Research Training Group 1100

Modelling, analysis and simulation in econometrics

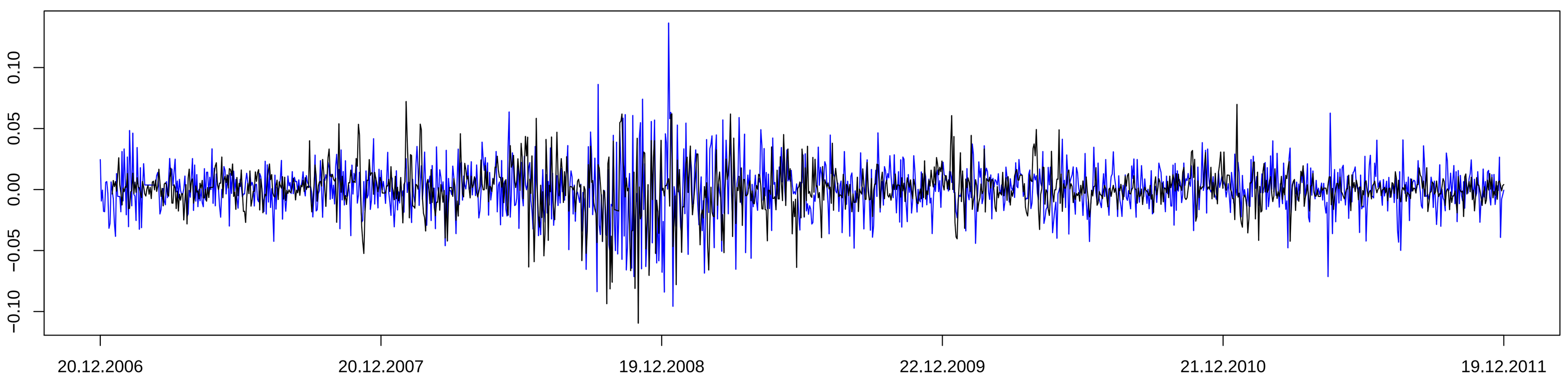
Deutsche Forschungsgemeinschaft
DFG

Time-Dynamic Copula Models

Local power comparisons of homogeneity tests in SCOMDY models

Heteroscedastic time series

We want to capture the stylized facts of financial time series in a multivariate set-up.



$d \geq 2$ $\quad \boldsymbol{Y}_t = \boldsymbol{\Sigma}_t^{1/2}\boldsymbol{\varepsilon}_t$ with $\boldsymbol{\varepsilon}_t \sim$ WN$(\mathbf{0},\mathbf{C})$ iid

$\boldsymbol{\Sigma}_t = \mathrm{Cov}\left(\boldsymbol{Y}_t \,|\mathcal{F}_{t-1}\right)$, where $\mathcal{F}_{t-1} = \sigma(\boldsymbol{Y}_{t-1}, \boldsymbol{Y}_{t-2}, \dots)$

According to [1] and [2] we choose $\boldsymbol{\Sigma}_t = \mathrm{diag}(\sigma_{1t}^2, \dots, \sigma_{dt}^2)$, (parsimonious) univariate Garch(p, q)-models for every $1 \leq j \leq d$

$$Y_{jt} = \sigma_{jt}\varepsilon_{jt}, \;\; \sigma_{jt}^2 = \alpha_{j0} + \sum_{i=1}^{p} \alpha_{ji} Y_{j,t-i}^2 + \sum_{k=1}^{q} \beta_{jk}\sigma_{j,t-k}^2$$

and a copula-based multivariate distribution with time-dynamic parameters θ_t for the residuals

$$\boldsymbol{\varepsilon}_t \sim F(\theta_t) = C(F_1, \dots, F_d; \theta_t),$$

such that $\boldsymbol{\varepsilon}_t$ are still independent, but not identically distributed.

Parameter stability. We aim at testing the null hypothesis

$$\mathbf{H}_0\colon\; \{\theta_t \equiv \theta_0 \;\forall\, t\}$$

against the alternative that the parameter changes over time

$$\mathbf{H}_A\colon\; \{\theta_1 = \ldots = \theta_{t_1} \neq \theta_{t_1+1} = \ldots = \theta_{t_k} \neq \theta_{t_k+1} = \ldots\ldots = \theta_T\}\,.$$

Consider the two-step estimation procedure.

