Goodness-of-fit tests for Archimedean copulas in large dimensions

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Abstract

A goodness-of-fit test for exchangeable Archimedean copulas is presented. In a large-scale simulation study it is shown that the test performs well according to the error probability of the first kind and the power under several alternatives, especially in large dimensions. The proposed test is compared to other known tests for Archimedean copulas. In contrast to the latter, the former is simple and easy to apply in any dimension. Commonly applied goodness-of-fit tests are numerically demanding according to precision and runtime, especially as the dimension increases, which is also investigated in this work. The presented goodness-of-fit test is based on a transformation originally intended for sampling random variates from exchangeable Archimedean copulas and its correctness is proven. The proposed goodness-of-fit test may therefore be interpreted as an analogon to the goodness-of-fit test based on Rosenblatt's transformation which is linked to the conditional distribution method for sampling random variates. Further, it complements in some sense another commonly used goodness-of-fit test based on the probability integral transformation.

Keywords

Archimedean copulas, bootstrap, goodness-of-fit tests, large dimensions, Rosenblatt transformation, sampling transformation.

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1 Introduction

In contrast to the most widely used class of elliptical copulas, *exchangeable Archimedean copulas*, simply referred to as *Archimedean copulas* in the sequel, are given explicitly by

$$C(\boldsymbol{u}) = \psi(\psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d)),\ \boldsymbol{u} \in I^d, \tag{1}$$

where an *(Archimedean) generator* $\psi : [0, \infty] \to I := [0, 1]$ is a continuous, decreasing function, which satisfies $\psi(0) = 1$, $\psi(\infty) := \lim_{t\to\infty} \psi(t) = 0$, and is strictly decreasing on $[0, \inf\{t : \psi(t) = 0\}]$. A necessary and sufficient condition under which (1) is indeed a

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proper copula is that ψ is *d-monotone*, i.e. ψ is continuous on $[0,\infty]$, admits derivatives up to the order $d-2$ satisfying $(-1)^k \frac{d^k}{dt^k}\psi(t) \geq 0$ for all $k \in \{0,\dots,d-2\}$, $t \in (0,\infty)$, and $(-1)^{d-2}\frac{d^{d-2}}{dt^{d-2}}\psi(t)$ is decreasing and convex on $(0,\infty)$, see McNeil and Nešlehová (2009).

Archimedean copulas build a famous class of copulas due to several reasons. Members of this class have an explicit form, given by (1), which often allows for easy handling in calculations. Further, relevant properties of Archimedean copulas can be expressed in terms of the generator ψ, a one-place function. Moreover, in contrast to elliptical copulas, Archimedean copulas may introduce radial asymmetry which allows to model different kinds of tail dependence, a desired feature for many applications.

Goodness-of-fit techniques for copulas only recently gained interest, see e.g. Genest and Rivest (1993), Breymann et al. (2003), Fermanian (2005), Berg and Bakken (2006), Dobrić and Schmid (2007), Genest et al. (2006), Berg and Bakken (2007), Genest et al. (2009), and references therein. This can also be seen by consulting the standard textbook of Cherubini et al. (2004, pp. 176) which only contains two pages that explicitly address goodness-of-fit tests for estimated copula families. Similarly, the famous textbook of McNeil et al. (2005, pp. 236) only gives short references to specific goodness-of-fit tests. Although usually presented in a d-dimensional setting, only some of the publications in this area actually apply goodness-of-fit tests in more than two dimensions, including Berg and Bakken (2007) and Savu and Trede (2008) up to dimension $d = 5$, Berg (2009) up to dimension $d = 8$, and Berg and Bakken (2006) up to dimension $d = 10$. The common deficiency of goodness-of-fit tests for copulas in general, but also for the class of Archimedean copulas, seems to be their limited applicability when the dimension increases. This is mainly due to the fact that goodness-of-fit techniques feasible in small dimensions lack a simple or at least numerically accessible form as the dimension increases. Further, parameter estimation usually becomes much more demanding in large dimensions.

