Policyholder Exercise Behavior for Variable Annuities including Guaranteed Minimum Withdrawal Benefits

PRELIMINARY VERSION*

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First version: February 2011
This version: April 2011

Abstract

In recent years, Guaranteed Minimum Withdrawal Benefits within Variable Annuities have become increasingly popular. However, thus far, there is no consensus on how policyholders should optimally exercise these options, which has repercussions for their pricing and their management. Specifically, while option pricing techniques applied in the actuarial literature point to policyholders optimally withdrawing the maximal guaranteed amount in most circumstances, insurers rely on fixed exercise probabilities combined with simple exercise rules for their calculations.

The present paper demonstrates that preferred tax treatment of Variable Annuities and the benefit from the embedded guarantees to risk-averse policyholders yield more nuanced optimal exercise patterns. More precisely, we show that a lifetime utility maximizing policyholder in a model with exogenous income and an outside account may frequently forgo guaranteed withdrawals. In particular, our numerical analyses indicate that typical market rates appear to sufficiently provide for the guarantees. Interestingly, we predict that the insurer can *increase* expected net profits by e.g. eliminating the excess withdrawal charge, as this would significantly alter optimal withdrawal behavior in the insurer's favor.

Keywords: Variable Annuities, Guaranteed Minimum Benefits, Optimal Policyholder Behavior, Lifecycle Theory.

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1 Introduction

Policyholder behavior is an important but little understood risk factor for insurance companies offering contracts that include exercise-dependent features. Nonetheless, it has an immediate effect on pricing and costly risk management strategies, so that a consistent analysis and anticipation may prove crucial for the firm's competitiveness and financial stability.

In this paper, we focus our attention on policyholder exercise behavior for Variable Annuity (VA) contracts with Guaranteed Minimum Withdrawal Benefits (GMWB). Here, a VA essentially is a unit-linked, tax-deferred savings plan potentially entailing guaranteed payment levels, for instance upon death (Guaranteed Minimum Death Benefit, GMDB) or survival until expiration (Guaranteed Minimum Living Benefits, GMLB). A GMWB, on the other hand, provides the policyholder with the right to withdraw the initial investment over a certain period of time, irrespective of investment performance, as long as annual withdrawals do not exceed a pre-specified amount. To finance these guarantees, most commonly insurers continuously deduct an option fee at a constant rate from the policyholder's account value.[1]

The combined net assets of U.S. variable annuities amounted to \$1.35 trillion at then end of 2009, and U.S. individual VA sales in the first three quarters of 2010 totaled over \$100 billion. These figures indicate the importance for insurers to understand how policyholders may utilize the embedded options. However, to date most liability models fail to capture this risk factor in adequate fashion. In particular, companies usually rely on historic exercise probabilities or static exercise rules, although some insurers indicate they use simple dynamic assumptions in their C3 Phase II calculations (cf. Society of Actuaries (2009)). Consequently, changes in economic or regulatory conditions have on occasion caused dramatic shifts in policyholder behavior that have caught the industry off-guard. For instance, rising interest rates in the 1970s led to the so-called *disintermediation*[2] process, which caused substantial increases in surrenders and policy loans in the whole life market. Many life insurers experienced negative cash flows for the first time since the Great Depression, which forced them to "liquidate bonds and other securities, usually at significant discounts from par" (cf. Black and Skipper (2000), p.111). Similarly, in 2000, the UK-based mutual life insurer Equitable Life was closed to new business due to problems arising from a misjudgment of policyholder behavior with respect to exercising guaranteed annuity options within individual pension policies (cf. Boyle and Hardy (2003)).

As a potential solution, the prevalent assumption in the actuarial literature is that policyholders may exercise optimally with respect to the value of the contract consistent with arbitrage pricing theory (see, among others, Bauer et al. (2008), Grosen and Jørgensen (2000), Milevsky and Posner (2001), Milevsky and Salisbury (2006), Ulm (2006), or Zaglauer and Bauer (2008)). Specifically, the value is characterized by an optimal control problem identifying the supremum of the contract value over all admissible exercise strategies. While such an approach may be defended in that it – in principle – identifies the unique superval-

[1]More recently, some insurers have started charging the guarantee fee as a percentage of the "guaranteed benefit base", i.e. the total outstanding guaranteed amount. While in this paper we consider the more traditional fee structure as a percentage of the account value, alternative policies can be easily implemented in our model framework. We leave the analysis of the consequences for future research.

[2]Life insurers had underwritten many whole life policies that permitted policy loans at interest rates below prevailing market rates.

uation and superhedging strategy robust to any policyholder behavior (cf. Bauer et al. (2010)), the resulting "fair" guarantee fees considerably exceed the levels encountered in practice. For example, Milevsky and Salisbury (2006) calculate the no-arbitrage hedging cost of a GMWB to range from 73 to 160 basis points, depending on parameter assumptions, although typically insurers charge about 30 to 45 bps. While from the authors' perspective these observed differences between theory and practice are a result of "suboptimal" policyholder behavior, we argue that these deviations can also be attributed to the policyholder foregoing certain privileges and protection when making a withdrawal, even in case of *rational* decision making. The key insight in this context is that, in contrast to financial derivatives, policyholders generally are not able to sell – or repurchase – their policy at its risk-neutral value.

In order to derive corresponding results in the case of GMWBs, we introduce a structural model that explicitly considers the problem of decision making under uncertainty faced by the holder of a VA policy. More specifically, the policyholder's state-contingent decision process is modeled using a lifetime utility model of consumption and bequests. We assume stochasticity in both the financial market and individual lifetime, and include appropriate tax treatments of all investments. We parametrize the model based on standard assumptions about risk aversion, the financial market, etc., and solve the decision making problem numerically using a recursive dynamic programming approach. When endowing the policyholder with deterministic income and the opportunity to save and invest in an outside account, we are able to identify a variety of aspects that factor into the policyholder's decision process. Most importantly, our numerical results indicate that withdrawals are usually only optimal if either the VA account value is below the tax base, or if the VA account far exceeds the outside account. While the latter observation is due to consumption smoothing and overall portfolio allocation and risk exposure, in most cases withdrawal behavior is determined by non-trivial interactions between tax considerations, the current value of the guarantee, and the excess withdrawal fee.

Our numerical results indicate that policyholders empty their guarantee account less than 6% of the time; the guarantee is being used in only 24% of all cases; and in more than two thirds of all cases, *no* early withdrawals are made. These findings are in stark contrast to the results based on arbitrage pricing theory, which find that withdrawing at least the guaranteed amount is optimal in almost all circumstances (cf. Milevsky and Salisbury (2006)). In particular, our results indicate that a guarantee fee of 50 basis points appears to sufficiently provide for the considered return-of-investment GMWB. Moreover, our results raise questions regarding the use of simple dynamic exercise rules based on the "in-the-moneyness" of the guarantees, as they are slowly adopted by some life insurance companies (cf. Society of Actuaries (2009)). More precisely, although for our base case scenario we predict a similar level of expected aggregate withdrawals as compared to "optimal" policyholder behavior, the respective withdrawal patterns are very different. It is therefore not guaranteed that the similarities in expected aggregate withdrawals will hold up under more extreme scenarios. Interestingly, we predict that the insurer can *increase* expected net profits by e.g. eliminating the excess withdrawal charge, as this would significantly alter optimal withdrawal behavior in the insurer's favor. More generally, we observe that changes to the policy can cause a dramatic shift in policyholder behavior and sometimes lead to counter-intuitive overall effects on the insurer's net profit. We therefore advise caution when offering long-term insurance products that are sensitive to policyholder behavior.

The withdrawal and lapsation decision problem of policyholders of VAs shows similarities to prepayment options within mortgages. Here, Stanton (1995) proposes a rational expectations model with heterogeneous transaction costs, which the author shows to "resemble many of the empirical features of mortgage prepayments". Similarly, De Giovanni (2010) develops a model to evaluate the surrender options in life insurance contracts, which also allows for *irrational* in addition to rational exercises. Others have proposed purely empirical approaches to forecast policyholder behavior, in which exercise probabilities are estimated from historical contract data controlling for business cycle, interest rates, and other economic variables (see e.g. Kim (2005) and Kuo et al. (2003)). However, aside from the problem that there is relatively scarce data on policyholder behavior once one controls for asset prices, macroeconomic factors, and evolving product characteristics, it is arguable whether such an approach will be able to provide reliable estimates in states or regimes with little or no behavioral data. In contrast, one advantage of our model is that it should allow insurers – upon proper calibration to the available data – to make out-of-sample predictions.

The remainder of the paper is structured as follows: The following section introduces a general lifetime utility model for variable annuities. Section 3 is dedicated to the implementation of the model in a Black-Scholes framework and to discussing computational details. We proceed by presenting our numerical results in Section 4. Finally, Section 5 concludes and briefly discusses possible extensions.

2 A Lifetime Utility Model for Variable Annuities with GMWBs

There exists a large variety of VA products available in the U.S.: The policies differ by how the premium is collected; policyholder investment opportunities, including whether the policyholder can reallocate funds during the contract phase; and guarantee specifics, for instance what type of guarantees are included, how the guarantees are designed and how they are paid for, etc. For a detailed description of VAs and the guarantees available in the market, we refer the interested reader to Bauer et al. (2008).

This section develops a lifetime utility model of variable annuities including (among others) a simple return-of-investment GMWB option, with stochasticity in policyholder lifetime and asset returns. The policyholder's state-contingent decision process entails annual choices over withdrawals from the VA account, consumption, and asset allocation in an outside portfolio.

In contrast to mutual funds, VAs grow tax deferred, which presents the primary reason for their popularity among individuals who exceed the limits of their qualified retirement plans. For instance, Milevsky and Panyagometh (2001) show that variable annuities outperform mutual funds for investments longer than 10 years, even when the option to harvest losses is taken into account for the mutual fund. Since the preferred tax treatment possibly also has a significant influence on policyholder exercise behavior, we briefly describe current U.S. taxation policies on variable annuities and the way these are captured in our model in Section 2.4.

2.1 Description of the Variable Annuity Policy

We consider an x-year old individual who has just (time $t = 0$) purchased a VA with finite integer maturity T against a single up-front premium P_0. The insurer will return the policyholder's concurrent account value – or some guaranteed amount, if eligible – at the end of the policyholder's year of death or at maturity, whichever comes first. In addition, the contract contains a GMWB option, which – following an accumulation phase of T^W years – grants the policyholder the right but not the obligation to withdraw the initial investment P_0 free of charge and independent of investment performance, as long as annual withdrawals do not exceed the guaranteed annual amount g^W. We define g_t^W as zero for $0 < t \leq T^W$, and as g^W for all t thereafter. Withdrawals in excess of g_t^W, and in particular any withdrawals prior to T^W, carry a (partial) surrender charge of $s_t \in (0, 1)$, as a percentage of the (excess) withdrawal amount. The remaining total amount to be withdrawn, i.e. the level of the GMWB account, is denoted by G_t^W. We model a "generic" contract that may also contain a GMDB or other GMLB options. In that case, we denote by G_t^D, G_t^I, and G_t^A the guaranteed minimum death, income, and accumulation benefit, respectively.[3] For simplicity of exposition and without much loss of generality, we assume that all included guarantees are return-of-investment options. In that case, all involved guarantee accounts have an identical *benefits base*

$$G_t^{\cdot} \equiv G_t^W = G_t^D = G_t^A = G_t^I.$$

If an option is not included, we simply set the corresponding guaranteed benefit to zero in the model. Hence this model allows us to include a variety of guarantees, without having to increase the state space, which makes the problem computationally feasible.[4] However, other contract designs could be easily incorporated at the cost of a larger state space. We refer to Bauer et al. (2008) for details.

