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Tests and Power Comparisons in Time-Dynamic Copula Models

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University of Mainz
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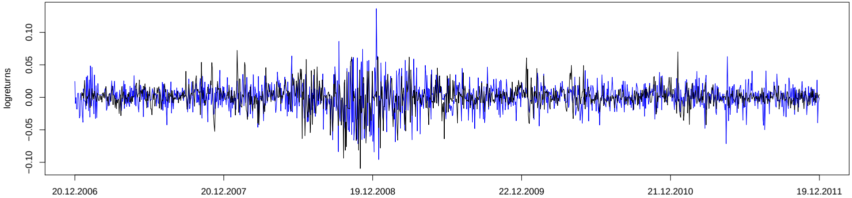
Outline

Heteroscedastic Time Series



$d \geqslant 2$ $\quad \mathbf{Y}_t = \mathbf{\Sigma}_t^{1/2} \varepsilon_t$ with $\varepsilon_t \sim \mathrm{WN}(\mathbf{0}, \mathbf{E}_d)$ iid

$$\mathbf{\Sigma}_t = \mathrm{Cov}\left(\mathbf{Y}_t \,|\mathcal{F}_{t-1}\right), \text{ where } \mathcal{F}_t = \sigma(\mathbf{Y}_t, \mathbf{Y}_{t-1}, \ldots)$$

$$\text{With} \quad \mathbf{\Sigma}_t = \mathrm{diag}(\sigma_{1t}^2, \ldots, \sigma_{dt}^2)$$

the dependence is captured by the multivariate distribution of the residuals

$$\varepsilon_t \sim F(\theta) = C(F_1, \ldots, F_d; \theta) \text{ independent for all } t.$$

Heterogeneous Dependence Structure

(Parsimonious) univariate Garch(p, q)-models for every $1 \leqslant j \leqslant d$

$$Y_{jt} = \sigma_{jt}\varepsilon_{jt}, \ \sigma_{jt}^2 = \alpha_{j0} + \sum_{i=1}^{p} \alpha_{ji} Y_{j,t-i}^2 + \sum_{k=1}^{q} \beta_{jk}\sigma_{j,t-k}^2$$

are linked through the dependence structure of the residuals.

Heterogeneity in the dependence via time-dynamic parameters θ_t for every t

$$\varepsilon_t \sim F(\theta_t),$$

such that ε_t are still independent, but not identically distributed.

Parameter Stability

Test the null hypothesis

$$\mathbf{H}_0\colon \{\theta_t \equiv \theta_0 \ \forall\, t\}$$

against the alternative that the parameter changes over time

$$\mathbf{H}_A\colon \{\theta_1 = \ldots = \theta_{t_1} \neq \theta_{t_1+1} = \ldots = \theta_{t_k} \neq \theta_{t_k+1} = \ldots\ldots = \theta_T\}.$$

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- estimation of parameters and construction of tests

1. DeGarching

Let the corresponding observations of $\mathbf{Y}_t = (Y_{1t}, \ldots, Y_{dt})'$ be

$$(y_{1t}, \ldots, y_{dt})', \; 1 \leqslant t \leqslant T.$$

The parameter vector $\gamma_j = (\alpha_{j0}, \alpha_{j1}, \ldots, \alpha_{jp}, \beta_{t1}, \ldots, \beta_{tq})'$ for every component $1 \leqslant j \leqslant d$ is estimated via quasi-maximum-likelihood (QMLE):

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- assume that $\varepsilon_{jt} \sim \mathcal{N}(0,1)$ for all $1 \leqslant t \leqslant T$
- maximize the likelihood function L_T to obtain $\hat{\gamma}_j$
  - L_T as function of observations and some starting values y_0, σ_0^2

Even if the normality assumption fails, but $\mathbb{E}\varepsilon_{jt}^4 < \infty$, it holds that for $T \to \infty$

1. $\hat{\gamma}_j \xrightarrow{a.s.} \gamma_j$
2. $\sqrt{T}(\hat{\gamma}_j - \gamma_j) \xrightarrow{d} \mathcal{N}(0,\mathrm{v})$