From the practitioner's point of view, however, there is an increasing interest in copula theory and applications in large dimensions. This is intuitively clear if one considers e.g. financial applications including a large number of assets, risks, and so forth. One of our goals is therefore to present and explore goodness-of-fit tests for Archimedean copulas applicable in a large-dimensional framework, say e.g. $d = 10$ or $d = 20$, or even much larger. Further, we investigate the influence of the dimension on the conducted goodness-of-fit tests and address the problems that specifically arise in large dimensions.

As a general goodness-of-fit test, the transformation of Rosenblatt (1952) is well-known. It is important to note that the inverse of this transformation leads to a popular sampling algorithm, the conditional distribution method, see e.g. Embrechts et al. (2001). In other words, for a bijective transformation which converts d independent and identically distributed (i.i.d.) standard uniform random variables to d random variables distributed according to some copula, the corresponding inverse transformation may be applied to obtain d i.i.d. standard uniform random variables from d random variables distributed according to some given copula. In this work we precisely use this idea based on a transformation originally proposed by Wu et al. (2006) for sampling Archimedean copulas. Based on the work of McNeil and Nešlehová (2009) we present a more elegant proof under weaker assumptions. We then apply the first $d-1$ components of the transformation to build a general goodness-of-fit test for Archimedean copulas. This complements goodness-of-fit tests based on the d-th component, the probability integral transformation, see e.g. Genest et al. (2006), Savu and Trede (2008), or Genest et al. (2009). Further, our proposed test can be interpreted as an Archimedean analogon to the goodness-of-fit tests based on the transformation of Rosenblatt (1952) for copulas in general, as it establishes a link between a sampling algorithm and a goodness-of-fit test. The appealing property of the transformation of Wu et al. (2006) for Archimedean copulas is that it is easily applied in any dimension, whereas the general transformation of Rosenblatt (1952), as well as the test based on the probability integral transformation, face several numerical difficulties.

This paper is organized as follows. In Section 2, commonly used goodness-of-fit tests are recalled. In Section 3, a new goodness-of-fit test for Archimedean copulas is presented. Section 4 contains details about the conducted large-scale simulation study, including the implementation in Section 4.1 and the experimental design in Section 4.2. The results of the simulation study are presented in Section 5. Finally, Section 6 concludes.

2 Goodness-of-fit tests for copulas

Let $\boldsymbol{X} = (X_1, \dots, X_d)^T$, $d \in \mathbb{N}\backslash\{1\}$, denote a random vector with common distribution function H and continuous marginals $F_1, \dots, F_d$. By Sklar's Theorem, see e.g. Sklar (1996), there exists a unique copula C which couples $F_1, \dots, F_d$ with H, i.e.

$$H(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d)).$$

In a copula model for $\boldsymbol{X}$, C is assumed to belong to a class $\mathcal{C}_0$ with

$$\mathcal{C}_0 := \{C_\vartheta \mid \vartheta \in \Theta\},$$

where, in our framework, ϑ denotes a single parameter in the set Θ of admissible parameters. After estimating ϑ by $\hat{\vartheta}$ based on $n \in \mathbb{N}$ realizations of i.i.d. random vectors $\boldsymbol{X}_i = (X_{i1}, \dots, X_{id})^T$, $i \in \{1, \dots, n\}$ of H, the goal is to test the null hypothesis