While for return-of-investment guarantees, the initial benefits base is $G_0^{\cdot} = P_0$, this equality will no longer be satisfied after funds have been withdrawn from the account. More precisely, following Bauer et al. (2008) we model the adjustments of the benefits base in case of a withdrawal prior to maturity based on the following assumptions: If the withdrawal does not exceed the guaranteed annual amount g_t^W, the benefits base will simply be reduced by the withdrawal amount. Otherwise, the benefits base will be the lesser of that and the so-called pro rata adjustment. Thus,

$$G_{t+1}^{\cdot} = \begin{cases} (G_t^{\cdot} - w)^+ & : w \leq g_t^W \\ \left(\min\left\{ G_t^{\cdot} - w \,,\, G_t^{\cdot} \cdot \frac{X_t^+}{X_t^-} \right\} \right)^+ & : w > g_t^W, \end{cases} \tag{1}$$

where w is the withdrawal amount, $X_t^{-/+}$ denote the VA account values immediately before and after the withdrawal is made, respectively, and $(a)^+ \equiv \max\{a, 0\}$. To finance the guarantees, the insurer continuously deducts an option fee at constant rate $\phi \geq 0$ from the policyholder's account value. Alternatively, later in the text we also consider specifications where the option fee is deducted as a fixed percentage of the

[3] A Guaranteed Minimum Accumulation Benefit (GMAB) guarantees a minimal (lump-sum) payout at maturity of the contract, provided that the policyholder is still alive. Under the same conditions, a Guaranteed Minimum Income Benefit (GMIB) guarantees a minimal annuity payout.

[4] In particular, we can also analyze contracts that do not contain a GMWB option at all.

benefit base $G_t^{\cdot}$.

We also assume that at $t = 0$, the policyholder has chosen his investment strategy for the VA, and the allocation remains constant subsequently. In the presence of a GMWB option this is not unusual since otherwise the policyholder may be incentivized to choose the most risky investment strategy in order to maximize the value of the guarantee. Finally, we suppose that these policyholder decisions are all made at policy anniversary dates, $t = 1, \ldots, T$. In particular, utilities are determined based on a discrete number of dates.

2.2 Consumer Preferences

The policyholder gains utility from consumption, while alive, and from bequesting his savings upon his death (if death occurs prior to retirement). We assume time-separable preferences with an individual discount factor β, and utility functions $u_C(\cdot)$ and $u_B(\cdot)$ for consumption and bequests, respectively.

The policyholder is endowed with an initial wealth W_0, of which he invests P_0 in the VA. The remainder is allocated in an "outside account". We suppose there exist d identical investment opportunities inside and outside the VA, the main difference being that adjustments to the investment allocations in the outside portfolio can be made every year at the anniversary date of the VA policy. We denote the time-t value of the VA account by X_t and the time-t value of the outside account by A_t, where the corresponding investment allocations are specified by the d-dimensional vectors ν^X and ν_t for the VA and outside account, respectively. In either case, short sales are not allowed, so that we require

$$\nu_t, \nu^X \geq 0, \text{ and } \sum_i \nu_t(i) = \sum_i \nu^X(i) = 1. \tag{2}$$

For the values at policy anniversaries, we add superscript $^-$ to denote the level of an account (state variable) at the beginning of a period, just prior to the policyholder's decision, and superscript $^+$ to indicate its value immediately afterwards. Note that guarantee accounts do not change *between* periods, i.e. between $(t)^+$ and $(t+1)^-$, but only through withdrawals at policy anniversary dates, so that no superscripts are necessary here. The policyholder receives annual (exogenous) net income I_t, which for the purpose of this paper is deterministic and paid in a single installment at each policy anniversary. Upon observing his current level of wealth, i.e. his outside account value A_t^-; the state of his VA account X_t^-; the level of his guarantees $G_t^{\cdot}$, and his VA tax basis H_t (see below); as well as deterministic factors such as g_t^W, I_t, mortality rates, etc., the policyholder chooses how much to withdraw from the VA account, how much to consume, and how to allocate his outside investments in the upcoming policy year.

For the reader's convenience, a comprehensive list of notation used in this paper can be found in Appendix A. Appendix B describes the assumed timeline of events leading up to and following the policyholder's decision process each period.

2.3 Mortality

For simplicity, we assume that all deaths become effective at the end of a period. Since the policyholder does not make decisions during the period, this means that benefits are payable at the end of the year of death, assuming accounts are carried on until then as if the policyholder were alive.

Relying on standard actuarial notation, we denote by ${}_tq_x$ the probability that (x) dies within t years, and by ${}_tp_x \equiv 1 - {}_tq_x$ the corresponding probability of survival. In particular, we express the one-year death and survival probabilities by q_x and p_x, respectively. Consequently, the probability that (x) dies in the interval $(t, t+1]$ is given by ${}_tp_x \cdot q_{x+t}$.

2.4 Tax Treatment of Variable Annuities

We model taxation of income and investment returns based on concurrent U.S. regulation, albeit with a few necessary simplifications. More precisely, we assume that all investments are post-tax and non-qualified, i.e. all investments have already been taxed – either as income or as previous capital gains, etc. As such, taxes will only be due on future investment gains, not the initial investment (principal) itself.

Investments into a variable annuity grow tax deferred. In other words, the policyholder will not be taxed on any earnings until he starts to make withdrawals from his VA account. However, all earnings from a VA will eventually be taxed as ordinary *income*. Moreover, withdrawals are taxed on a last-in first-out basis, meaning that earnings are withdrawn *before* the principal. Specifically, early withdrawals after an investment gain are subject to income taxes. Only if the account value lies below the tax base will withdrawals be tax free. In addition, withdrawals prior to the age of $59\frac{1}{2}$ are subject to an early withdrawal tax of s^g (typically 10%). At maturity, denoting the concurrent VA account value by X_T, and the tax base by H_T, if he chooses the account value to be paid out as a lump-sum, the policyholder is required to pay taxes on the (remaining) VA earnings

$$\max\{X_T - H_T, 0\}$$

immediately (if applicable, we substitute G_T^A for X_T). However, if the policyholder chooses to annuitize his account value – e.g. in level annual installments, as we assume in this paper – his annual tax-free amount is his current tax base H_T divided by his life expectancy e_{x+T}, as computed from the appropriate actuarial table. In other words, the policyholder will need to declare any annuity payments from this VA in excess of H_T/e_{x+T} per year as ordinary income.[5] In case of the policyholder's death prior to maturity, the same taxation rules would apply to the beneficiaries.

For the initial tax base, we obviously have $H_1 = P_0$. The subsequent evolution of the tax base depends on both the evolution of the account value and withdrawals, where H_t essentially denotes the part of the account value that is left from the original principle. More precisely, the tax base remains unaffected by withdrawals smaller or equal to $X_t^- - H_t$, i.e. those withdrawals that are fully taxed (because they come from earnings), whereas tax-free withdrawals reduce the tax base dollar for dollar. Hence, algebraically we

[5] See IRS Publication 939 (*General Rule for Pensions and Annuities*, available at `http://www.irs.gov/pub/irs-pdf/p939.pdf`).

have

$$H_{t+1} = H_t - \left(w_t - \left(X_t^- - H_t\right)^+\right)^+. \tag{3}$$

In contrast, returns from a mutual fund are not tax deferrable. While in practice parts of them are ordinary dividends and thus taxed as income, others are long term capital gains and subject to the (lower) capital gains tax rate. We simplify taxation of mutual fund earnings to be at a constant annual rate, denoted by κ, which for future reference we call the *capital gains tax*, although it may be chosen a little higher than the actual tax on capital gains to reflect income from dividends or coupon payments, which are taxable at a higher rate. The income tax rate is also assumed to be constant over taxable money and time, at rate τ.[6]

2.5 Policyholder Optimization During the Lifetime of the Contract

The setup, as described in this section, requires four state variables: A_t^-, the value of the outside account just before the t-th policy anniversary date; X_t^-, the value of the VA account just before the t-th policy anniversary date; $G_t^{\cdot}$, the value of the benefits base (and thus all guarantee accounts) in period t; and H_t, the tax base in period t. At the t-th policy anniversary, given withdrawal of w_t, we define next-period benefits base and tax base by equations (1) and (3), respectively.

2.5.1 Transition from $(t)^-$ to $(t)^+$

Upon withdrawal of w_t, consumption C_t, and new outside portfolio allocation level ν_t, we update our state variables as follows:

$$\begin{aligned} X_t^+ &= \left(X_t^- - w_t\right)^+, \text{ and} \\ A_t^+ &= A_t^- + I_t + w_t - C_t - \text{fee}_I - \text{fee}_G - \text{taxes}, \end{aligned} \tag{4}$$

where

$$\text{fee}_I = s \cdot \max\left\{w_t - \min(g_t^W, G_t^W)\right\}$$

denotes the excess withdrawal fees the policyholder pays to the insurer,

$$\text{fee}_G = s^g \cdot (w_t - \text{fee}_I) \cdot \mathbf{1}_{\{x+t<59.5\}}$$

are the early withdrawal penalty fees the government collects on withdrawals prior to age 59.5, and

$$\text{taxes} = \tau \cdot \min\{w_t - \text{fee}_I - \text{fee}_G, (X_t^- - H_t)^+\}$$

are the (income) taxes the policyholder pays upon withdrawing w_t.

In our basic model, we update the guaranteed withdrawal account by (1). If the contract specifies guarantees to evolve differently (e.g. step-up or ratchet-type guarantees), the updating function must be modified accordingly. In that case we may also need to carry along an additional (binary) state variable to

[6] We believe this to be a reasonable simplification as holders of variable annuities are typically relatively wealthy, so that brackets over which the applicable marginal income tax rate is constant are fairly large, and it is likely that withdrawals will be taxed at a constant marginal rate in practice as well. Moreover, we want to avoid withdrawal behavior being affected unpredictably by "fragile" tax advantages.

keep track of whether the policyholder has previously made a withdrawal. We refer to Bauer et al. (2008) for details.

2.5.2 Transition from $(t)^+$ to $(t+1)^-$

In our model, the only state variables changing stochastically between $(t)^+$ and $(t+1)^-$ are the account values inside and outside of the VA, both driven by the evolution of the financial assets. One can think of them as a (column) vector of stochastic processes, $(S_t)_{t\geq 0}$.[7] Here, ν_t denotes the (row) vector capturing the shares of his wealth A_t^+ the policyholder wants to invest in each asset. Taking into account appropriate tax treatments, as described in section 2.4, we can update the account values as follows:

$$\begin{aligned} A_{t+1}^- &= A_t^+ \cdot \left[\nu_t \cdot \tfrac{S_{t+1}}{S_t} - \kappa \cdot \left(\nu_t \cdot \tfrac{S_{t+1}}{S_t} - 1\right)^+\right], \text{ and} \\ X_{t+1}^- &= X_t^+ \cdot e^{-\phi} \cdot \left[\nu^X \cdot \tfrac{S_{t+1}}{S_t}\right], \end{aligned} \tag{5}$$

where $\frac{S_{t+1}}{S_t}$ denotes the component-wise quotient.