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Since $\sigma_{jt}^2 = w_t(\gamma_j)$, we obtain empirical residuals $\tilde{\varepsilon}_t$ with components

$$\tilde{\varepsilon}_{jt} = \frac{y_{jt}}{\hat{\sigma}_{jt}}, \quad \text{where} \quad \hat{\sigma}_{jt} = \sqrt{\hat{w}_t(\hat{\gamma}_j)}.$$

2. Canonical Maximum Likelihood

Semi-parametric estimation on $(\tilde{\varepsilon}_{1t}, \ldots, \tilde{\varepsilon}_{dt})' \sim C(F_1, \ldots, F_d; \theta)$

- rank-transformation for the marginal distributions $\tilde{u}_{jt} = \frac{1}{T+1} \sum_{s=1}^{T} \mathbb{1}_{\{\tilde{\varepsilon}_{js} \leqslant \tilde{\varepsilon}_{jt}\}}$
- maximum likelihood estimation

$$\hat{\theta}_t = \underset{\theta}{\operatorname{argmax}} \sum_{s=t-b+1}^{t} \log c(\tilde{u}_{1s}, \ldots, \tilde{u}_{ds}; \theta)$$

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According to Chan *et al.* (2009) it holds that for $T \to \infty$ and if $b = o(T)$

1. $\hat{\theta}_t \xrightarrow{P} \theta$ for all t
2. $\sqrt{b}\,(\hat{\theta}_t - \theta) \xrightarrow{d} \mathcal{N}(0,\Sigma)$

with variance due to Genest *et al.* (1995)

$$\Sigma = \frac{\operatorname{Var}\left(\frac{\partial}{\partial\theta} \log c(F_1(\varepsilon_1), \ldots, F_d(\varepsilon_d); \theta) + \sum_{j=1}^{d} W_j(\varepsilon_j)\right)}{\mathbb{E}\left([\frac{\partial}{\partial\theta} \log c(F_1(\varepsilon_1), \ldots, F_d(\varepsilon_d); \theta)]^2\right)^2} =: \frac{\sigma^2}{\beta^2}$$

where $W_j(\varepsilon_j) = \int_{[0,1]^d} \mathbb{1}_{\{F_j(\varepsilon_j) \leqslant u_j\}} \frac{\partial^2}{\partial\theta\,\partial u_j} \log c(u_1, \ldots, u_d; \theta) dC(u_1, \ldots, u_d; \theta)$.

Simple Test - Construction and Power

Under the simple hypothesis $\mathbf{H}_{0,k}: \{\theta_{t_k} = \theta_0\}$ consider the statistic

$$T_{bk} = \frac{\sqrt{b}(\hat{\theta}_{t_k} - \hat{\theta})}{\sqrt{\hat{\Sigma}}} \xrightarrow[(T \to \infty)]{d} \mathcal{N}(0,1)$$

with $\hat{\theta}$ a global estimate and $\hat{\Sigma}$ a consistent variance estimate.

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Under the local alternative $\mathbf{H}_{A,k}\colon \{\theta_{t_k} = \theta_b^{\pm}(x) = \theta_0 \pm \frac{x}{\sqrt{b}}\}$ and for $x \neq 0$ it holds

$$\mathbb{P}\left(|\hat{T}_{bk}| > q_{\alpha/2} \,\middle|\, \mathbf{H}_{A,k} \right) \xrightarrow[(T\to\infty)]{} \gamma(x) > \alpha$$

with $q_{\alpha/2} = \mathcal{N}_{0,1}^{-1}(1 - \alpha/2)$.

The local test $|\hat{T}_{bk}| > q_{\alpha/2}$ has an asymptotic local power γ of size $\sqrt{b}$.