$$H_0: \ C \in \mathcal{C}_0. \tag{2}$$

In practical applications, one usually either assumes that the marginals $F_1, \dots, F_d$ belong to some known parametric families which are estimated beforehand or the marginals are treated as nuisance parameters and are replaced by their (usually slightly scaled) empirical counterparts, see Genest et al. (2009). Following the latter approach one ends up with rank-based pseudo-observations $\boldsymbol{U}_i = (U_{i1}, \dots, U_{id})^T$, $i \in \{1, \dots, n\}$, which are interpreted as observations of C and are therefore used for parameter estimation and goodness-of-fit techniques to test H_0. As e.g. Dobrić and Schmid (2007) describe, there are two problems with this approach. First, the pseudo-observations are neither realizations of perfectly independent random vectors nor are the components perfectly following a univariate standard uniform distribution. This affects the null distribution of the test statistic under consideration. However, all goodness-of-fit approaches suffer from these effects and a solution may be a bootstrap procedure to access the exact null distribution. Note that particularly in large dimensions, conducting a bootstrap is often too time-consuming, especially for the commonly used goodness-of-fit tests. For a comparison with our proposed goodness-of-fit test, we therefore assume that we have data from C directly. Second, using estimated copula parameters additionally affects the null distribution. In our large-scale simulation study, we take care of this problem by investigating the behavior of the tests under some variations of the tested parameters, described in Section 4.2. Since our proposed goodness-of-fit test turns out to be fast enough, even in large dimensions, we conduct a bootstrap procedure for this approach, see Section 4.2.

In order to conduct a goodness-of-fit test, the given data $\boldsymbol{U}_i$, $i \in \{1, \dots, n\}$, is usually first transformed to some data $\boldsymbol{U}_i'$, $i \in \{1, \dots, n\}$, so that the distribution of the latter is known under the null hypothesis; for Rosenblatt's transformation, see Section 2.1, the data $\boldsymbol{U}_i'$, $i \in \{1, \dots, n\}$, is also d-dimensional; for the testing approach based on the probability integral transformation, described in Section 2.2, it is one-dimensional; and for the goodness-of-fit approach we propose in Section 3, it is $d-1$-dimensional. If not already one-dimensional, after such a transformation, the data $\boldsymbol{U}_i'$, $i \in \{1, \dots, n\}$, is usually mapped to one-dimensional data Y_i, $i \in \{1, \dots, n\}$, such that the corresponding distribution F_Y is again known under the null hypothesis. So indeed, instead of (2), one usually considers some adjusted hypothesis

$$H_0^* : \ F_Y \in \mathcal{F}_0$$

under which a goodness-of-fit test can be carried out in a one-dimensional setting. For this, different approaches exist, see Section 2.2. The reasoning behind this procedure is that if H_0^* is rejected, so is H_0. However, due to informational loss by reducing the dimension, H_0 is a subset of H_0^* and hence the two hypotheses are not equivalent. So we keep in mind that in fact there is already some informational loss inherent in commonly used goodness-of-fit test as they are carried out in a one-dimensional setting.

2.1 Rosenblatt's transformation

The transformation introduced by Rosenblatt (1952) is a standard approach for obtaining realizations of standard uniform random vectors $\boldsymbol{U}_i'$, $i \in \{1, \dots, n\}$, given the random vectors $\boldsymbol{U}_i$, $i \in \{1, \dots, n\}$, which can then be tested or further mapped to one-dimensional variates for testing purposes. Consider a representative d-dimensional random vector $\boldsymbol{U}$ distributed according to the copula C. To obtain a standard uniform random vector $\boldsymbol{U}'$ on I^d, Rosenblatt (1952) proposed the transformation $R : \boldsymbol{U} \to \boldsymbol{U}'$, given by

$$\begin{aligned} U_1' &:= U_1, \\ U_2' &:= C_2(U_2 \,|\, U_1), \\ &\ \vdots \\ U_d' &:= C_d(U_d \,|\, U_1, \dots, U_{d-1}), \end{aligned}$$

where for $j \in \{2, \dots, d\}$, $C_j(u_j \,|\, u_1, \dots, u_{j-1})$ denotes the conditional distribution function of U_j given $U_1 = u_1, \dots, U_{j-1} = u_{j-1}$. This transformation is quite general in that it applies to any copula.

Note that the inverse transformation R^{-1} of Rosenblatt's transformation leads to a general sampling algorithm for copulas, the so-called conditional distribution method, see e.g. Embrechts et al. (2001). In fact, this link brings rise to the general idea of using sampling algorithms based on transformations of d i.i.d. standard uniform random variables to construct goodness-of-fit tests. This is done in Section 3 to construct a goodness-of-fit test for Archimedean copulas based on a transformation originally proposed by Wu et al. (2006) for sampling random variates.