2.5.3 Bellman Equation

Denoting the policyholder's time-t value function by $V_t : \mathbb{R}^4 \to \mathbb{R}$, $y_t \equiv (A_t^-, X_t^-, G_t^\cdot, H_t) \mapsto V_t(y_t)$, where we call y_t the vector of state variables, we can describe his optimization problem at each policy anniversary date recursively by

$$V_t(y_t) = \max_{C_t, w_t, \nu_t} u_C(C_t) + e^{-\beta} \cdot E_t\left[q_{x+t} \cdot u_B(b_{t+1}|S_{t+1}) + p_{x+t} \cdot V_{t+1}(y_{t+1}|S_{t+1})\right], \tag{6}$$

subject to (1), (3), (2), (4), (5), the bequest amount

$$b_{t+1} = A_{t+1}^- + \max\{X_{t+1}^-, G_{t+1}^D\}, \tag{7}$$

and the choice variable constraints

$$\begin{aligned} 0 &\leq C_t \leq A_t^- + I_t + w_t - \text{fee}_I - \text{fee}_G - \text{taxes}, \\ 0 &\leq \nu_t \leq 1, \quad \text{and} \\ 0 &\leq w_t \leq \max\left\{ X_t^-, \min\left\{ g_t^W, G_t^\cdot \right\}\right\}. \end{aligned} \tag{8}$$

2.6 Policyholder Behavior upon Maturity of the Variable Annuity

If the policyholder is alive when the Variable Annuity matures at time T, we assume that he retires immediately and he will no longer receive any outside income. He will live off his concurrent savings which consist of the time-T value of his outside portfolio, plus the maximum of his VA account value and any remaining GMLB benefits. More precisely, we assume he uses these savings to purchase a single-premium whole life

[7] As usual in this context, underlying our consideration is a complete filtered probability space $(\Omega, F, \mathbb{P}, \mathbf{F} = (F_t)_{t\geq 0})$, where $\mathbf{F}$ satisfies the usual conditions and $\mathbb{P}$ denotes the "physical" probability measure.

annuity, and that he no longer has a bequest motive; his consumption preferences, on the other hand, are the same as before.

We model the taxation of annuities following our discussion in Section 2.4. The outside account value A_T^- is already net of taxes, thus only future earnings (i.e. interest) need to be taxed. Therefore, A_T^- acts as the tax base for the whole life annuity. The outside account can thus be converted into net annuity payments of

$$c_A \equiv \frac{A_T^-}{e_{x+T}} + (1-\tau)\cdot\left(\frac{A_T^-}{\ddot{a}_{x+T}} - \frac{A_T^-}{e_{x+T}}\right) = \tau\cdot\frac{A_T^-}{e_{x+T}} + (1-\tau)\cdot\frac{A_T^-}{\ddot{a}_{x+T}} \tag{9}$$

at the beginning of every year as long as the policyholder is alive. Here, $\ddot{a}_{x+T}$ denotes the actuarial present value of an annuity due paying 1 at the beginning of each year while $(x+T)$ is alive.

At maturity, the policyholder can withdraw the remainder of the account value (or some guaranteed level, if applicable) from the VA. That is:

$$w_T = \max\{X_T^-, \max\left[G_T^A, \min\left(G_T^W, g_T^W\right)\right]\}. \tag{10}$$

This results in life-long annual payments of

$$\begin{aligned} c_X &\equiv \min\left\{\tfrac{w_T}{\ddot{a}_{x+T}}, \tau\cdot\tfrac{H_T}{e_{x+T}} + (1-\tau)\cdot\tfrac{w_T}{\ddot{a}_{x+T}}\right\} \\ &= \tfrac{w_T}{\ddot{a}_{x+T}} - \tau\cdot\max\left\{\tfrac{w_T}{\ddot{a}_{x+T}} - \tfrac{H_T}{e_{x+T}}, 0\right\}. \end{aligned} \tag{11}$$

If a GMIB is included in the contract, the policyholder can also choose to annuitize the guaranteed amount G_T^I at a guaranteed annuity factor $\ddot{a}_{x+T}^{guar}$, and thus receive annual payouts

$$c_I \equiv \frac{G_T^I}{\ddot{a}_{x+T}^{guar}} - \tau\cdot\max\left\{\frac{G_T^I}{\ddot{a}_{x+T}^{guar}} - \frac{H_T}{e_{x+T}}\ ,\ 0\right\} \tag{12}$$

Overall, the policyholder can therefore consume $c_A + \max\{c_X, c_I\}$ every year during his retirement. The time-T expected lifetime utility for the policyholder is thus

$$V_T(A_T^-, X_T^-, G_T^\cdot, H_T) = \sum_{t=0}^{\omega-x-T} \exp(-\beta t)\cdot {}_t p_x \cdot u_C(c_A + \max\{c_X, c_I\}), \tag{13}$$

subject to equations (9) to (12).

3 Implementation in a Black-Scholes Framework

For our implementation, we consider two investment possibilities only, namely a risky asset $(S_t)_{t\geq 0}$ and a risk-free asset $(B_t)_{t\geq 0}$. More specifically, akin to the well-known Black-Scholes-Merton model, we assume that the risky asset evolves according to the Stochastic Differential Equation (SDE)

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dZ_t\ ,\ \ S_0 > 0, \tag{14}$$

where $\mu, \sigma > 0$, and (Z_t) is a standard Brownian motion, while the risk-free asset (savings account) follows

$$\frac{dB_t}{B_t} = r\,dt\,,\ \ B_0 = 1 \quad \Rightarrow \quad B_t = \exp(r\,t).$$

In this setting, our optimization problem (6) takes the form

$$V_t^-(y_t) = \max_{C_t, w_t, \nu_t} u_C(C_t) + e^{-\beta} \int_{-\infty}^{\infty} \psi(\gamma) \left[q_{x+t} \cdot u_B(b_{t+1}|S'(\gamma)) + p_{x+t} \cdot V_{t+1}^-(y_{t+1}|S'(\gamma)) \right] d\gamma, \quad (15)$$

where $\psi(\gamma) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{\gamma^2}{2})$ is the standard normal probability density function, and $S'(\gamma) = S_t \cdot e^{\sigma\gamma+\mu-\frac{1}{2}\sigma^2}$ is the annual gross return of the risky asset, subject to

$$\begin{aligned}
y_t &= \{A_t^-, X_t^-, G_t^{\cdot}, H_t\}, \\
X_t^+ &= \left(X_t^- - w_t\right)^+, \\
A_t^+ &= A_t^- + I_t + w_t - \text{fee}_I - \text{fee}_G - \text{taxes} - C_t, \\
\text{fee}_I &= s \cdot \max\left\{w_t - \min(g_t^W, G_t^W)\right\}, \\
\text{fee}_G &= s^g \cdot (w_t - \text{fee}_I) \cdot \mathbf{1}_{\{x+t<59.5\}}, \\
\text{taxes} &= \tau \cdot \min\{w_t - \text{fee}_I - \text{fee}_G, (X_t^- - H_t)^+\}, \\
G_{t+1}^{W/A/D/I} &= \left(G_t^{W/A/D/I} \cdot \frac{X_t^+}{X_t^-}\right)^+, \\
A_{t+1}^- &= A_t^+ \cdot \left[\nu_t \cdot e^{\sigma\gamma+\mu-\frac{1}{2}\sigma^2} + (1-\nu_t) \cdot e^r - \kappa \cdot \left(\nu_t \cdot e^{\sigma\gamma+\mu-\frac{1}{2}\sigma^2} + (1-\nu_t)\cdot e^r - 1\right)^+\right], \\
X_{t+1}^- &= X_t^+ \cdot e^{-\phi} \cdot \left[\nu^X \cdot e^{\sigma\gamma+\mu-\frac{1}{2}\sigma^2} + (1-\nu^X)\cdot e^r\right], \\
b_{t+1} &= A_{t+1}^- + \max\{X_{t+1}^-, G_{t+1}^D\}, \\
0 \leq C_t &\leq A_t^- + I_t + w_t - \text{fee}_I - \text{fee}_G - \text{taxes}, \\
0 \leq w_t &\leq \max\left\{\, X_t^-\ ,\ \min\left\{\, g_t^W\ ,\ G_t^W\,\right\}\,\right\}, \text{and} \\
0 \leq \nu_t &\leq 1.
\end{aligned}$$

In the remainder of this section, we present a recursive dynamic programming approach for the solution of this optimization problem. In particular, we address practical implementation problems arising from the complexity associated with the high dimensionality of the state space.

3.1 Estimation Algorithm

The key idea underlying our algorithm is a discretization of the state space. More specifically, our approach to derive the optimal consumption, allocation, and – particularly – withdrawal policies consists of the following steps.

Algorithm 3.1.

(I) Discretize the state space of the values for $A, X, G^{\cdot}$, and H appropriately to create a grid.

(II) For $t = T$: for all grid points $(A, X, G^{\cdot}, H)$, compute $V_T^-(A, X, G^{\cdot}, H)$ via Equation (13).

(III) For $t = T-1, T-2, \ldots, 1$*:*

(1) Given V^{-}_{t+1}*, calculate* $V^{-}_{t}(A, X, G^{\cdot}, H)$ *recursively for each* $(A, X, G^{\cdot}, H)$ *on the grid via the (approximated) solution given by Equation* (15).

(2) Store the state-contingent withdrawal, consumption, and allocation choices for further analyses.

(IV) For $t = 0$*: For the given starting values* $A_0 = W_0 - P_0$*,* $X_0 = P_0$*,* $G^{\cdot}_0 = G^{\cdot}_1 = P_0$ *and* $H_0 = H_1 = P_0$*, compute* $V^{-}_0(W_0 - P_0, P_0, P_0, P_0)$ *recursively from equation* (15).

Storing the optimal choices in step $(III.2)$ not only allows us to analyze if and when a representative policyholder makes use of the withdrawal guarantee, which is the primary goal of our paper, but we may also determine the time zero value of the collected fees, as well as the time zero value of payouts to the policyholder or his beneficiaries via their expected present value under the risk-neutral measure $\mathbb{Q}$.[8] In particular, by comparing these values we can make inferences whether or not the contracted fee percentage within our representable contract sufficiently provides for the offered guarantees. However, before discussing our results in Section 4, the remainder of this section provides the necessary details about the implementation of the steps in Algorithm 3.1 as well as the choice of the underlying parameters.

3.2 Evaluation of the Integral Equation (15)

Since within step (III) of Algorithm 3.1 the (nominal) value function is only given on a discrete grid, it is clearly not possible to directly evaluate the integral in Equation (15). We consider two different approaches for its approximation by discretizing the underlying return space.