Multiple Test - Construction

For the local alternative $\mathbf{H}_{A,k}\colon \{\theta_{t_k} = \theta_b^{\pm}(x)\}$ we consider

$$X_k := \mathbb{1}_{\{|\hat{T}_{bk}| > q_{\alpha/2}\}} \xrightarrow{d} \begin{cases} \mathrm{Bin}(1, \alpha) & \text{under } \mathbf{H}_{0,k}, \\ \mathrm{Bin}(1, \gamma) & \text{under } \mathbf{H}_{A,k}. \end{cases}$$

For independent $\mathbf{H}_{0,k}$ the global null hypothesis $\mathbf{H}_0 = \bigcap_{k=1}^{n} \mathbf{H}_{0,k}$ corresponds to

$$\mathbf{H}_0\colon \{\theta_{t_1} = \ldots = \theta_{t_n} \equiv \theta_0 \;\; \text{for } |t_i - t_j| > b, i \neq j\}.$$

Under $\mathbf{H}_0$ a suitable test statistic $S_n = \sum_{k=1}^{n} X_k$ is asymptotically $\mathrm{Bin}(n, \alpha)$ and the test has proper size $\alpha^\star$ if

$$\mathbb{P}\left(S_n \geqslant \kappa \,\middle|\, \mathbf{H}_0\right) = \sum_{j=\kappa}^{n} \binom{n}{j} \alpha^j (1-\alpha)^{n-j} \leqslant \alpha^\star.$$

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Fix $\alpha, \alpha^\star \implies$ some $\kappa \notin \mathbb{N}$ in general, need to randomize the test.

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Fix $\kappa, \alpha^\star \Longrightarrow$ solution $\alpha \in (0,1)$, choose $c_{\alpha^\star} = \kappa - 1$ as critical value.

Multiple Test - Power

Consider the multiple alternative

$$\mathbf{H}_A\colon \left\{\theta_k = \theta_b^{\pm}(x), 1 \leqslant k \leqslant m\right\} \cap \left\{\theta_k = \theta_0, k > m\right\} = \bigcap_{k=1}^{m} \mathbf{H}_{A,k} \cap \bigcap_{k=m+1}^{n} \mathbf{H}_{0,k}.$$

Then under $\mathbf{H}_A$ the test statistic is a convolution of binomial distributions with different probabilities. In particular, it holds that

$$S_n \,\middle|\, \mathbf{H}_A = S_m^{(\gamma)} + S_{n-m}^{(\alpha)} \overset{st}{\geqslant} S_m^{(\alpha)} + S_{n-m}^{(\alpha)} = S_n^{(\alpha)} = S_n \,\middle|\, \mathbf{H}_0.$$

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Therefore, for $x \neq 0$ it holds that

$$\Gamma(x) = \lim_{T\to\infty} \mathbb{P}\left(S_n > c_{\alpha^\star} \,\big|\, \mathbf{H}_A\right) \geqslant \lim_{T\to\infty} \mathbb{P}\left(S_n > c_{\alpha^\star} \,\big|\, \mathbf{H}_0\right) = \alpha^\star.$$

The multiple test $S_n > c_{\alpha^\star}$ has also an asymptotic local power Γ of size $\sqrt{b}$.

Omnibus Test - Construction

The global null hypothesis $\mathbf{H}_0\colon \{\theta_1 = \ldots = \theta_T \equiv \theta_0\}$ implies $\mathbf{H}_0\colon \{\max_t \theta_t = \theta_0\}$ and can be tested against the alternative

$$\mathbf{H}_A^{\star}\colon \{\max_t \theta_t \neq \theta_0\}.$$

For this, define a thinned estimator series for all $k = 1, \ldots, N = \frac{T-b}{cb}$ with a thinning constant $c \in (1/b, 1]$

$$\xi_{t_k} := \frac{\sqrt{b}(\hat{\theta}_{t_k} - \hat{\theta})}{\sqrt{\hat{\Sigma}}} \xrightarrow{d} \mathcal{N}(0,1).$$

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Theorem (asymptotic extremal distribution)

For $N = N(T) \xrightarrow[(T\to\infty)]{} \infty$, $M_N = \max_k \xi_{t_k}$ *and if* $b = o(T^{7/9})$, *it holds that*

$$\lim_{T\to\infty} \mathbb{P}\left((M_N - d_N)/a_N \leqslant x\right) = \Lambda(x) = e^{-e^{-x}} \quad \forall\, x \in \mathbb{R},$$

where $a_N = \sqrt{2\log N}$ *and* $d_N = a_N - \frac{\log\log N - \log 2\pi}{2a_N}$.