To find the quantities $C_j(u_j \,|\, u_1, \dots, u_{j-1})$, $j \in \{2, \dots, d\}$, for a specific copula C, the following connection between conditional distributions and partial derivatives is usually applied, a proof of which can be found in Schmitz (2003, p. 20). Assuming C admits continuous partial derivatives with respect to the first $d-1$ arguments, then

$$C_j(u_j \,|\, u_1, \dots, u_{j-1}) = \frac{\mathrm{D}_{j-1,\dots,1}\, C^{(1,\dots,j)}(u_1, \dots, u_j)}{\mathrm{D}_{j-1,\dots,1}\, C^{(1,\dots,j-1)}(u_1, \dots, u_{j-1})},\ j \in \{2, \dots, d\}, \tag{3}$$

where $C^{(1,\dots,k)}$ denotes the k-dimensional marginal copula of C corresponding to the first k arguments and $\mathrm{D}_{j-1,\dots,1}$ denotes the mixed partial derivative of order $j-1$ with respect to the first $j-1$ arguments.

The problem in applying (3) in large dimensions is that it is usually quite difficult to access the involved derivatives, the price which one has to pay for such a general transformation. Further, numerically evaluating the derivatives is often time-consuming and prone to numerical errors.

Rosenblatt's transformation as a method for constructing goodness-of-fit tests is denoted by R in the sequel.

2.2 Reducing the dimension

In order to apply a goodness-of-fit test in a one-dimensional setting one has to summarize the d-dimensional data $\boldsymbol{U}_i$ or $\boldsymbol{U}_i'$ to one-dimensional quantities Y_i, $i \in \{1, \dots, n\}$, for which the distribution is known under the null hypothesis. In what follows, some popular mappings achieving this task are described.

N_d: Under H_0, the one-dimensional random variables $Y_i = F_{\chi_d^2}\big(\sum_{j=1}^d (\Phi^{-1})^2(U_{ij}')\big)$, $i \in \{1, \dots, n\}$, should be i.i.d. according to a standard uniform distribution, where $F_{\chi_d^2}$ denotes the distribution function of a χ_d^2 distribution, i.e. a χ^2 distribution with d degrees of freedom, and Φ denotes the standard normal distribution function. This transformation is denoted by N_d in the sequel.

$S_{n,d}^B$: Another test statistic is proposed by Genest et al. (2009). As an overall result, they recommend to use a distance between the distribution under H_0, assumed to be standard uniform on I^d, and the empirical distribution, namely

$$S_{n,d}^B := n \int_{I^d} (D_n(\boldsymbol{u}) - \Pi(\boldsymbol{u}))^2 \, d\boldsymbol{u},$$

where $\Pi(\boldsymbol{u})$ denotes the independence copula and $D_n(\boldsymbol{u}) := \frac{1}{n}\sum_{i=1}^n \mathbb{1}_{\{\boldsymbol{U}_i' \le \boldsymbol{u}\}}$, i.e. the empirical distribution function based on the random vectors $\boldsymbol{U}_i'$, $i \in \{1,\dots,d\}$. This transformation is referred to as $S_{n,d}^B$.

K_C: For a copula C let K_C denote the *probability integral transformation* (or *Kendall's transformation*), i.e. $K_C(t) := \mathbb{P}(C(\boldsymbol{U}) \le t)$, $t \in I$, where $\boldsymbol{U} \sim C$, see Barbe et al. (1996) or McNeil and Nešlehová (2009). Under H_0 and if K_C is continuous, the random variables $Y_i := K_C(C(\boldsymbol{U}_i))$ should be i.i.d. according to a standard uniform distribution. This approach for goodness-of-fit testing will be referred to as K_C. Note that in this case, no multidimensional transformation of the data is performed beforehand. Further, it is well known, that $K_C(t)$ is not unique, in the sense that there can be more than one copula admitting the same $K_C(t)$, see Genest and Boies (2003). This fact implies that a test based on $K_C(t)$ cannot distinguish between two copulas having the same $K_C(t)$.