Since the integral entails the standard normal density function, one prevalent approach is to rely on a Gauss-Hermite Quadrature. This can be a very effective method in the presence of a smooth value function, as it typically requires only about 15 to 20 nodes to achieve a good approximation. A possible drawback, however, is that the Gauss-Hermite Quadrature method assigns relatively high weights to only a few nodes close to the distribution mean, while it leaves relatively many nodes with more extreme values and near-zero probability weights. In our case, this appears to be causing (minor) inaccuracies in some instances, which could potentially be due to irregularities in the value function. For instance, if close to maturity the policyholder's VA account value is significantly below the remaining guarantee, a small increase in the account value will have a very minor effect on the value function. However, ceteris paribus, as the VA account value becomes larger, changes in the account value increase the value function much more significantly.

Hence, to ascertain the accuracy of our approximation, we additionally consider a second approach. Note that our integral equation is of the form

$$K \equiv \int_{-\infty}^{\infty} \phi(u) F(\lambda(u))\, du, \tag{16}$$

[8] As is well-known, there exists a unique risk-neutral measure $\mathbb{Q}$ in our underlying Black-Scholes framework, under which the asset process will again evolve according to a Geometric Brownian Motion (as for the "physical" probability measure $\mathbb{P}$), where the drift is given by the risk-free rate r.

where $\phi(u) = \frac{1}{\sqrt{(2\pi)}} \exp(\frac{1}{2}u^2)$ is the standard normal density function, and $\lambda(u) = \exp(\sigma u + \mu - \frac{1}{2}\sigma^2)$ corresponds to the annual stock return S_{t+1}/S_t, and

$$F(x) \equiv q_{x+t} \cdot u_B(b_{t+1}|\frac{S_{t+1}}{S_t} = x) + p_{x+t} \cdot V^-_{t+1}(y_{t+1}|\frac{S_{t+1}}{S_t} = x).$$

We now divide the return space $(-\infty, \infty)$ into $M > 0$ subintervals $[u_k, u_{k+1})$, for $k = 0, 1, \ldots, M-1$, where we set $-\infty = u_0 < u_1 < \ldots < u_{M-1} < u_M = \infty$. Then, a consistent approximation of the integral (16) is given by

$$K \approx \sum_{k=0}^{M-1} \Phi(u_{k+1}) \cdot [a_k - a_{k+1}] + exp(\mu) \cdot \Phi(u_{k+1} - \sigma) \cdot [b_k - b_{k+1}], \tag{17}$$

where $\Phi(.)$ is the standard normal cdf, $a_k \equiv \frac{x_{k+1} \cdot \psi_k - x_k \cdot \psi_{k+1}}{x_{k+1} - x_k}$, $b_k \equiv \frac{\psi_{k+1} - \psi_k}{x_{k+1} - x_k}$ for $k = 0, \ldots, M-1$, $a_M = b_M \equiv 0$, $x_k = \lambda(u_k)$ represent the gross returns, and $\psi_k \equiv F(x_k)$ are the function values evaluated at returns x_k (see Appendix C for a derivation of (17)). With this approach, we have the freedom to choose number $(M-1)$ and location (x_k) of all nodes. This provides us with more flexibility than the Gauss-Hermite Quadrature method, and we will use it to allocate relatively more nodes in the proximity of $u = 0$, so as to spread out the probability weights more equally. It is important to note, however, that the values ψ_k cannot be calculated directly, but need to be derived from the value function grid at time t. Here, we rely on multilinear interpolation for function values in between grid points, and exponential extrapolation for values beyond the boundaries of the grid.

3.3 State Variable Grids

As discussed above, the choice-dependent state variables in the basic setup are A^-_t, X^-_t, $G^{\cdot}_t$ and H_t, where only the first two evolve stochastically over time. Since policyholders are more sensitive to nominal differences in account values when account values are small, and since it is crucial for the computational feasibility of the program to keep the grid sizes manageable, we refrain from choosing an equidistant grid for the account values A^-_t and X^-_t. Instead, we use a nonlinear grid that increases according to triangular numbers. For instance, using an initial step size of 1000, the grid points are $0, 1000, 3000, 6000, 10000, 15000$, etc. When trying to “locate”, for instance, a value A on the A^-_t grid, we can then make use of the fact that the n-th triangular number is given by $n \cdot (n+1)/2$. The model is implemented with an initial step size of $2,000$ for X^-_t and $5,000$ for A^-_t, up to maximum values of $1,980,000$ and $2,325,000$, respectively.

The guarantee account $G^{\cdot}_t$ and the tax base H_t are bounded from above by their starting value $G^{\cdot}_0 = P_0$ and $H_0 = P_0$, respectively. For the guarantee account, we divide the interval $[0, G^{\cdot}_0]$ into 11 equidistant grid points, including the boundaries. Now note that the tax base can never fall below the benefits base: Both start off at the same level, namely the principal, and both are only affected by withdrawals. The benefits base, however, is reduced by *at least* the withdrawal amount (and possibly more if the withdrawal amount exceeds the annual guaranteed amount); the tax base, on the other hand, is reduced *at most* by the withdrawal amount (namely if withdrawals come from the principal, not earnings). Therefore, we only need to consider state vectors for which $H_t \geq G_t^{\cdot}$.

Our numerical analysis yields very similar results – across methods (Quadrature versus (17)) and minuteness of the state space grid – for all variables of interest, thus ensuring the accuracy of our approximations.

3.4 Parameter Assumptions

For our numerical analysis, we consider a male policyholder, who at contract initiation is 55 years old, and whose mortality follows the U.S. actuarial mortality table.[9] We assume an initial wealth level of $W_0 = 200,000$, of which he invests half ($P_0 = 100,000$) in the VA with a maturity of $T = 15$ years. His annual net income is $I_t = 40,000$, collected at the beginning of each year.

Furthermore, we suppose that the policyholder is an expected (discounted) utility maximizer with a CRRA utility function with risk aversion parameter $\gamma = 3$, and subjective discount rate $\beta = r$. Upon the policyholder's death, his bequested amount is converted into a risk-free perpetuity (reflecting that upon the beneficiary's death, remaining funds will be passed on to his own beneficiaries, and so on) at constant rate r. We assume the beneficiaries have the same preferences as the policyholder, and that his bequest motive is $B = 1$. That is, if he leaves bequest amount x (net of taxes), the discounted expected bequest utility for the policyholder is given by:

$$\frac{1}{1 - \exp(-\beta)} \cdot B \cdot u_C([1 - e^{-r}] \cdot x).$$

Taxes are assumed constant at $\tau = 0.3$ and $\kappa = 0.15$. Early (or excessive) withdrawals are subject to a surrender fee of $s = 5\%$ (collected by the insurer) of the excessive withdrawal amount, and any withdrawals prior to age 59.5 are subject to an early withdrawal penalty of $s^g = 10\%$ (collected by the government). During the first five years of the VA contract, no free withdrawals can be made. Starting at the end of year 6, the policyholder can withdraw up to 20% of the initial investment P_0 for free, regardless of his current account value. That is:

$$g_t^W = \begin{cases} 0 & : t \leq 5 \\ 20,000 & : t > 5. \end{cases}$$

Finally, for the financial market we assume $r = 4\%$, $\mu = 8\%$, and $\sigma = 15\%$.

3.5 Monte Carlo Simulation to Determine Appropriate Grids and Analyze Results

Using Algorithm 3.1, we can determine (and record) the policyholder's optimal decision variables at all times and for all state space grid points. However, to obtain an idea about the necessary size of the grids, we also conduct Monte Carlo simulations of $N = 10,000,000$ paths for the evolution of the risky asset from $t = 0$ to $T = 15$, and also N draws of whether the policyholder will die prior to maturity of the VA and, if so, in what year.

Our simulations indicate that the 99.9th percentile of X_T^- is $1,400,000$. We therefore deem a maximum grid value of 1.98 million to be sufficient. Next, we find the 99.9th percentiles of A_T^- and $A_T^- + X_T^-$ to be around $600,000$ and $1,560,000$, respectively. We hence choose the grid for the outside account to go up to

[9] We use the "2006 period life table" by the U.S. Social Security Agency to model mortality, cf. `http://www.ssa.gov/OACT/STATS/table4c6.html`.

2.3 million, to ensure that we can accurately value the policyholder's lifetime utility without having to rely heavily on extrapolation, even in case he chooses to fully surrender his filled-up VA account and add it on to his outside portfolio.

	Mean	95%-ile	99%-ile	99.9%-ile	max. grid point
A_T^-	$282,000$	$378,090$	$495,560$	$606,650$	$2,325,000$
X_T^-	$272,590$	$606,920$	$902,650$	$1,411,300$	$1,980,000$
$A_T^- + X_T^-$	$554,580$	$876,180$	$1,127,600$	$1,562,300$	n/a

In addition, the simulation procedure allows us to derive various result statistics such as the distribution of benefits and tax base at maturity, which we report in Section 4. Moreover, by adjusting the drift parameter of the risky asset to r (cf. Footnote 8), the simulation allows us to determine the time-zero value of the collected guarantee fees, and to compare that to the overall payouts that are strictly due to the presence of the guarantees.

4 Results

4.1 Withdrawal Behavior

The primary research question of this paper is whether it is optimal for policyholders to withdraw prematurely from their VA and, if so, how much. In this regard, we find that optimal early withdrawals are an exception rather than the norm. More specifically, in roughly 67% of all cases, our representative policyholder will not withdraw anything from his VA account prior to maturity, and in only 5.8% of all scenarios will he empty his guarantee account.[10] Moreover, we observe very few withdrawals exceeding the guaranteed amount (only in 0.7% of all cases will it be optimal to make excessive withdrawals over the lifetime of the policy), which appears to be caused by the 5% charge on (excess) withdrawals.

In contrast, when analyzing the policyholder's optimal behavior with arbitrage pricing theory and in the absence of taxation, the existing literature predicts relatively frequent early surrenders of VA policies (see e.g. Chen et al. (2008)), especially for very high account values. We do not observe such patterns with our approach. In addition, the referenced literature estimates equilibrium guarantee fees significantly above concurrent market rates. Based on our analysis, however, an annual guarantee fee of $\phi = 50$ bps seems sufficient to cover the expected costs of the guarantee: We find that the time zero value of the collected guarantee fees is $\mu_F = 6,252$ (plus an additional 19 in excess withdrawal charges), while the risk-neutral value of payments attributable to the GMWB is only $\mu_G = 4,558$.

We do not record any withdrawal activity during the five-year accumulation period of the contract. Afterwards, when the policyholder is allowed to withdraw $20,000$ p.a. due to the GMWB, we observe withdrawals primarily when they are tax-free and the guarantee is in the money (or when the VA account significantly exceeds the outside account, which we will analyze below). More specifically, Figures 1 and 2

[10] The results presented in this section are based on an approximation of the return space as described in (17). Applying a Gauss-Hermite Quadrature method yields very similar numbers.

display the optimal withdrawal behavior w_t as a function of the VA account value X_t^-, at different points in time and for different levels of the benefits base (that is, the guarantee account). The figures also include the maximum possible withdrawal amount as a reference. In all cases, the policyholder can withdraw $20,000$ free of charge. For each graph, withdrawals for VA account values above $150,000$ are mostly zero, so that we exclude them for reasons of clarity. We also find withdrawal patterns in these regions to be rather independent of the level of the outside account, A_t^-.

