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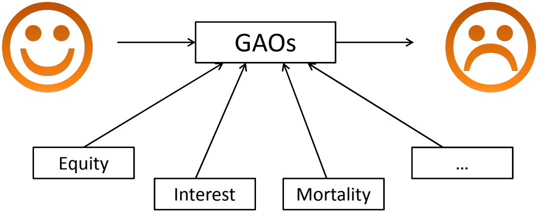
Decomposition of life insurance liabilities into risk factors – theory and application

Joint work with Daniel Bauer, Marcus C. Christiansen, Alexander Kling

Motivation

British insurance companies during the 1980s vs. 1990s:



Question: Which are the most relevant risk drivers?

Why is that important?

To be able to take adequate risk management strategies such as

- Product modifications
- Hedging

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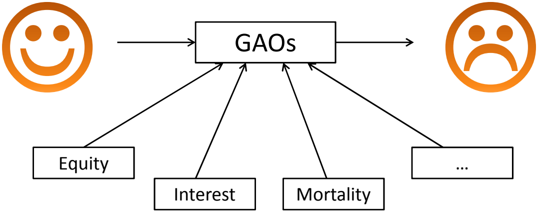

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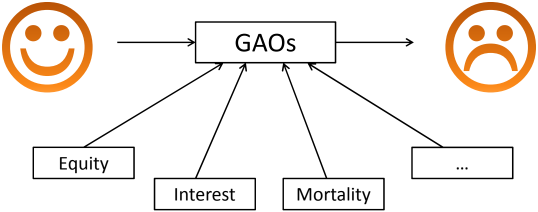

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Research objectives

Situation:

- It is common to measure the total risk by advanced stochastic models.
- The question of how to determine the most relevant risk driver is not very well understood.

Our paper

(1) **Theory:**
How to allocate the randomness of liabilities to different risk sources?

(2) **Application:**
What is the dominating risk in annuity conversion options?

Note: we focus on the distribution under the real-world measure $\mathbb{P}$.

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General setting

- Insurance product with maturity T
- Insurer's liability as from time 0: $\mathbf{L_0}$
- Two risk drivers: $\mathbf{X_1} := (X_1(t))_{0\leq t\leq T^*}$ and $\mathbf{X_2} := (X_2(t))_{0\leq t\leq T^*}$

$$\begin{tikzcd}
 & \boxed{\text{Liability } L_0} & \\
\boxed{\text{Risk } X_1} \arrow[ur] & & \boxed{\text{Risk } X_2} \arrow[ul]
\end{tikzcd}$$

<u>Question:</u> How to decompose L_0 with respect to X_1 and X_2?

Variance decomposition approach

Step 1: $L_0 = \underbrace{\mathrm{E}\left(L_0 \middle| X_1 \right)}_{=:R_1} + \underbrace{\left[L_0 - \mathrm{E}\left(L_0 \middle| X_1 \right)\right]}_{=:R_2}$

- R_1 represents the randomness of L_0 caused by X_1
- R_2 represents the randomness of L_0 caused by X_2

Step 2: $\mathrm{Var}\left(L_0\right) = \mathrm{Var}\left(R_1\right) + \mathrm{Var}\left(R_2\right)$

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Desirable property: full distribution of R_1 and R_2

- Bühlmann (1995): annual loss = financial loss + technical loss
- Example: $L_0 = X_1(T)X_2(T)$, X_1, X_2 independent Brownian motions
  - $L_0 = \mathrm{E}(L_0|X_1) + [L_0 - \mathrm{E}(L_0|X_1)] = \underbrace{0}_{=R_1} + \underbrace{X_1(T)X_2(T)}_{=R_2}$
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  - $L_0 = \mathrm{E}(L_0|X_2) + [L_0 - \mathrm{E}(L_0|X_2)] = 0 + X_1(T)X_2(T)$

Desirable property: symmetric definition (uniqueness)

Further approaches

Sensitivity analysis

- Analyzing the effect of changes in the input parameters/variables on the insurer's liability
- Usually based on derivatives

Desirable property: comparability of the risk contributions

Taylor expansion approach

- Function of random variables ≈ first-order Taylor expansion

Desirable property: $L_0 - \mathrm{E}(L_0) = R_1 + \ldots + R_n$

- Local method: expansion point is relevant

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Risk driving processes (1)

1.) **State process** $X(t)$: financial and demographic factors

- Risky assets
- Short rate
- Mortality intensity

Assumption

$X = (X_1(t), \ldots, X_n(t))_{0 \leq t \leq T^*}$ is an n-dimensional **diffusion process** satisfying

$$dX_i(t) = \theta_i(t, X(t))dt + \sum_{j=1}^{d} \sigma_{ij}(t, X(t))dW_j(t), \;\; i = 1, \ldots, n,$$

with deterministic initial value $X(0) = x_0 \in \mathbb{R}^n$.