Sketch of proof I

Consider the joint distribution of $\hat{\theta}_{t_1}, \ldots, \hat{\theta}_{t_N}$

$$\Rightarrow \quad S_b = \frac{1}{\sqrt{b}} \left(\mathbf{X}^{(1)} + \ldots\ldots + \mathbf{X}^{(b)} \right) = \frac{1}{\sqrt{b}} \left(\mathbf{Y}^{(1)} + \ldots + \mathbf{Y}^{(cb)} \right)$$

with

$$\mathbf{C}_N^2 = \mathrm{Cov}\, \mathbf{Y}^{(i)} = \begin{pmatrix} \kappa & \kappa - 1 & \ldots & 1 & & \mathbf{0} \\ & \ddots & \ddots & & \ddots & \\ & & \ddots & \ddots & & 1 \\ & & & \ddots & \ddots & \vdots \\ & & & & \ddots & \kappa - 1 \\ & & & & & \kappa \end{pmatrix}$$

symmetric, positive definite, $(\kappa - 1)$-banded, Toeplitz with decaying elements.

Sketch of proof II

Due to Berenhaut & Bandyopadhyay (2005) the Cholesky square root

$$\mathbf{C}_N = \begin{pmatrix} c_{11} & & & & \\ \vdots & \ddots & & 0 & \\ c_{\kappa 1} & & \ddots & & \\ & \ddots & & \ddots & \\ 0 & & c_{N\kappa} & \dots & c_{NN} \end{pmatrix}$$

is lower triangular, $(\kappa - 1)$-banded, with monotone entries $c_{ij} \geqslant c_{i+1,j} \geqslant 0$.

$\Longrightarrow$ $\mathbf{C}^2 = \lim_{N\to\infty} \mathbf{C}_N^2$ and $\mathbf{C} = \lim_{N\to\infty} \mathbf{C}_N$ are bounded operators on $\ell^2(\mathbb{Z})$

Due to Demko *et al.* (1984) the inverse matrix $\mathbf{C}^- = (c_{ij}^-)$ is lower triangular and exhibits off-diagonal exponential decay, i.e.

$$|c_{ij}^-| \leqslant const \cdot \lambda_1^{|i-j|}$$

$\Longrightarrow$ matrix entries are absolutely summable

Sketch of proof III

Lyapunov-type Berry-Esseen bounds in a mv CLT due to Bentkus (2004)

$$\Delta_b = \sup_{\mathbf{x}\in\mathbb{R}^N} |F_{\mathbf{S}_b}(\mathbf{x}) - \Phi_{N,C^2}(\mathbf{x})| \leqslant const \cdot N^{1/4} \cdot \beta,$$

where $\beta = \beta_1 + \ldots + \beta_n$ with $\beta_k = \mathbb{E}\left|C^{-}\mathbf{Y}^{(k)}\right|^3$.

For $b = o(T^{7/9})$ it holds that

$$\Delta_b \xrightarrow[(T\to\infty)]{} 0.$$

The proof concludes with classical extreme value theory for weakly dependent normal random variables (e.g. Leadbetter *et al.* , 1983). □

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With $\lambda_{\alpha^\star} = \Lambda^{-1}(1 - \alpha^\star)$, the quantile of the Gumbel extremal distribution, it holds that

$$\mathbb{P}\left((M_N - d_N)/a_N > \lambda_{\alpha^\star} \,\middle|\, \mathbf{H}_A^\star \right) \xrightarrow[(T \to \infty)]{} \Gamma^\star(x) > \alpha^\star.$$

The omnibus test $(M_N - d_N)/a_N > \lambda_{\alpha^\star}$ has a local power $\Gamma^\star$ of size $\sqrt{b/\log N}$.

Simulation Study

- $M = 1000$, $T = 500$, $b = \frac{T}{10}$, $\alpha^{\star} = 5\%$
- Clayton Copula $C(u_1, \ldots, u_d; \theta) = (u_1^{-\theta} + \ldots + u_d^{-\theta} - d + 1)^{-\frac{1}{\theta}}, \theta > 0$
- parameters $\theta \in \{0.1, 0.5, 1.5, 6\}$, dimensions $d \in \{2, 5, 10\}$

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Nominal level of the Binomial test?