1. DeGarching

Let the corresponding observations of $\boldsymbol{Y}_t = (Y_{1t}, \dots, Y_{dt})^\top$ be

$$(y_{1t}, \dots, y_{dt})^\top, \;\; 1 \leq t \leq T.$$

The parameter vector $\gamma_j = (\alpha_{j0}, \alpha_{j1}, \dots, \alpha_{jp}, \beta_{j1}, \dots, \beta_{jq})^\top$ for every component $1 \leq j \leq d$ is estimated via quasi-maximum-likelihood (QMLE):

- assume that $\varepsilon_{jt} \sim \mathcal{N}(0,1)$ for all $1 \leq t \leq T$
- L_T as function of observations and some starting values y_0, σ_0^2 is maximized to obtain $\hat{\gamma}_j$

Even if the normality assumption fails, but $\mathbb{E}\,\varepsilon_{jt}^4 < \infty$, it holds that for $T \to \infty$

$$\hat{\gamma}_j \xrightarrow{a.s.} \gamma_j \quad \text{and} \quad \sqrt{T}(\hat{\gamma}_j - \gamma_j) \xrightarrow{d} \mathcal{N}(0, \mathrm{v}).$$

Since $\sigma_{jt}^2 = w_t(\gamma_j)$, we obtain empirical residuals $\tilde{\varepsilon}_t$ with components

$$\tilde{\varepsilon}_{jt} = \frac{y_{jt}}{\hat{\sigma}_{jt}}, \quad \text{where} \quad \hat{\sigma}_{jt} = \sqrt{\hat{w}_t(\hat{\gamma}_j)}.$$

2. Canonical maximum likelihood

Semi-parametric estimation on $(\tilde{\varepsilon}_{1t}, \dots, \tilde{\varepsilon}_{dt})^\top \sim C(F_1, \dots, F_d; \theta)$

- rank-transformation for the marginal distributions $\tilde{u}_{jt} = \frac{1}{T+1}\sum_{s=1}^{T} \mathbb{1}\{\tilde{\varepsilon}_{js} \leq \tilde{\varepsilon}_{jt}\}$
- maximum likelihood estimation

$$\hat{\theta}_t = \underset{\theta}{\mathrm{argmax}}\; \sum_{s=t-b+1}^{t} \log c(\tilde{u}_{1s}, \dots, \tilde{u}_{ds}; \theta)$$

For $T \to \infty$ and if $b = o(T)$ we have for all t

$$\hat{\theta}_t \xrightarrow{P} \theta \quad \text{and} \quad \sqrt{b}\,(\hat{\theta}_t - \theta) \xrightarrow{d} \mathcal{N}(0, \Sigma)$$

with variance due to [3]

$$\Sigma = \frac{\mathrm{Var}\left(\frac{\partial}{\partial\theta}\log c(F_1(\varepsilon_1), \dots, F_d(\varepsilon_d); \theta) + \sum_{j=1}^{d} W_j(\varepsilon_j)\right)}{\mathbb{E}\left([\frac{\partial}{\partial\theta}\log c(F_1(\varepsilon_1), \dots, F_d(\varepsilon_d); \theta)]^2\right)^2} =: \frac{\sigma^2}{\beta^2}$$

where $W_j(\varepsilon_j) = \int_{[0,1]^d} \mathbb{1}\{F_j(\varepsilon_j) \leq u_j\} \frac{\partial^2}{\partial\theta\partial u_j} \log c(u_1, \dots, u_d; \theta) dC(u_1, \dots, u_d; \theta)$.

Bootstrapping the variance. An extensive simulation study was carried through in order to asses a suitable variance estimate. Despite the theoretical result, we need to non-parametrically bootstrap the variance.

Moving window estimators

With $n^{(c)} = \lfloor (T-b)/cb \rfloor$ with a constant $c \in [1/b, 1]$ we consider the points in time $t_k = T - (n^{(c)} - k)cb$ for all $k = 1, \dots, n^{(c)}$ and

$$\xi_{t_k} := \frac{\sqrt{b}(\hat{\theta}_{t_k} - \hat{\theta})}{\sqrt{\hat{\Sigma}}} \xrightarrow{d} \mathcal{N}(0,1)$$

with $\alpha \in (0,1)$, a global estimate $\hat{\theta}$, a bootstrapped variance estimate $\hat{\Sigma}$ and local estimates $\hat{\theta}_{t_k}$.

Multiple Binomial test

By choosing $c = 1$, we arrive at independent local estimates. Consider

$$S_{n^{(1)}} := \sum_{k=1}^{n^{(1)}} \mathbb{1}\{|\xi_{t_k}| > \mathcal{N}_{0,1}^{-1}(1 - \alpha/2)\} \sim \text{Bin}(n^{(1)}, \alpha)$$

as a multiple test statistic.

The test $S_{n^{(1)}} > \text{Bin}_{n^{(1)},\alpha}^{-1}(1 - \alpha^*)$, $\alpha^* \in (0,1)$ has a local power of order $\sqrt{b}$.