- $W = (W_1(t), \ldots, W_d(t))_{0 \leq t \leq T^*}$ d-dimensional standard Brownian motion
- $\mathbb{G} = (\mathcal{G}_t)_{0 \leq t \leq T^*}$ augmented natural filtration generated by W

Risk driving processes (2)

2.) **Counting process** $N(t)$: actual occurrence of death

- Portfolio of m homogeneous policyholders of age x at time 0
- τ_x^i: remaining lifetime of the i-the policyholder as from time 0
  - first jump time of a doubly stochastic process with intensity $\mu = (\mu(t))_{0 \leq t \leq T^*}$
  - μ is assumed to be continuous, $\mathbb{G}$-adapted, and non-negative
- $N(t) = \sum_{i=1}^{m} \mathbb{1}_{\{\tau_x^i \leq t\}}$: number of policyholders who died until time t
- $\mathbb{I}^i = (\mathcal{I}_t^i)_{0 \leq t \leq T^*}$ augmented natural filtration generated by $(\mathbb{1}_{\{\tau_x^i > t\}})_{0 \leq t \leq T^*}$

We assume: $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with $\mathbb{F} = \mathbb{G} \vee \bigvee_{i=1}^{m} \mathbb{I}^i$

Insurer's net liability

The life insurance contract implies:

- Cash flows $C(t_k)$, **independent** of the policyholder's survival
- Cash flows $C_a(t_k)$, in case the policyholder **survives** until time t_k
- Cash flows $C_{ad}(t)$, in case the policyholder **dies** at time t

The **insurer's time-t net liability** is given by the sum of the (possibly discounted) future cash flows as from time t:

$$L_t = \sum_{k:\ t_k \geq t} C(t_k) + \sum_{k:\ t_k \geq t} (m - N(t_k)) C_{\mathrm{a}}(t_k) + \int_t^{T^*} C_{\mathrm{ad}}(v) dN(v).$$

<u>In what follows</u>: we focus on the insurer's net liability $\mathbf{L_0}$ at time 0.

MRT decomposition

Idea

Decompose $L_0 - \mathrm{E}^{\mathbb{P}}(L_0)$ into **Itô integrals with respect to the compensated risk driving processes**, i.e.

$$L_0 - \mathrm{E}^{\mathbb{P}}(L_0) = \sum_{i=1}^{n} \underbrace{\int_0^T \psi_i^W(t)dM_i^W(t)}_{=:R_i} + \underbrace{\int_0^T \psi^N(t)dM^N(t)}_{=:R_{n+1}} \qquad (1)$$

for some $\mathbb{F}$-predictable processes $\psi_i^W(t)$ and $\psi^N(t)$, where

- $dM_i^W(t) = \sum_{j=1}^{d} \sigma_{ij}(t, X(t))dW_j(t)$
- $dM^N(t) = dN(t) - (m - N(t-))\mu(t)dt.$

Existence and uniqueness

Assume that $n = d$, $\det \sigma(t,x) \neq 0$ for all $(t,x) \in [0, T^*] \times \mathbb{R}^n$, and L_0 is $\mathcal{F}_T$-measurable. Then the MRT decomposition in eq. (1) exists and is unique.

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Properties of the MRT decomposition

MRT decomposition

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List of desirable properties:

- ✓ Full distribution of each risk contribution R_i
- ✓ Symmetric definition
- ✓ No problem-specific choices
- ✓ It holds: $L_0 - \mathrm{E}(L_0) = R_1 + \ldots + R_n$
- ✓ Comparability of the risk contributions
- ✓ Unsystematic mortality risk is diversifiable
- ✓ Appropriate dealing with correlations

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Specification of the MRT decomposition

Exemplarily, we decompose $L_0 = (m - N(T))C_a(T)$.

Special case

Let the assumptions for existence and uniqueness hold.
If $\mathrm{E}^{\mathbb{P}}(e^{-\int_t^T \mu(s)ds} C_a(T)|\mathcal{G}_t) = f(t, X(t)),\ 0 \leq t \leq T$, for some sufficiently smooth function f, then Itô's lemma yields

$$L_0 - \mathrm{E}^{\mathbb{P}}(L_0) = \sum_{i=1}^{n} \int_0^T (m - N(t-)) \frac{\partial f}{\partial x_i}(t, X(t))\, dM_i^W(t) - \int_{0+}^T f(t, X(t))\, dM^N(t).$$

<u>Existence of f:</u>

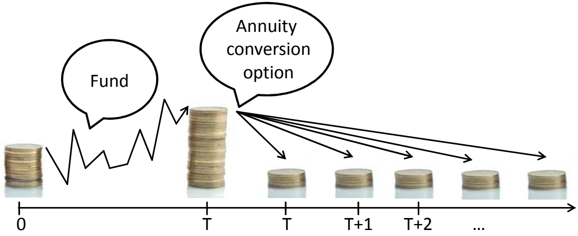
- $C_a(T) = h(X(T))$ for some Borel-measurable function $h : \mathbb{R}^n \to \mathbb{R}$

<u>Smoothness of f:</u>

- Conditions from Theorem 1 in Heath and Schweizer (2000)

Guaranteed annuity option (GAO)

- Special type of annuity conversion option (cf. UK)