- pure copula data & MLE
  - variance estimate: $\hat{\mathrm{I}}(\hat{\theta})^{-1}$ ✓
- multivariate data with standard normal margins & CML
  - variance estimate: $\hat{\Sigma}$ ✓
- Garch$(1, 1)$ models & deGarching & CML
  - variance estimate: $\hat{\Sigma}$ ↯

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- bootstrap: $M^{(BS)} = 200$, $T^{(BS)} = b$

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  - parametric bootstrap with $\hat{\theta}$ ↯
  - non-parametric bootstrap from empirical residuals $\tilde{\varepsilon}_t$ ✓

Commodity Contracts

- 2nd front month future contracts on 11 different commodities (coal, gas, oil, electricity)
- time horizon of 5 years (Dec 06 – Dec 11)
- univariate AR(1)-eGARCH(1,1) processes for log returns (cf. Chevallier, 2011)

$$Y_{jt} = \mu + \rho Y_{j,t-1} + \sigma_{jt}\varepsilon_{jt}$$
$$\log(\sigma_{jt}^2) = \omega_j + \alpha_j \varepsilon_{j,t-1} + \gamma(|\varepsilon_{j,t-1}| - \mathbb{E}(\varepsilon_{j,t-1})) + \beta_j \log(\sigma_{j,t-1}^2)$$

- pairwise empirical residuals coupled by a bivariate Clayton copula

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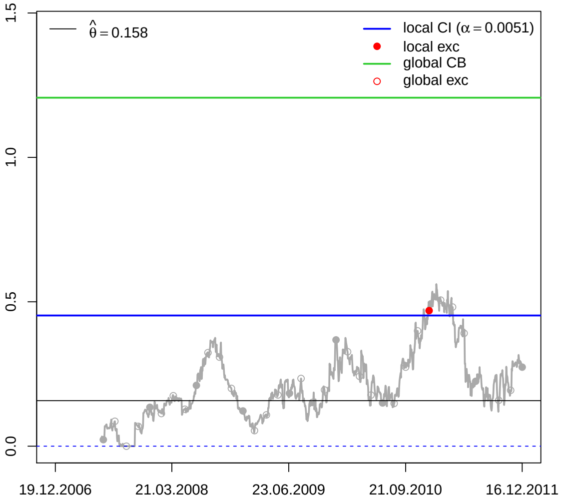
Results

- estimation succeeded for 47 pairs, R-packages `rugarch`, `copula`
- rejections of $\mathbf{H}_0$: 25 Binomial tests (53%), 3 Extremal tests (6%)

	$\hat{\theta}$	$\hat{\Sigma}$	$\hat{\Sigma}^{(BS)}$	$\hat{\tau} = \hat{\theta}/\hat{\theta}+2$	$\hat{\lambda}_L = 2^{-1/\hat{\theta}}$
∅	0.41403	0.00336	0.07040	0.17151	0.18747
min	0.00998	0.00067	0.00405	0.00497	0.00000
max	7.30908	0.10148	2.71164	0.78516	0.90952

Coal vs. German Electricity



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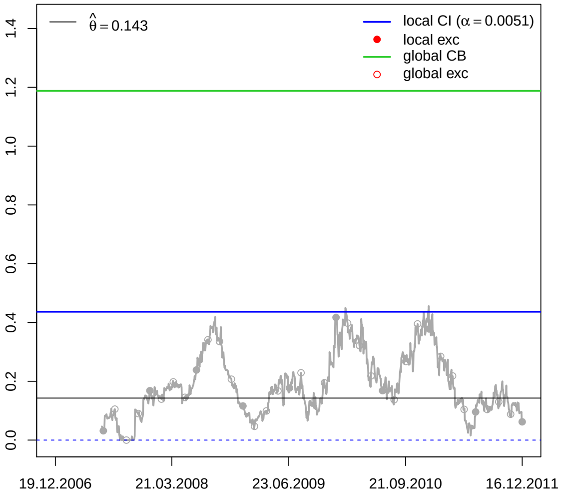
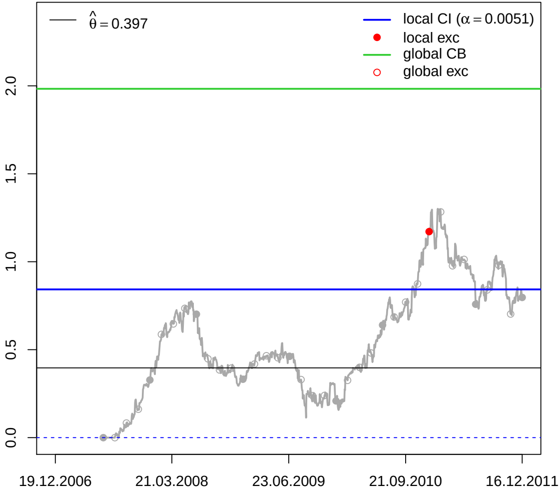