Extremal test

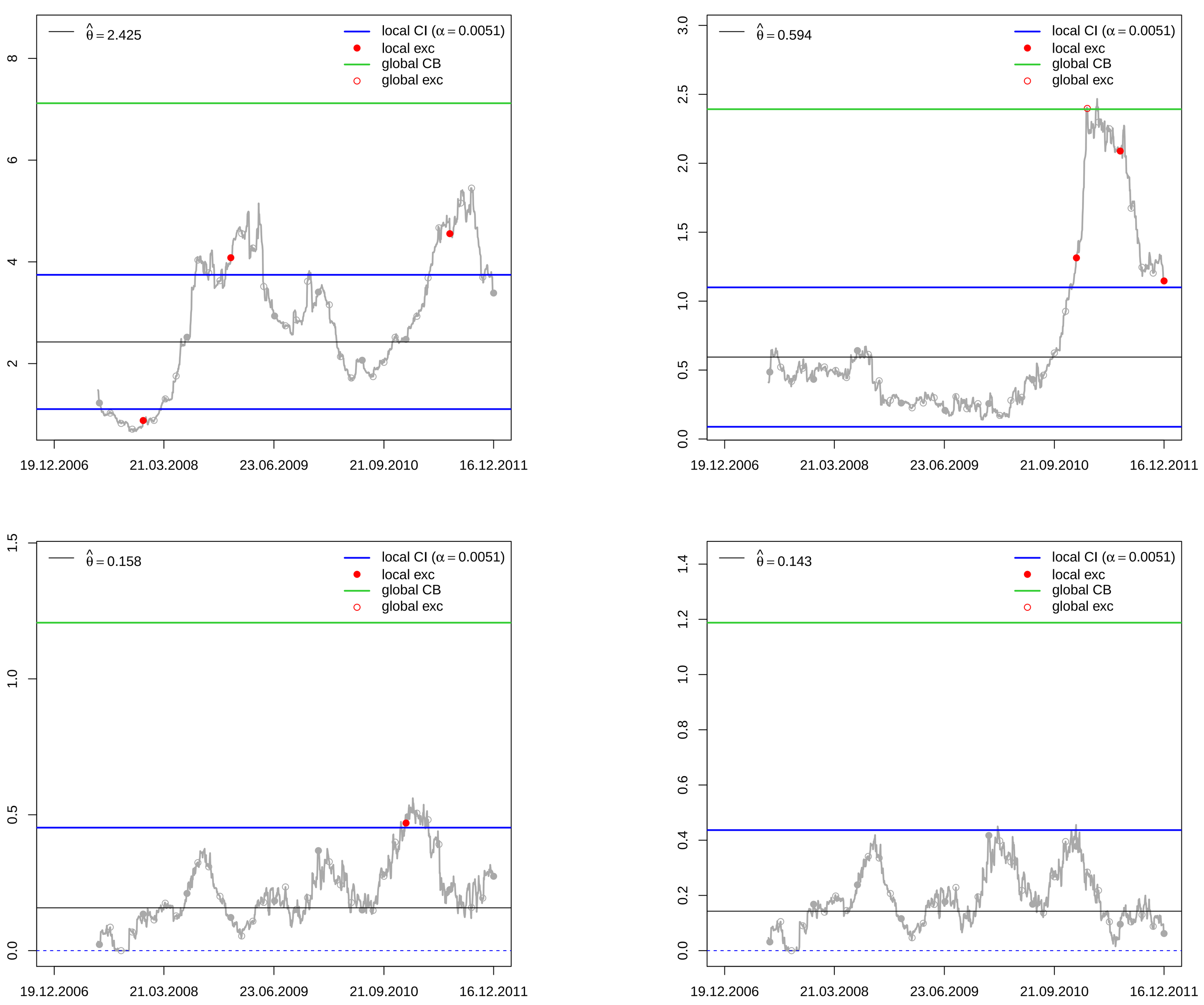
With $c \in (1/b, 1)$ we arrive at a thinned estimator series, and if $b > T^{7/9}$ and as $n^{(c)} \xrightarrow[(T\to\infty)]{} \infty$ we have

$$\lim_{T\to\infty} \mathbb{P}\left((M_{n^{(c)}} - d_{n^{(c)}})/a_{n^{(c)}} \leq x\right) = \Lambda(x) = \mathrm{e}^{-\mathrm{e}^{-x}} \quad \forall\, x \in \mathbb{R},$$

where $M_{n^{(c)}} = \max_k \xi_{t_k}$, $a_{n^{(c)}} = \sqrt{2\log n^{(c)}}$ and $d_{n^{(c)}} = a_{n^{(c)}} - \frac{\log\log n^{(c)} - \log 2\pi}{2a_{n^{(c)}}}$.

The test $(M_{n^{(c)}} - d_{n^{(c)}})/a_{n^{(c)}} > \Lambda^{-1}(1 - \alpha^*)$ has a local power of order $\sqrt{b/\log n^{(c)}}$.

Empirical results



- Binomial test needs independent subsamples, but hints at change points.
- Extremal test has low local power, but is the most omnibus test.
- In empirical analyses there are only few exceedances; often the deviations are extreme in unusual market situations.

References

[1] Chen, X. & Fan, Y. (2006). *Estimation of copula-based semiparametric time series models*, Journal of Econometrics, 130, 307–335.

[2] Chan, N.-H., Chen, J., Chen, X., Fan, Y. & Peng, L. (2009). *Statistical inference for multivariate residual copula for GARCH models*, Statistica Sinica, 19, 53–70.

[3] Genest, C., Ghoudi, K. & Rivest, L.-P. (1995). *A semiparametric estimation procedure of dependence parameters in multivariate families of distributions*, Biometrika, 82, 543–552.

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Contact

Magda Mroz
magda.mroz@uni-ulm.de
+49 (0)731 50 31083

ulm university universität ulm