- Guaranteed annual annuity $= g$ (conversion rate) $\times A_T$ (account value)
- **Insurer's liability (= option payoff at time T):**

$$L_0^{\text{GAO}} = \mathbb{1}_{\{\tau_x > T\}} \max\{gA_T a_T - A_T, 0\}$$
$$= \mathbb{1}_{\{\tau_x > T\}} gA_T \max\left\{a_T - \frac{1}{g}, 0\right\}$$

  - τ_x: remaining lifetime of a policyholder aged x at time 0
  - a_T: time-T value of an immediate annuity of unit amount per year

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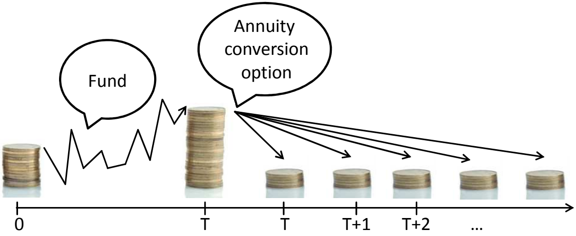

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Stochastic model

Insurer's total liability (= option payoff at time T for a portfolio)

$$L_0^{\mathrm{GAO}} = \underbrace{\sum_{i=1}^{m} \mathbb{1}_{\{\tau_x^i > T\}}}_{=m-N(T)} \underbrace{g A_T \max\left\{a_T - \frac{1}{g}, 0\right\}}_{=C_a(T)}$$

Risk	Process	Model
Fund risk	$S(t)$	GBM
Interest risk	$r(t)$	CIR model
Systematic mortality risk	$\mu(t)$	time-inhomogeneous CIR model
Unsystematic mortality risk	$N(t)$	Binomial distribution

Assumption: Processes S, r and μ are independent

MRT decomposition of GAO (1)

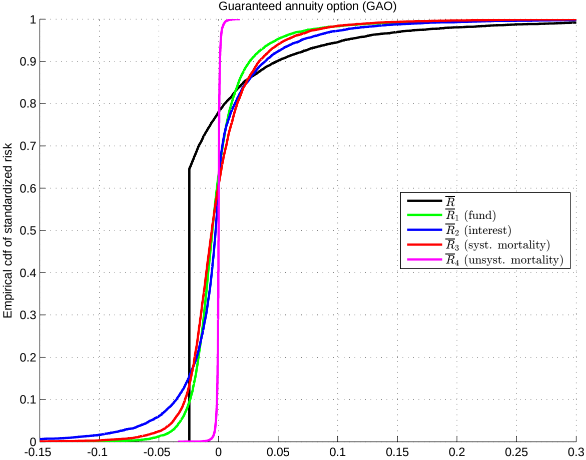
It can be shown:

- $C_a(T) = h(S(T), r(T), \mu(T))$ for some measurable function h
- $f(t, X(t)) := \mathrm{E}^{\mathbb{P}}\left(e^{-\int_t^T \mu(s)ds} C_a(T) \middle| \mathcal{G}_t\right)$ is sufficiently smooth

This yields:

$$
\begin{aligned}
&L_0^{\mathrm{GAO}} - \mathrm{E}^{\mathbb{P}}\left(L_0^{\mathrm{GAO}}\right) && \Big\} =: R \\
&= \int_0^T (m - N(t-)) \frac{\partial f}{\partial x_1}(t, X(t)) \sigma_S S(t) dW_S(t) && \Big\} =: R_1 \\
&\quad + \int_0^T (m - N(t-)) \frac{\partial f}{\partial x_2}(t, X(t)) \sigma_r \sqrt{r(t)} dW_r(t) && \Big\} =: R_2 \\
&\quad + \int_0^T (m - N(t-)) \frac{\partial f}{\partial x_3}(t, X(t)) \sigma_\mu(t) \sqrt{\mu(t)} dW_\mu(t) && \Big\} =: R_3 \\
&\quad + \int_{0+}^T f(t, X(t)) dM^N(t). && \Big\} =: R_4
\end{aligned}
$$

MRT decomposition of GAO (2)



- Unsystematic mortality plays a minor role ($m = 100$)
- Distributions of fund, interest and systematic mortality risk are comparable

Introduction

Risk decomposition methods from literature

Life insurance modeling framework

MRT approach

Application to annuity conversion options

Outlook

Future research

- **Model**
  - Extension to Lévy processes (instead of Brownian motions)
- **Application**
  - Further annuity conversion options, e.g. modified GAOs
  - Taking into account hedging

Contact

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Thank you very much for your attention!

Literature

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Model parameters

Description	Parameter	Value
Age	x	50
Term to maturity	T	15
Single premium	P_0	1
Conversion rate	g	0.07
Limiting age	ω	121
Number of realizations (outer)	N	10,000
number of realizations (inner)	M	100
Number of discretization steps per year	n	100
Number of contracts	m	100
GBM drift	μ_S	0.06
GBM volatility	σ_S	0.22
CIR initial value	$r(0)$	0.0029
CIR speed of reversion	κ ($\tilde{\kappa}$)	0.2 (0.2)
CIR mean level	θ ($\tilde{\theta}$)	0.025 (0.025)
CIR volatility	σ_r ($\tilde{\sigma}_r$)	0.075 (0.075)
Correlation	ρ	0