{123} & \mu_{223} & \mu_{233} \\ \mu_{113} & \mu_{123} & \mu_{133} \\ \mu_{123} & \mu_{223} & \mu_{233} \\ \mu_{133} & \mu_{233} & \mu_{333} \end{pmatrix}$$

Third moment: Linear transformations

$$x \in \Re^d \qquad A \in \Re^k \times \Re^d$$

$$\mu_3(Ax) = (A \otimes A)\mu_3(x)A^T$$

Third moment: Symmetry

The third moment of a centrally symmetric random vector is a null matrix

$$x-\mu \equiv \mu - x \Rightarrow \mu_3\left(x-\mu\right) = O_{d^2\times d}$$

Third moment: Sample counterpart

$$m_3(X) = \frac{1}{n}\sum_{i=1}^{n} x_i \otimes x_i^{\;T} \otimes x_i$$

$$X = \text{ data matrix}$$

$$x_i = \text{ i-th column of } X^T$$

Third moment: Statistical applications

- Multivariate Edgeworth expansions;
- Skewness of financial data;
- Other measures of skewness.

Third moment: Essential references

- Franceschini, C., Loperfido, N., (2009). A Skewed GARCH-Type Model for Multivariate Financial Time Series. In *"Mathematical and Statistical Methods for Actuarial Sciences and Finance* XII", Corazza M. and Pizzi C. (Eds.), ISBN: 978-88-470-1480-0

- Kollo, T. (2008). Multivariate skewness and kurtosis measures with an application in ICA. Journal of Multivariate Analysis 99, 2328-2338

- Kollo, T. & von Rosen, D.(2005). Advanced Multivariate Statistics with Matrices. Springer, Dordrecht, The Netherlands

Mardia's skewness: Definition

$$x, y \in \Re^d \quad E(x) = \mu \quad V(x) = \Sigma > 0 \quad x, y \quad are \quad i.i.d.$$

$$\beta_{1,d}^{M}(x) = E\left[(x-\mu)\Sigma^{-1}(y-\mu)\right]^3$$

Mardia's skewness: Invariance

Mardia's skewness is invariant with respect to one-to-one affine transformations

$$\left.\begin{array}{c} A \in \Re^d \times \Re^d \\ \det(A) \neq 0 \\ b \in \Re^d \end{array}\right\} \Rightarrow \beta_{1,d}^M(x) = \beta_{1,d}^M(Ax + b)$$

Mardia's skewness: Standardization

Mardia's skewness is the squared (Frobenius) norm of the third standardized moment

$$\Downarrow$$

$$\beta_{1,d}^{M}(x) = \left\| \mu_3(z) \right\|^2 = tr\left[\mu_3^T(z) \mu_3(z) \right]$$

Mardia's skewness: Symmetry

Mardia's skewness equals zero for centrally symmetric random vectors:

$$x-\mu \equiv \mu - x \Rightarrow \beta_{1,d}^{M}\left(x\right) = 0$$

Mardia's skewness: Sample counterpart

$$b_{1,d}(X) = \frac{1}{n^2} \sum_{i,j}^{n} \left[(x_i - m)^T S^{-1} (x_j - m) \right]^3$$

Mardia's skewness: Statistical applications

- Multivariate normality testing;
- Assessing robustness in MANOVA;
- Parametric point estimation.

Mardia’s skewness: Essential references

- Baringhaus, L. and Henze, N. (1992). Limit distributions of Mardia's measures of multivariate skewness. Ann. Statist. 20, 1889-1902
- Mardia, K.V. (1970). Measures of multivariate skewness and kurtosis with applications. Biometrika 57, 519-530
- Mardia, K.V. (1975). Assessment of Multinormality and the Robustness of Hotelling's T^2 Test. Appl. Statist. 24, 163-171

Directional skewness: Definition

$$\beta_{1,d}^{D}(x) = \max_{c \in \Re_0^d} \beta_1\left(c^T x\right)$$

Directional skewness: Invariance

Directional skewness is invariant with respect to one-to-one affine transformations

$$\left.\begin{array}{c} A \in \Re^d \times \Re^d \\ \det(A) \neq 0 \\ b \in \Re^d \end{array}\right\} \Rightarrow \beta_{1,d}^{D}(x) = \beta_{1,d}^{D}(Ax+b)$$

Directional skewness: Standardization

Directional skewness is a function of the third standardized moment

$$\beta_{1,d}^{D}(x) = \sup_{c^T c = 1} \left[\left(c^T \otimes c^T \right) \mu_3(z) c \right]^2$$

Directional skewness: Sample counterpart

$$b^{D}{}_{1,d}(X) = \max_{c \in \Re_0^d} \left\{ \frac{1}{n} \sum_{i=1}^{n} \left(\frac{\mathbf{c}^T x_i - \mathbf{c}^T m}{\sqrt{\mathbf{c}^T S \mathbf{c}}} \right)^3 \right\}^2$$

Directional skewness: Statistical applications

- Multivariate normality testing;
- Projection pursuit;
- Parametric point estimation.

Directional skewness: Essential references

- Malkovich, J.F. and Afifi, A.A. (1973). On tests for multivariate normality. Journal of the American Statistical Association 68, 176-179
- Huber, P.J. (1985). Projection pursuit (with discussion). Ann. Statist. 13, 435-475
- Baringhaus, L. and Henze, N. (1991). Limit distributions for measures of multivariate skewness and kurtosis based on projections. J. Multiv. Anal. 38, 51-69

Hints for future research

- Interpretation
- Modelling
- Inference