K_Π: One can also consider the random vectors $\boldsymbol{U}_i'$, $i \in \{1,\dots,n\}$, in conjunction with the independence copula, i.e. define $\tilde{Y}_i := \prod_{j=1}^d U_{ij}'$, where $\tilde{Y}_i$ has distribution function $K_\Pi(t) = t\sum_{k=0}^{d-1}\frac{1}{k!}(-\log t)^k$. Under H_0, the sample $Y_i := K_\Pi(\tilde{Y}_i)$, $i \in \{1,\dots,n\}$, should indicate a uniform distribution on the unit interval. This approach is referred to as K_Π.

In the approaches N_d, K_C, and K_Π we have to test the hypothesis that realizations of the random variables Y_i, $i \in \{1,\dots,n\}$, follow a uniform distribution on the unit interval. This may be achieved in several ways, the following two of which are applied in the sequel.

χ^2: Pearson's χ^2 test, see Rao (2001, p. 391), shortly referred to as χ^2.

AD: The so-called Anderson-Darling test, a specifically weighted Cramér-von Mises test, see Anderson and Darling (1952) and Anderson and Darling (1954). This method is referred to as AD.

3 A goodness-of-fit test for Archimedean copulas

In this section we present a goodness-of-fit test for Archimedean copulas in large dimensions. This test is based on the following transformation originally presented in Wu et al. (2006) for generating random variates from Archimedean copulas. Note that we present a rather short proof of this interesting result, under weaker assumptions.

Theorem 3.1 (The main transformation)
Let $\boldsymbol{U} \sim C$, $d \in \mathbb{N}\backslash\{1\}$, where C is an Archimedean copula with d-monotone generator ψ and continuous probability integral transformation K_C. Then $\boldsymbol{U'} \sim \mathrm{U}(I^d)$, where

$$U_j' = \left(\frac{\sum_{k=1}^j \psi^{-1}(U_k)}{\sum_{k=1}^{j+1} \psi^{-1}(U_k)}\right)^j,\ j \in \{1,\dots,d-1\},\ U_d' = K_C(C(\boldsymbol{U})). \tag{4}$$

Proof
As shown in McNeil and Nešlehová (2009), $(\psi^{-1}(U_1),\dots,\psi^{-1}(U_d))^T$ has an l_1-norm

symmetric distribution with survival copula C and radial distribution $F_R = \mathcal{W}_d^{-1}[\psi]$, where $\mathcal{W}_d[\cdot]$ denotes the Williamson d-transform. Hence, $(\psi^{-1}(U_1), \dots, \psi^{-1}(U_d))^T \overset{\mathrm{d}}{=} R\boldsymbol{S}$, where $R \sim F_R$ and $\boldsymbol{S} \sim \mathrm{U}(\{\boldsymbol{x} \in \mathbb{R}_+^d \mid ||\boldsymbol{x}||_1 = 1\})$ are independent. For $Z_{(0)} := 0$, $Z_{(d)} := 1$, and $(Z_1, \dots, Z_{d-1}) \sim \mathrm{U}(I^{d-1})$, it follows from Devroye (1986, p. 207) that $S_j \overset{\mathrm{d}}{=} Z_{(j)} - Z_{(j-1)}$, $j \in \{1, \dots, d\}$, independent of R. This implies that $\psi^{-1}(U_j) \overset{\mathrm{d}}{=} R(Z_{(j)} - Z_{(j-1)})$, $j \in \{1, \dots, d\}$, and hence that

$$\boldsymbol{U}' \overset{\mathrm{d}}{=} \left(\left(\frac{Z_{(1)}}{Z_{(2)}} \right)^1, \dots, \left(\frac{Z_{(d-1)}}{Z_{(d)}} \right)^{d-1}, K_C(\psi(R)) \right) =: \boldsymbol{X}.$$

Since K_C is continuous and $\psi(R) \sim K_C$, $K_C(\psi(R))$ is uniformly distributed in I. Further, as a function in R, $K_C(\psi(R))$ is independent of $(X_1, \dots, X_{d-1})^T$. It therefore suffices to show that $(X_1, \dots, X_{d-1})^T \sim \mathrm{U}(I^{d-1})$, a proof of which can be found in Devroye (1986, p. 212). □

The proof of Theorem 3.1 implies the following result.