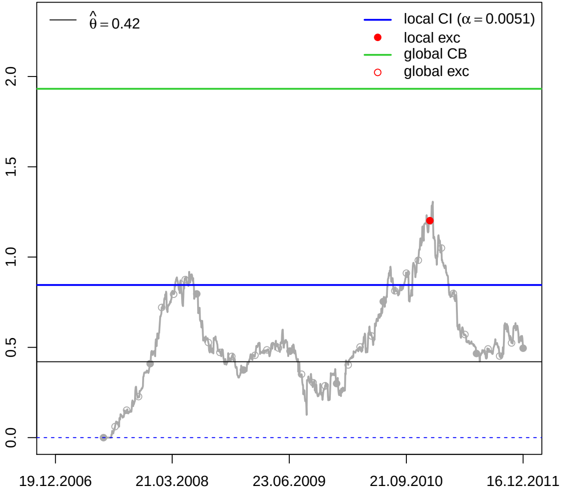
Gas vs. German Electricity

Gas vs. German Electricity

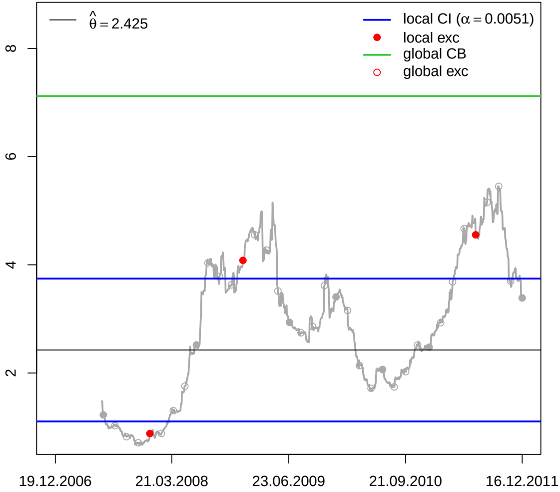
UK vs. German Baseload Electricity



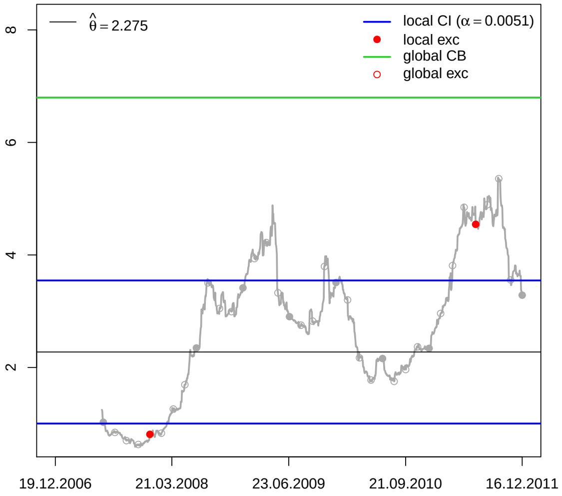
UK vs. German Baseload Electricity



Coal at different delivery places



Coal at different delivery places



Summary & Outlook

- In most cases $\mathbf{H}_0$ isn't rejected (especially for shorter time series)
- Some of the exceedances can be remedied by univariate regimes
- Some can't
    - develop a model that accounts for spikes
    - detect a regime change in copula parameter via Change Point test

Summary & Outlook

- In most cases $\mathbf{H}_0$ isn't rejected (especially for shorter time series)
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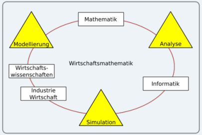
Regime change at $T/2$