Figure 1(a) shows the optimal withdrawal behavior one year prior to maturity in case both guarantee benefits and tax base are at their original level of $100,000$, i.e. no previous withdrawals have been made. At this time, only two opportunities for withdrawing remain, namely at $t = 14$ and $t = 15$. Since the policyholder is only guaranteed $20,000$ each time, the guarantee "strike" price is only $40,000$ (despite the higher level of the benefits base). As the graph shows, we can divide the optimal withdrawal rule into four sections.

For low account values ($X_{14}^- \leq 20,000$), the guarantee is deep in the money and the policyholder is limited in the amount he can withdraw; he thus optimally withdraws at the guaranteed level, because he might otherwise not receive the underlying funds at all. In this scenario he makes full use of his guarantees: the VA account drops to $X_{14}^+ = 0$, and the benefits base to $G_{15}^\cdot = 80,000$, so that he will withdraw another $20,000$ at maturity directly from the guarantee.

As we increase the account value, it becomes optimal to withdraw a substantial part of the funds – in fact, the policyholder withdraws around 80% of his VA account value. Since $X_{14}^- < G_{14}^\cdot$, the benefits base will then also be reduced by 80%, which results in $G_T^\cdot \approx 20,000$, i.e. the guaranteed annual amount for that year. The policyholder now holds around $10,000$ in his account, but is guaranteed a withdrawal of $20,000$ at the end of next year, so that one year away from retirement, his guarantee is now deep in the money. In other words, even though the "strike price" of the guarantee is only $40,000$ due to limited withdrawal opportunities, the policyholder can still make use of his much higher benefits base. Since at maturity there is no value to the benefits base in excess of $20,000$, the policyholder has an incentive to reduce it to that point, i.e. the optimal strategy is to withdraw whatever he can, while leaving $20,000$ in the guarantee account for the final period. However, this strategy comes at the cost of a 5% charge on all withdrawals exceeding $20,000$. Therefore, at some point (in our case around $X_t^- = 70,000$) withdrawing substantial amounts becomes too expensive and the fees outweigh the benefits of making full use of the remaining guarantee and having downside protection in the final period.

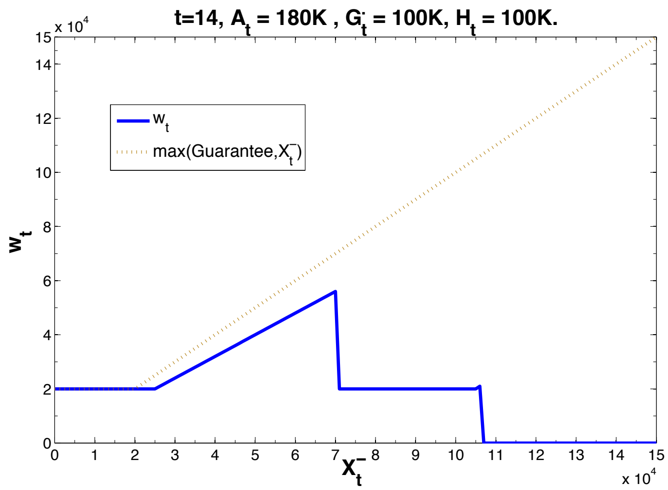
For account values above ca. $110,000$, the policyholder does not withdraw anything. This appears to be due to the consideration that the policyholder would have to pay 30% taxes on all withdrawals when the account value is above the tax base of $100,000$. Since he cannot earn interest on the taxes he pays today and since the guarantee is deep out of the money, he prefers to leave the money inside the VA account for another year.

Finally, for VA account values between ca. $70,000$ and $110,000$ the policyholder withdraws the guaranteed amount. In this range, the guarantee is deep out of the money, and bringing it into the money for the final period (through excessive withdrawals) would be too costly. On the other hand, withdrawals below the tax base of $100,000$ are tax free, and the policyholder wants to take advantage of the lower tax rate on returns from the outside account, therefore withdrawing at the guaranteed level.

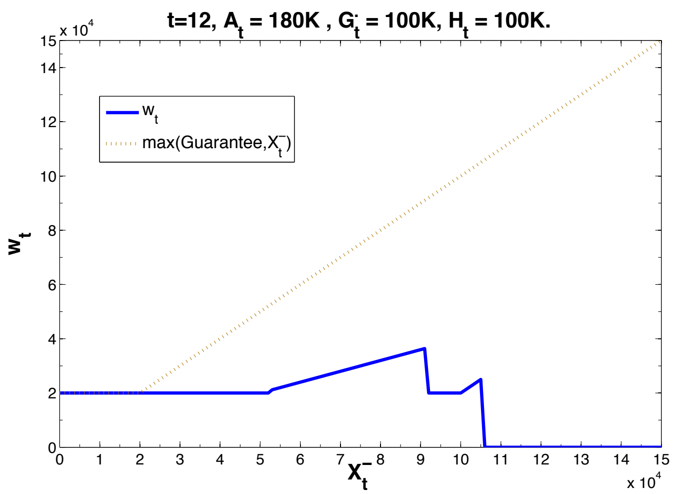
Figure 1(b) displays the optimal withdrawal behavior for the same levels of state variables, but two years earlier. Therefore, the account value has more time to grow and the incentive to withdraw due to the guarantee being in the money is reduced considerably. There are now four chances to subsequently withdraw the guaranteed $20,000$, making the guarantee "strike" $80,000$. This is still below the benefits base of $100,000$, and for appropriate account values (in this case between around $50,000$ and $90,000$), the policyholder has incentives to withdraw excessively, in order to reduce the benefits base to around $60,000$, which allows him to withdraw the guaranteed amount of $20,000$ in each of the remaining three years ($t = 13, 14, 15$). Note that because the policyholder now only withdraws 40% of his account value (instead of 80% as in Figure 1(a)), his excess withdrawal charges are lower and he thus continues to withdraw at higher account values than he would at $t = 14$.

Finally, at $t = 10$, there are six withdrawal opportunities remaining, so that the guarantee "strike" equals the benefits base. In other words, the policyholder has time to withdraw 100% of his original investment. Therefore, as shown in Figure 1(c), he will withdraw at the guaranteed level because he does not want to incur excess withdrawal charges given that he can withdraw the money "for free" in the following periods.

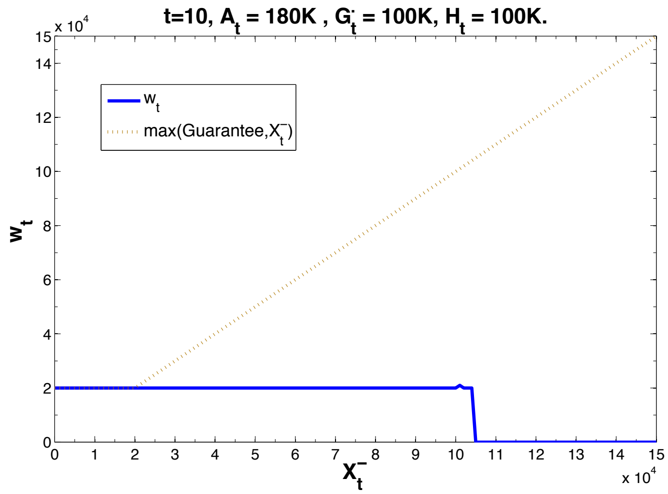
Figures 2(a) and 2(b) describe the optimal withdrawal rule at $t = 14$ when the benefits base is at $60,000$ and $20,000$, respectively. We observe that in comparison to Figure 1(a), where $G_t^{\cdot} = 100,000$, the optimal withdrawal strategy remains the same when the VA account value is below $20,000$ or above $70,000$. In between these values, however, the policyholder's incentives to withdraw decline with the benefits base. In Figure 2(a), the excess withdrawals are less pronounced: as before, the policyholder wants to reduce the guarantee account to $20,000$ for the final period. Since the benefits base is now at $60,000$, this corresponds to withdrawing $2/3$ of the VA account X_{14}^{-}, as compared to the 80% withdrawal rate when the benefits base was $100,000$. In addition, we find that for account values close to the benefits base, no withdrawals are made. This may be explained by the opportunity for tax-free growth: Since the next-period account value X_{15}^{-} will most likely fall below the tax base of $100,000$, the policyholder can withdraw his account value tax-free at maturity. Conversely, any amount withdrawn could be invested equally well in the outside account, but returns will be taxed at 15%. If the time-14 account value exceeds $70,000$, this seems to be outweight by the higher taxation rate of 30% (above the tax base) inside the VA account. On the other hand, for account values below $50,000$ the policyholder prefers to protect himself against downside market risk by bringing the guarantee in the money. This allows him to secure as much of the "guaranteed" amount (i.e. the current benefits base) as possible over the remaining period(s). However, for account values closer to the benefits base, excessive withdrawals become unneccessary as it will be increasingly likely that this guaranteed amount can be withdrawn free of charge at maturity (if the market performs regularly in the final period). Therefore, the policyholder will prefer to stop withdrawing excessively as the account value approaches the benefits base. In Figure 2(b), this "ditch" of zero withdrawals covers the entire interval between benefits base and $X_{14}^{-} = 70,000$, which is consistent with our interpretation. Intuitively, when $G_t^{\cdot} = 20,000$, any excessive withdrawals will automatically deplete the guarantee account and not provide any protection for the future. We thus observe very limited withdrawal activity when the benefits base is low.