Corollary 3.2
Let $\boldsymbol{U} \sim C$, $d \in \mathbb{N}\backslash\{1\}$, where C is an Archimedean copula with d-monotone generator ψ. Then $(U_1', \dots, U_{d-1}')^T \sim \mathrm{U}(I^{d-1})$, where

$$U_j' = \left(\frac{\sum_{k=1}^{j} \psi^{-1}(U_k)}{\sum_{k=1}^{j+1} \psi^{-1}(U_k)} \right)^j, \; j \in \{1, \dots, d-1\}.$$

The transformation $T : \boldsymbol{U} \to \boldsymbol{U'}$ addressed in (4) can be interpreted as an analogon to Rosenblatt's transformation R specifically for Archimedean copulas. Both T and R map d random variables to d random variables and can therefore be used in both directions, for generating random variates and goodness-of-fit testing; the latter approach for T is proposed in this paper. The advantage of this approach for obtaining the data $\boldsymbol{U}_i' \sim \mathrm{U}(I^d)$, $i \in \{1, \dots, n\}$, from $\boldsymbol{U}_i \sim C$, $i \in \{1, \dots, n\}$, in comparison to Rosenblatt's transformation lies in the fact that it may be much easier to compute the quantities in (4) than accessing the derivatives in (3). One can then proceed as for Rosenblatt's transformation and use one of the transformations listed in Section 2.2 to transform the data $\boldsymbol{U}_i'$, $i \in \{1, \dots, n\}$, to the one-dimensional data Y_i, $i \in \{1, \dots, n\}$, for testing H_0^*. A test involving the transformation T to obtain the random vectors $\boldsymbol{U}_i' \sim \mathrm{U}(I^d)$, $i \in \{1, \dots, n\}$, is referred to as approach T_d in the sequel.

One may argue that the evaluation of K_C involves knowledge of the derivatives of ψ, see Barbe et al. (1996) or McNeil and Nešlehová (2009), so evaluating K_C for computing T_d may be as difficult as computing the derivatives in (3). However, first note that evaluation of numerically complicated quantities is only required for computing U_d' in (4), whereas computing U_j', $j \in \{1, \dots, d-1\}$, is easily achieved for any Archimedean copula, even under no additional assumption on ψ, see Corollary 3.2. Further, if K_C is continuous, which holds e.g. if ψ is $d-1$ times continuously differentiable, see McNeil and Nešlehová (2009), it can be written as $K_C(t) = \sum_{k=0}^{d-1} (0 - \psi^{-1}(t))^k \psi^{(k)}(\psi^{-1}(t))/k!$.

For fixed $t \in I$, K_C can therefore be interpreted as a Taylor polynomial of order $d-1$ about $\psi^{-1}(t)$ evaluated at 0. Since computing Taylor polynomials is numerically well investigated, evaluating K_C is generally a good-natured problem. Still, the computation of K_C may be numerically demanding in large dimensions.