Test: $\mathbf{H}_0 : \{\theta_t \equiv \theta_0\}$ vs. $\mathbf{H}_A : \{\theta_t = \theta_0 \pm \frac{x}{\sqrt{T}}$ for $t > T/2\}$

Statistic: $\bar{S}_{Tb} = \frac{1}{\sqrt{T}} \sum_{t=b}^{T/2} (\hat{\theta}_t - \hat{\theta}_{t+T/2})$ based on $f(x_t) = \begin{cases} x_t, & t \leqslant T/2 \\ -x_t, & t > T/2 \end{cases}$

- Based on forecast regression (West & McCracken, 1998) and bias correction (Yaskov, 2010)
- The test $\left| \frac{\bar{S}_{Tb} - \hat{\mu}_{Tb}}{\hat{\sigma}_{Tb}} \right| > q_{\alpha/2}$ has an asymptotic local power of size $\sqrt{T}$

Contact



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Thank you for your attention

References I

BENTKUS, VIDMANTAS. 2004. A Lyapunov-Type Bound in $\mathbb{R}^d$. *Theory of Probability and its Applications*, **49**(2), 311–323.

BERENHAUT, KENNETH, & BANDYOPADHYAY, DIPANKAR. 2005. Monotone convex sequences and Cholesky decomposition of symmetric Toeplitz matrices. *Linear Algebra and Its Applications*, **403**, 75–85.

CHAN, NGAI-HANG, CHEN, JIAN, CHEN, XIAOHONG, FAN, YANQIN, & PENG, LIANG. 2009. Statistical Inference for Multivariate Residual Copula for GARCH Models. *Statistica Sinica*, **19**(1), 53–70.

CHEVALLIER, JULIEN. 2011. *Econometric Analysis of Carbon Markets: The European Union Emissions Trading Scheme and the Clean Development Mechanism*. Springer. ISBN 978-94-007-2411-2.

DEMKO, STEPHEN, MOSS, WILLIAM F., & SMITH, PHILIP W. 1984. Decay Rates for Inverses of Band Matrices. *Mathematics of Computation*, **43**, 491–491.

GENEST, CHRISTIAN, GHOUDI, KILANI, & RIVEST, LOUIS-PAUL. 1995. A semiparametric estimation procedure of dependence parameters in multivariate families of distributions. *Biometrika*, **82**(3), 543–552.

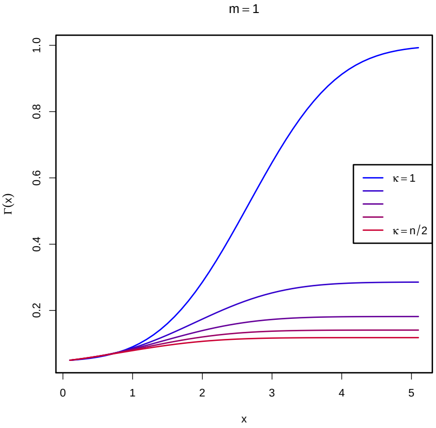
References II

Leadbetter, M. R., Lindgren, Georg, & Rootzen, Holger. 1983. *Extremes and Related Properties of Random Sequences and Processes*. Springer.

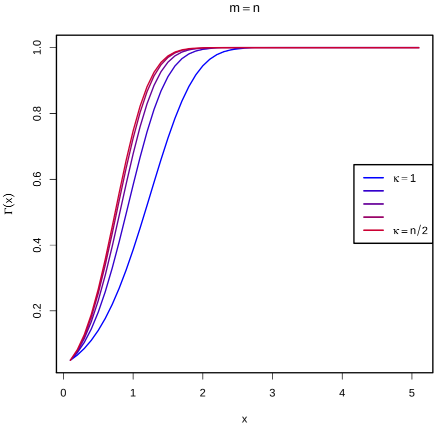
West, Kenneth D., & McCracken, Michael W. 1998. Regression-Based Tests of Predictive Ability. *International Economic Review*, 817–840.

Yaskov, Pavel. 2010. Testing for predictive ability in the presence of structural breaks (in Russian). *Quantile*, 127–135.

Local Power



(a) "One false" alternative



(b) "All false" alternative

Figure: Binomial multiple test for $n = 10$, one-sided local alternatives