(a) $t = 14$

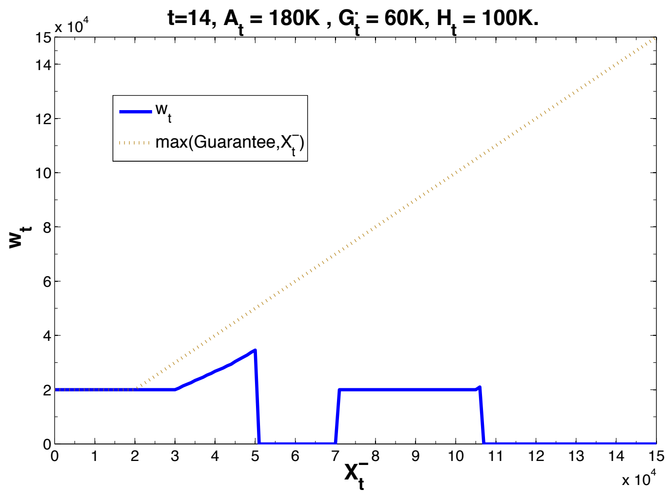


(b) $t = 12$

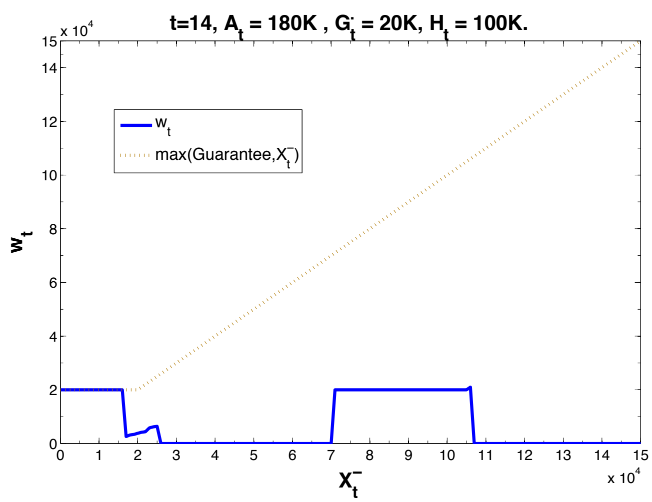


(c) $t = 10$

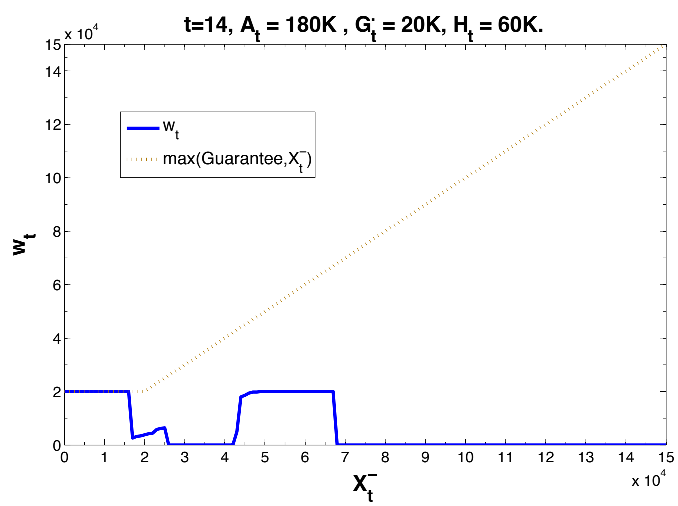
Figure 1: Withdrawal behavior as a function of X_t^-, when $G_t^{\cdot} = 100,000 = H_t$.



(a) $t = 14, G_t^{\cdot} = 60,000, H_t = 100,000$



(b) $t = 14, G_t^{\cdot} = 20,000, H_t = 100,000$



(c) $t = 14, G_t^{\cdot} = 20,000, H_t = 60,000$

Figure 2: Withdrawal behavior as a function of X_t^-.

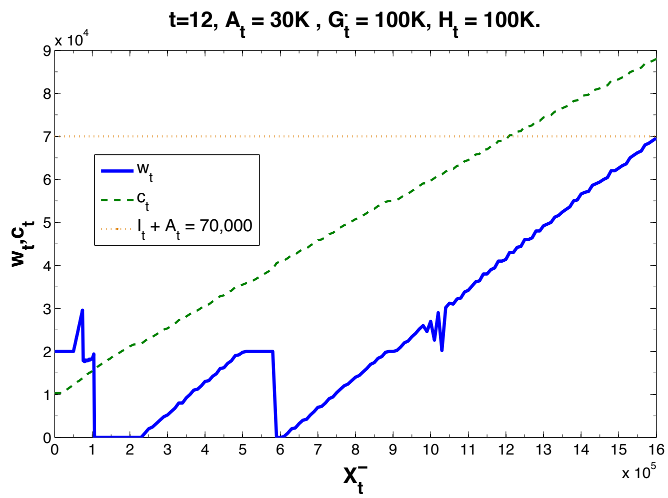
Therefore the optimal withdrawal strategy close to maturity for account values below the tax base can be summarized as follows: for account values below the annual guarantee, the policyholder withdraws at the guaranteed level. As the account value increases (and if the benefits base is sufficiently large), excessive withdrawals can be observed, followed by a period of no withdrawals (if the benefits base is sufficiently low) and finally a "hump" where he withdraws at the guaranteed level. Furthermore, Figure 2(c) shows that this "hump" moves with the tax base. This graph demonstrates that withdrawals are zero when the account value exceeds the *tax base*, indicating that the preferred tax treatment of VAs is the primary motivation for not withdrawing for high account values.

Our analysis shows that the excess withdrawal fee, the value of the guarantee and tax treatments are crucial determinants of individual withdrawal behavior. They jointly lead to withdrawing not being optimal for large values of the VA account, and a very nuanced policyholder behavior when the account value is low. These results turn out to be fairly independent of the outside account value, with one exception: As shown in Figure 3(a), when the VA account is considerably higher than the outside account, the policyholder will start withdrawing. We attribute this to consumption smoothing and the policyholder's desire to limit his overall exposure in the risky investment. Specifically, the graph shows consumption (dashed line) and withdrawal choices when the outside account is rather low ($A_t^- = 30,000$). As a reference, the horizontal (dotted) line represents the amount available for consumption in the absence of withdrawals ($A_t^- + I_t = 70,000$). We see that for very large X_t^-, consumption exceeds this level so that the policyholder is required to withdraw from the VA in order to meet his consumption needs. However, the corresponding withdrawals are much higher than the excess consumption, and we also observe withdrawals when the outside account value would be sufficient to meet the policyholder's demand for consumption. This can be explained by the policyholder's preference for a balanced overall portfolio. Recall that inside the VA the entire investment is allocated to the risky asset. When the discrepancy in account values is high, it appears optimal to withdraw from the VA and to invest the funds risk-free in the outside account, in order to obtain a more desirable overall risk allocation (cf. Figure 3(b)). As the value in the outside account increases, these effects disappear (cf. Figure 3(c)). Our Monte Carlo analysis reveals that consumption smoothing is in fact a very rare event. A substantially larger fraction of policyholders withdraws when $X_t^- > H_t$, indicating that risk allocation has a more significant effect on withdrawal behavior. In fact, our analysis indicates that in our base case scenario, a policyholder withdraws on average $5,030$ (that is over 40% of his average aggregate withdrawals) in order to adjust his investment portfolio, but only 2 for the purpose of consumption smoothing.

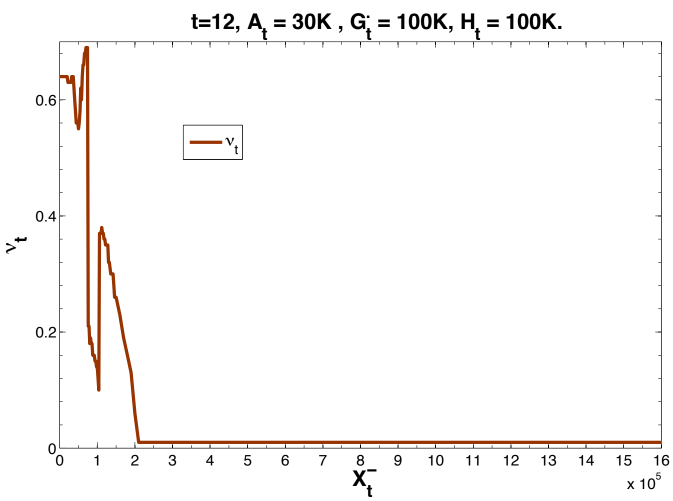
4.2 Overall Portfolio Allocation

According to Merton (1969), an expected utility maximizer with a CRRA utility function will optimally choose the proportion of his investment allocated in the risky asset constant over time and wealth levels. For our parameters, this so-called *Merton Ratio* is given by

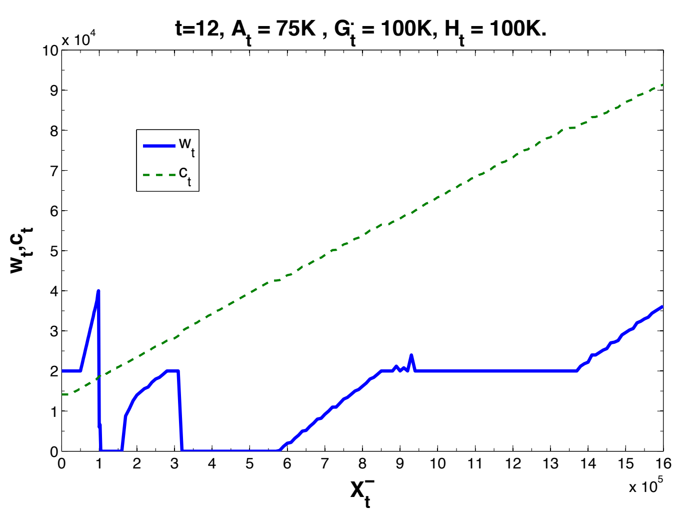
$$\frac{\mu - r}{\sigma^2 \cdot \gamma} = \frac{0.08 - 0.04}{0.15^2 \cdot 3} \approx 0.5926,$$



(a) $t = 12, A_t^- = 30,000$



(b) ν_t



(c) $t = 12, A_t^- = 75,000$

Figure 3: Withdrawal, consumption and risk allocation behavior as a function of X_t^-.

which comes close to the *rule of thumb* that an investor should optimally hold 60% of his assets in stocks and 40% in bonds (cf. Gerber and Shiu (2000)). When accounting for (current and future) income, which essentially presents a risk-free bond, and considering that the outside account is valued post-tax, we find that the overall risk exposure by the policyholder in our model also comes close to the Merton Ratio, as illustrated by the following numerical example: Let $t = 11$, $A_{11}^- = 180,000$, $X_{11}^- = 306,000$, and $G_{11} = H_{11} = 100,000$. At this point of the state space grid, the policyholder optimally chooses $w_{11} = 0$, $c_{11} \approx 37,087$, and $\nu_{11} \approx 0.25$. He receives immediate income $I_{11} = 40,000$, and will also receive three more installments of future income ($t = 12, 13, 14$). At risk-free rate $r = 4\%$, the time 11 net present value of his income stream is therefore approximately $111,000$. Hence, the net present value of the outside account is approximately $180,000 + 40,000 - 37,087 + 0 + 111,000 = 293,913$. Observe that the guarantee is very deep out of the money and nearly worthless, but the policyholder owes roughly 30% taxes on $X_{11}^- - H_{11} = 206,000$ of his VA account. Therefore the "net" value of the VA account is $306,000 - 0.3 \cdot 206,000 = 244,200$. Hence, the total (approximate) risk exposure is

$$\frac{100\% \cdot 244,200 + 25\% \cdot 293,913}{244,200 + 293,913} = 0.5904,$$

which is close to our Merton Ratio of 0.5926.