Now assume the dimension to be large. For $d \to \infty$, K_C converges pointwise to the distribution function which jumps from 0 to 1 at zero. Further, for large d, evaluation of K_C is often numerically complicated, except for specific cases such as Clayton's Archimedean copula family where all involved derivatives of ψ are directly accessible, see Wu et al. (2006). Moreover, note that applying T_d for obtaining the transformed data $\boldsymbol{U}_i'$, $i \in \{1, \dots, n\}$, requires n-times the evaluation of the probability integral transformation K_C, which can be computationally intense, especially in simulation studies. With the informational loss inherent in the goodness-of-fit tests following the approaches addressed in Section 2.2 in mind, one may therefore suggest to omit the last component T_d of T and only consider $T_1, \dots, T_{d-1}$, i.e. using the data $(U_{i1}', \dots, U_{id-1}')$, $i \in \{1, \dots, n\}$, for testing purposes if d is large. Our main goal is to show that this leads to fast goodness-of-fit tests for Archimedean copulas in large dimensions, such that the informational loss due to omitting T_d is negligible. A goodness-of-fit test based on omitting the last component of the transformation T is referred to as approach T_{d-1} in the sequel.

4 A large-scale simulation study

4.1 A word concerning the implementation

All numerical experiments are run on a compute node which consists of eight cores (two four-core Intel Xeon E5440 Harpertown CPUs with 2.83GHz and 6MB second level cache) and 16GB memory. The node is part of the bwGRiD Cluster Ulm, see bwGRiD. All algorithms are implemented in C/C++ and compiled using GCC 4.2.4 with option `-O2` for code optimization. Moreover, we use the algorithms of the Numerical Algorithms Group, see NAG, the GNU Scientific Library 1.12, see GSL, and the OpenMaple interface of Maple 12, see Maplesoft. For generating uniform random variates an implementation of the Mersenne Twister by Wagner (2003) is used. For the Anderson-Darling test, the procedures suggested in Marsaglia and Marsaglia (2004) are used.

4.2 The experimental design

In this section the experimental design for the large-scale simulation study to compare various goodness-of-fit tests, including our proposed goodness-of-fit transformation, is described. Focus is put on two features, the error probability of the first kind, i.e. if a test maintains its nominal level, and the power under several alternatives. To distinguish between the different approaches we use either pairs or triples, e.g. the approach (T_{d-1}, N_{d-1}, AD) denotes a goodness-of-fit test based on first applying our proposed transformation T without the last component T_d, then using the approach based on the χ^2_{d-1} distribution to transform the data to one dimension, and then applying the

Anderson-Darling statistic to test H_0^*; similarly, $(T_{d-1}, S^B_{n,d-1})$ denotes a goodness-of-fit test which uses the approach $S^B_{n,d-1}$ for reducing the dimension and testing H_0^*.

In the conducted Monte Carlo simulation, the following nine different goodness-of-fit approaches are tested:

$$(T_{d-1}, N_{d-1}, AD),\ (T_{d-1}, S^B_{n,d-1}),\ (K_C, \chi^2),\ (K_C, AD),\ (T_d, N_d, AD),\\ (T_d, K_\Pi, AD),\ (T_d, S^B_{n,d}),\ (R, N_d, AD),\ (R, S^B_{n,d}). \tag{5}$$

We investigate all possible combinations of two sample sizes ($n = 150$ and $n = 500$), two different dependence levels, measured with Kendall's τ ($\tau = 0.25$ and $\tau = 0.5$), and three dimensions ($d = 5$, $d = 10$, and $d = 20$). For every scenario, we simulate the corresponding Archimedean copulas of Ali-Mikhail-Haq (A), Clayton (C), Frank(F), Gumbel (G), and Joe (J), see Nelsen (2007, pp. 116), as well as the Gaussian (Ga) and t copula with four degrees of freedom (t_4). For the scenarios (R, N_d, AD) and $(R, S^B_{n,d})$, all of the seven copula families are tested; otherwise, only the Archimedean among these copula families can be considered as at least one component of the transformation T is involved. Due to the fact that it is not possible for an Ali-Mikhail-Haq copula to attain Kendall's tau equal to 0.5, this copula family is only involved if $\tau = 0.25$. All in all, this amounts to 3498 goodness-of-fit testing scenarios.