4.3 Comparison to "Simple" Withdrawal Strategies

In order to compare our results to "simple", reduced-form withdrawal strategies, we may compare the actuarial present values of the collected fees as well as payments made through to the GMWB option (cf. Table 1). In general, withdrawing money sooner rather than later reduces the VA account value and, thus, future fee payments. The effect of earlier withdrawals on the payouts from the GMWB option is somewhat ambiguous: Suppose, for example, in our model that the VA account has fallen to $X_6^- = 60,000$, without any prior withdrawals (i.e. $G_6^W = 100,000$). Withdrawing $w_6 = 20,000$ would reduce both account values to $X_6^- = 40,000$ and $G_6^W = 80,000$, respectively. The GMWB guarantee takes over in case $0 = X_t^- < G_t^W$. This is more likely to occur when the VA account and Guarantee account are in a $1 : 2$ ratio, as compared to the $6 : 10$ ratio from before the withdrawal of w_6. Hence withdrawing early *increases* the interim value of the GMWB option if the guarantee is *in* the money. Conversely, consider the same example, but with $X_6^- = 140,000$. If $w_6 = 0$, the VA to Guarantee account ratio stays at $1.4 : 1$ while withdrawing the annual guaranteed amount $w_6 = 20,000$ yields $X_6^- = 120,000$ and $G_6^W = 80,000$, and thus a ratio of $1.5 : 1$. The higher ratio indicates that more extreme financial market scenarios are required to bring the guarantee in the money at some point prior to maturity. This shows that withdrawing early will *reduce* the interim value of the GMWB option if the guarantee is *out of* the money. We conclude that withdrawing early moves the guarantee further in or out of the money, respectively. Therefore, we expect that the time zero value of collected guarantee fees, $E^{\mathbb{Q}}[F]$, is lower for high-withdrawal strategies, and that the risk-neutral value of withdrawals through the GMWB option, $E^{\mathbb{Q}}[G]$, increases with the extent of in-the-money withdrawals and decreases with out-of-the-money withdrawals, respectively.

As the first intuitive strategy, we consider a policyholder withdrawing the maximal possible amount without incurring withdrawal fees:

$$w_t \equiv \min\{g_t^W, \max\{X_t^-, G_t^W\}\}.$$

We obtain risk-neutral values of $\mu_F = 4,559$ and $\mu_G = 4,262$. While the insurer can expect roughly the same payout through the guarantee as for our "optimal" strategy, much less fees will be collected. And although the fees are still sufficient to provide for the guarantee under the considered parameter values, the resulting "fair" guarantee fee will be higher than when assuming "optimal" policyholder behavior.

A second strategy, that is related to strategies employed in the life insurance practice, is to assume that policyholders withdraw if and only if the guarantee is "in the money", that is:

$$w_t \equiv \min\{g_t^W, G_t^W\} \cdot \mathbf{1}_{\{X_t^- <= G_t^W\}}.$$

This results in time zero values of $\mu_F = 5,925$ and $\mu_G = 4,761$. While these values are similar to the results for "optimal" behavior, we should clarify that the withdrawal patterns are not. For instance, while the average aggregate withdrawal amounts are fairly close ($12,000$ for the optimal and $14,000$ for the deterministic strategy), under the latter all withdrawals happen when the account value lies below the tax base. In particular, since all withdrawals are fully taxable and non-excessive, guarantee account and tax base are identical at all times. However, under the "optimal" strategy we find that more than 40% of the withdrawals (on average $5,000$ of the $12,000$) can be attributed to overall risk balancing when the account value exceeds the tax base, while on average nearly $7,000$ is withdrawn when the account value lies below the benefits base. These deviations are important to consider as they might result in different net present values as well, once we allow for a more complex framework to explain withdrawal behavior. We conclude that neither of the presented deterministic strategies appear to sufficiently approximate the nuances of optimal withdrawal behavior.

4.4 Sensitivity Analysis

We test the sensitivity of our estimation results with respect to the underlying core parameters. Results are displayed in Table 1. In our analysis of the base case scenario, the excess withdrawal fee of $s = 5\%$ appeared to be a decisive deterrent to early (excessive) withdrawals. We now observe that decreasing s to 2% yields similar overall results, although in a few instances the policyholder will now make some withdrawals during the accumulation phase of the contract. The latter is likely caused by the reduction of penalty fees from $s_g + s = 15\%$ in the base case to 12%. Overall, changes in withdrawal behavior are minimal as the altered parameter only affects excess withdrawals, which were a rather rare occurence in the base case. In other words, if a policyholder prefers not to withdraw (at least) the guaranteed amount, a lower excess withdrawal fee will not change that. In that spirit, when the fee is abandoned completely ($s = 0$), withdrawals during the payout phase of the contract are similar to the base case, but the policyholder withdraws on average an additional $4,400$ during the accumulation phase. As a consequence, the insurer's liabilities (i.e. the average guarantee payout) will be reduced by more than half, while collected fees are nearly as high as in the base case. Therefore, the insurer would *increase* his expected net profit when eliminating the guarantee fee. In

Table 1: Sensitivity Analyses

	Base Case	$s = 2\%$	$s = 0$	$\nu_X = 60\%$	$\sigma = 25\%$	No Taxes	Fee as % of $G^{\cdot}_t$	incl. GMDB	$w_t =$ guarantee	w/d if $X_t^- <= G^{\cdot}_t$
$\mathbb{E}^{\mathbb{Q}}$[Fees] [a]	6,252	6,205	5,907	6,256	5,133	4,734	4,629	6,353	4,559	5,925
$\mathbb{E}^{\mathbb{Q}}$[Excess-Fee]	19	54	0	18	991	64	53	29	0	0
$\mathbb{E}^{\mathbb{Q}}$[Guarantee]	4,558	4,506	2,136	4,564	11,558	4,746	4,661	4,441	4,262	4,761
$\mathbb{E}^{\mathbb{Q}}$[Death Benefit]	–	–	–	–	–	–	–	806	–	–
$\%$(Guarantee > 0)	24.34%	26.15%	31.94%	24.03%	46.84%	28.16%	26.37%	19.93%	29.54%	33.97%
$\mathbb{E}[G^{\cdot}_T]$ [b]	87,847	86,794	81,731	92,955	32,965	11,949	86,147	89,727	8,770	85,820
$\%(G^{\cdot}_T > 0)$	94.13%	93.69%	93.41%	94.07%	51.26%	53.47%	93.51%	96.05%	11.65%	89.68%
$\%(G^{\cdot}_T < P_0)$	32.68%	34.47%	35.19%	9.04%	81.39%	93.99%	36.25%	30.78%	93.94%	20.69%
$\mathbb{E}$[H]	93,063	92,466	89,597	93,110	68,468	71,426	92,370	94,934	67,994	85,820
$\%(Lapses)$ [c]	0	0.11%	1.14%	0	0.32%	0	0.10%	0	0	0
$\mathbb{E}[\max w_t]$	4,665	5,225	11,065	1,862	43,672	18,520	5,519	4,126	18,788	4,137
$\mathbb{E}$[agg. w/d & $t \leq 6$] [d]	0	435	4,374	0	13,169	82	426	0	0	0
$\mathbb{E}$[agg. w/d]	12,084	12,967	16,374	7,012	83,193	88,791	13,625	10,371	91,230	14,180
$\mathbb{E}$[w/d & c_t]	2	1	1	0	0	2	2	2	0	0
$\%(w_t > 0 \| X_t^- <= H_t)$ [e]	6.59%	6.80%	5.60%	6.65%	29.13%	35.12%	6.91%	5.97%	41.78%	20.68%
$\mathbb{E}[w_t \| X_t^- <= H_t]$	1,317	1,382	1,806	1,314	6,410	5,580	1,404	885	8,356	4,137
$\mathbb{E}$[agg. w/d & $X_t^- <= H_t$]	6,953	7,142	6,047	6,917	29,235	19,500	7,249	5,222	21,393	14,119
$\%(w_t > 0 \| X_t^- > H_t)$ [f]	5.70%	5.73%	5.74%	0.00169%	17.18%	75.21%	6.25%	5.45%	85.97%	86.00%
$\mathbb{E}[w_t \| X_t^- > H_t]$	564	597	661	0	12,083	13,671	661	561	17,195	0
$\mathbb{E}$[agg. w/d & $X_t^- > H_t$]	5,030	5,297	5,816	0	32,579	69,033	5,856	4,998	69,809	0

Explanations:
[a] All under risk-neutral measure $\mathbb{Q}$: Actuarial present value (APV) of fees collected by insurer; APV of excessive withdrawal fees charged to policyholder; APV of payouts made to policyholder only due to guarantee(s) (i.e. when $X_t^- = 0$); APV of payout due to death benefit guarantee (if GMDB is included); Probability that guarantee will become effective.
[b] Average level of benefits base at maturity; Percentage that the benefits base is not depleted; Percentage at least one withdrawal is made; Average tax base at maturity.
[c] Percentage of lapsed policies; Average biggest (single) withdrawal.
[d] Average (aggregate) withdrawal amount during accumulation phase; average aggregate withdrawal amount; average (aggregate) withdrawal amount due to consumption smoothing.
[e] All during payout phase: Probability a withdrawal is made when VA account below tax base; average (single) withdrawal when VA account below tax base; average *aggregate* withdrawal amount while VA account below tax base.
[f] Same as [e], but for VA account *above* tax base.

our model and with the specified parameter assumptions, $s = 0$ thus appears to be Pareto optimal.

The next column displays withdrawal behavior when only 60% of the VA account is allocated in the risky asset. We see that when the account value exceeds the tax base, withdrawing is never optimal. As explained earlier, this is likely due to the tax-deferred growth opportunities inside the VA. The result also implicitly supports our argument that in the base case withdrawals above the tax base are caused by the policyholder's desire to balance his overall risk portfolio.

We find that withdrawal behavior is highly sensitive to the volatility of the market, as column five illustrates. This can be partially explained by the reduced Merton and Sharpe ratios (due to the increase in σ, ceteris paribus), as the policyholder now desires a lower risk exposure. Since the allocation inside the VA cannot be changed, he has now more incentives to withdraw prematurely. In addition, we observe a sizable increase in both frequency and severity of withdrawals for account values below the tax base. More work is required in this area, which we leave for future research.

Column six illustrates that when excluding taxes from our consideration, withdrawals would be optimal in most circumstances, leaving the policyholder to roughly break even on average. This might in parts explain why the previous literature finds a guarantee fee of $\phi = 0.5\%$ to be too low to cover the insurer's expenses.

Next we analyze the case where the guarantee fees are proportional to the benefits base (i.e. annual fee $= 0.005 \cdot G_t^{\cdot}$). We find that optimal withdrawal behavior is close to the base case, indicating that in the base case our model provides little incentives for policyholders to withdraw early when the VA account value is high in order to avoid paying substantial guarantee fees in the future.

Finally, we observe that including a GMDB option (without raising the overall guarantee fee) leads to a small reduction in withdrawals ($5,200$ versus $7,000$ in the base case) when the account value lies below the tax base. Intuitively, withdrawing below the benefits base will negatively impact the value of the GMDB as it reduces the level of protection for the beneficiaries. And although these changes are not sufficient to outweigh the additional cost of the death benefit guarantee (ca. 800), our findings suggest that adding a GMDB rider to a VA should be cheaper when a GMWB is already included. Moreover, note that the guarantee payouts due to the GMWB are a multiple of the expected payouts from the GMDB.

In addition, we find that overall withdrawals are not sensitive to the level of initial wealth, W_0. While this result may be different for policyholders with non-CRRA preferences and in the presence of more complex financial market structures, it is a sign that it *might* be possible to predict policyholder behavior without having to know their respective levels of initial wealth, which in practice would be very hard to observe.

5 Conclusions and Future Research

In this paper, we present a lifetime utility model to analyze the withdrawal behavior for a representative holder of a variable annuity with a GMWB, and possibly other guarantees. We numerically solve the policyholder's decision problem for a return-of-investment GMWB in the classical Black-Scholes framework, by relying on recursive dynamic programming techniques.

Our numerical results indicate that policyholders mostly withdraw when their account value is below the tax base. However, even for account values below the tax base, we find the optimal withdrawal patterns and the interaction of the underlying drivers to be quite multifarious. In particular, excess withdrawals are possible in select cases. In addition, we observe optimal withdrawals when the outside account is very low and the VA account is very high. These patterns appear to be consistent with the risk-averse policyholder's desire to smooth consumption and to balance his overall risk allocation in VA and his outside portfolio.

Overall, we find that in most scenarios (more than two thirds), the policyholder will not make any withdrawal during the contract phase, and only in about 6% of all cases will the guarantee be exhausted. This leads us to conclude that the assumed guarantee fee of 0.5% p.a. appears to sufficiently provide for the expected payouts from the GMWB, contrary to findings in the existing literature that are based on No-Arbitrage pricing techniques.

In addition we find that abandoning the excess withdrawal fee will induce policyholders to increase their excessive withdrawal activities, to the extent that the insurer's liabilities (through the guarantee) are drastically reduced and net profits more than double. Lastly, we observe that optimal withdrawing behavior appears to be highly sensitive to the underlying market volatility. A more thorough analysis of this phenomenon is beyond the scope of this paper, and will be left for future research.

Further interesting extensions of our model could include more advanced policyholder preference specifications, e.g. by considering *Epstein-Zin preferences* via a recursive utility function (cf. Epstein and Zin (1989)). Another improvement would be the introduction of *unemployment risk* and *subjective mortality risk*: How does withdrawal behavior change when the policyholder is unemployed (terminally ill), and how does the policyholder react to the possibility of becoming unemployed (terminally ill) in the future? Finally, the model developed here can be used to ultimately compute the competitive equilibrium premium (i.e. guarantee fee) for any VA policy, just as long as policyholder characteristics are well-understood.

Some of these suggested extensions seem rather straightforward to implement, at least in their simplest form, while others appear to significantly increase the complexity of the model and make their numerical implementation computationally (nearly) unfeasible. Nonetheless, we believe that understanding policyholder behavior is a critical component to the future of the life insurance business, and further research is required.

References

Bauer, Daniel, Alexander Kling and Jochen Russ, 2008. A Universal Pricing Framework for Guaranteed Minimum Benefits in Variable Annuities. ASTIN Bulletin 38, 621-651.

Bauer, Daniel, Daniela Bergmann and Rüdiger Kiesel, 2010. On the risk-neutral valuation of life insurance contracts with numerical methods in view. ASTIN Bulletin 40, 6595.

Black, Kenneth, Jr. and Harold D. Skipper Jr., 2000. Life and Health Insurance, 13th Edition. Upper Saddle River, NJ: Prentice Hall.

Boyle, Phelim and Mary Hardy, 2003. Guaranteed Annuity Options. ASTIN Bulletin 33, 125-152.

Chen, Zhang, Ken Vetzal and Peter A. Forsyth, 2008. The Effect of Modelling Parameters on the Value of GMWB Guarantees. Insurance: Mathematics and Economics 43, 165-173.

De Giovanni, Domenico, 2010. Lapse Rate Modeling: a Rational Expectations Approach. Scandinavian Actuarial Journal 2010, 56-67.

Epstein, Larry G. and Stanley E. Zin, 1989. Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework. Econometrica 57, 937-969.

Gerber, Hans U. and Elias S.W. Shiu, 2000. Investing for Retirement: Optimal Capital Growth and Dynamic Asset Allocation. North American Actuarial Journal 4, 42-62.

Grosen, Anders and Peter Løchte Jørgensen, 2000. Fair Valuation of Life Insurance Liabilities: The Impact of Interest Rate Guarantees, Surrender Options, and Bonus Policies. Insurance: Mathematics and Economics 26, 37-57.

Kim, Changki, 2005. Modeling Surrender and Lapse Rates with Economic Variables. North American Actuarial Journal 9, 56-70.

Kuo, Weiyu, Chenghsien Tsai and Wei-Kuang Chen, 2003. An Empirical Study on the Lapse Rate: The Cointegration Approach. The Journal of Risk and Insurance 70, 489-508.

Merton, Robert C., 1969 Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case The Review of Economics and Statistics 51, 247-257

Milevsky, Moshe A. and Kamphol Panyagometh, 2001. Variable Annuities versus Mutual Funds: A Monte Carlo Analysis of the Options. Financial Services Review 10, 145-161.

Milevsky, Moshe A. and Steven E. Posner, 2001. The Titanic Option: Valuation of the Guaranteed Minimum Death Benefit in Variable Annuities and Mutual Funds. The Journal of Risk and Insurance 68, 93-128.

Milevsky, Moshe A. and Thomas S. Salisbury, 2006. Financial Valuation of Guaranteed Minimum Withdrawal Benefits. Insurance: Mathematics and Economics 38, 21-38.

Society of Actuaries, 2009. Policyholder Behavior in the Tail – Variable Annuity Guaranteed Benefits Survey/C3 Phase II, 2009 Results

Stanton, Richard, 1995. Rational Prepayment and the Valuation of Mortgage-Backed Securities. The Review of Financial Studies 8, 677-708.

Ulm, Eric, 2006 The Effect of the Real Option to Transfer on the Value of Guaranteed Minimum Death Benefits. The Journal of Risk and Insurance 73, 43-69.

Zaglauer, Katharina and Daniel Bauer, 2008. Risk-Neutral Valuation of Participating Life Insurance Contracts in a Stochastic Interest Rate Environment Insurance: Mathematics and Economics 43, 29-40.

Appendices

A Notation

A_t	Outside account value
b_t	Bequest amount if policyholder dies in period t
β	Subjective discounting parameter
C_t	Consumption in period t
G_t	Benefits base in period t
$G_t^{D/I/A}$	Guaranteed minimum death / income / accumulation benefit in period t
G_t^W	Remaining (aggregate) guaranteed withdrawal amount
g_t^W	Guaranteed withdrawal amount for period t
I_t	Income in period t
κ	Capital gains tax rate
ν_t	Ratio of mutual fund value invested in risky asset in period t
ν^X	Ratio of VA account invested in risky asset
P_0	Single up-front premium for VA
p_{x+t}	Probability that policyholder survives $t+1$, given he was alive at time $(t)^+$
ϕ	Annual guarantee fee
q_{x+t}	Probability that policyholder dies in period $t+1$, i.e. at time $(t+1)^-$
r	Short-term interest rate
s	Excess withdrawal fee
S_t	Time t value of stock
s^g	Early withdrawal penalty rate for Variable Annuities
T	Maturity of VA
T^W	Starting year for "free" withdrawals
τ	Income tax rate
$u_B(.)$	Utility of bequests
$u_C(.)$	Utility of consumption
V_t	policyholder's value function over lifetime utility
W_0	Policyholder's initial level of wealth, prior to purchasing VA
w_t	Withdrawal amount from VA account at time t
X_t	VA account value
y_t	Vector of state variables for period t
Z_t	Standard Brownian Motion
ω	Maximum possible age of policyholder

B Timeline

Starting at the end of policy year $t \in \{1, \ldots, T-1\}$, just prior to the t-th policy anniversary date, the timeline of events is as follows:

1. The policyholder observes the annual asset returns S_t/S_{t-1}, and thus the level of his current state variables, A_t^- and X_t^-.
2. The policyholder dies with probability q_{x+t-1}.
3. In case of death, he leaves bequest $b_t = A_t^- + \max\{X_t^-\ ,\ G_t^D\}$, and no further actions are taken. If he survives:
4. The policyholder receives income I_t for the new period.
5. Based on this information, he chooses how much to withdraw from the VA account, w_t, consume, C_t, and how to allocate his outside portfolio, ν_t.
6. The policyholder consumes C_t. Account values (A_t^+, X_t^+), the benefit base (G_{t+1}) and the tax base (H_{t+1}) are updated accordingly.
7. Between t and $t+1$, the account values evolve in line with the asset(s).
8. (1.) The policyholder observes S_{t+1}/S_t, etc.

C Derivation of Approximation (17)

For given M and $u_k, k = 0, \ldots, M$, we can compute $x_k \equiv \lambda(u_k) \equiv exp(\sigma \cdot u_k + r - \frac{1}{2}\sigma^2)$ and $\psi_k \equiv F(x_k)$, if necessary by interpolating and/or extrapolating $F(.)$ over the state space grid. Note that the domain of gross return variable x is by definition $(0, \infty)$.

For arbitrary $0 < x < \infty$, we can approximate the corresponding function value linearly by

$$\begin{aligned} F(x) &\approx \sum_{k=0}^{M-1} \left(\psi_k + \tfrac{x - x_k}{x_{k+1} - x_k} \cdot (\psi_{k+1} - \psi_k) \right) \cdot \mathbf{1}_{[x_k, x_{k+1})}(x) \\ &= \sum_{k=0}^{M-1} (a_k + b_k \cdot x) \cdot \mathbf{1}_{[x_k, x_{k+1})}(x), \end{aligned}$$

where for $k = 0, \ldots, M-1$

$$\begin{aligned} a_k &\equiv \tfrac{x_{k+1} \cdot \psi_k - x_k \cdot \psi_{k+1}}{x_{k+1} - x_k}, \quad \text{and} \\ b_k &\equiv \tfrac{\psi_{k+1} - \psi_k}{x_{k+1} - x_k}. \end{aligned}$$

In addition, we define $a_M = b_M \equiv 0$.

Plugging this into equation (16), we obtain

$$\begin{aligned} K &= \int_{-\infty}^{\infty} \phi(u) F(\lambda(u))\, du \\ &\approx \int_{-\infty}^{\infty} \phi(u) \cdot \sum_{k=0}^{M-1} (a_k + b_k \cdot x) \cdot \mathbf{1}_{[x_k, x_{k+1})}(x)\, du \\ &= \sum_{k=0}^{M-1} \int_{u_k}^{u_{k+1}} \phi(u) \cdot (a_k + b_k \cdot \lambda(u))\ du \\ &= \sum_{k=0}^{M-1} a_k \cdot \int_{u_k}^{u_{k+1}} \phi(u)\ du + b_k \cdot \int_{u_k}^{u_{k+1}} \phi(u) \cdot \lambda(u)\ du, \end{aligned}$$

and since

$$\phi(u) \cdot \lambda(u) = \frac{1}{\sqrt{2\pi}} \cdot \exp(-\frac{1}{2}u^2) \cdot \exp(\sigma u + \mu - \frac{1}{2}\sigma^2) = e^{\mu} \frac{1}{\sqrt{2\pi}} \exp(\frac{1}{2}(u-\sigma)^2) = e^{\mu} \phi(u - \sigma),$$

we obtain

$$K \approx \sum_{k=0}^{M-1} a_k \cdot [\Phi(u_{k+1}) - \Phi(u_k)] + \exp(\mu) \cdot b_k \cdot [\Phi(u_{k+1} - \sigma) - \Phi(u_k - \sigma)].$$

Finally, reordering of the summation terms yields the desired equation (17). □