For all testing scenarios, we assume the random vectors $\boldsymbol{U}_i \sim C$, $i \in \{1, \ldots, n\}$, to be independent with standard uniform marginals, so there is no need to consider ranks, which would additionally affect the null distribution. For estimating the copula parameter there exist several procedures including maximum likelihood estimation or the inversion of Kendall's tau based on averaged pairwise Kendall's taus; the latter approach is followed e.g. by Berg (2009) for estimating copula parameters. Note that parameter estimation may also change the null distribution. This generally requires a bootstrap procedure for accessing the correct null distribution. However, note that a bootstrap can be quite time-consuming in large dimensions, even parameter estimation already turns out to be computationally demanding. In the dimensions we work, this would have exceeded reasonable amounts of computational time. For comparing the goodness-of-fit tests listed in (5) in the first part of our simulation study, we therefore proceed as follows. We imitate an estimation procedure and test the known and correct parameter, i.e. the one which is used to simulate the random variates, but also two further parameters, denoted by ϑ_- and ϑ_+. These are determined in the following way. Given the correct value of τ, we consider minus and plus one standard deviation of the averaged pairwise Kendall's tau estimator and take the parameters ϑ_- and ϑ_+ corresponding to these Kendall's taus, respectively. For all involved dimensions d, sample sizes n, Kendall's taus τ, and copula families, the standard deviation of the averaged pairwise Kendall's tau estimator is determined by simulation beforehand, based on 500 000 replications. So indeed, for each of the 3498 scenarios, the three parameters ϑ_-, ϑ, and ϑ_+ are tested.

In contrast to all other investigated goodness-of-fit tests, our proposed goodness-of-fit test works fast even in large dimensions. We are therefore able to investigate our test with a bootstrap procedure. This is done in the second part of our simulation study.

We now address the chosen parameters in our simulation study. For all testing approaches listed in (5), $N = 1\,000$ replications are used for computing the empirical level and power, except for the tests involving Rosenblatt's transformation for the Gaussian and the t_4 H_0 copulas, where $N = 100$ replications are used due to extensive computational effort. The significance level is arbitrarily fixed at $\alpha = 5\%$. For the χ^2-test, 20 cells were used. For each n and d, $m = 10\,000$ values of the test statistic $S^B_{n,d}$ are computed via simulated vectors of $\mathrm{U}(I^d)$-distributed random variates beforehand. These values are used in goodness-of-fit tests involving $S^B_{n,d}$ for deciding whether H_0^* is rejected or not.

Concerning the use of Maple, we proceed as follows. For computing the first $d-1$ components $T_1, \dots, T_{d-1}$ of the transformation T involved in the first two and the fifth to seventh approach listed in (5), Maple is only used if working under double precision in C/C++ leads to errors. With errors, non-float values including `nan`, `-inf`, and `inf`, as well as float values less than zero or greater than one are meant. For computing the last component T_d involved in the fifth to seventh approach, Maple is used for all tested Archimedean families except Clayton's where an explicit form of all derivatives and hence K_C is known, see Wu et al. (2006). The same holds for computing K_C for the third and fourth approach in (5). For the approaches involving Rosenblatt's transform, a computation in C/C+ is tried for Clayton's family, the Gaussian, and the t_4 copula, whereas Maple is used for all other copula families or if the computation for the former three copulas leads to errors. If Rosenblatt's transformation produces errors even after computations in Maple, we disregard the corresponding goodness-of-fit test and use the remaining test results of the simulation for computing the empirical level and power.

As mentioned several times, one usually performs a bootstrap procedure in order to get rid of the unrealistic assumption of perfectly standard uniformly distributed marginals or a perfect estimation procedure. It will become clear from the results in Section 5 that our proposed goodness-of-fit test is fast enough to be applied in conjunction with a bootstrap procedure, described in the remaining part of this section. Bootstrap versions of all other goodness-of-fit tests in our investigations are presented in Genest et al. (2009), however, only applied in a two-dimensional setting where computations are usually much simpler and faster.

Note that for our proposed approach (T_{d-1}, N_{d-1}, AD) it is not clear whether the bootstrap procedure is valid from a theoretical point of view, see e.g. Dobrić and Schmid (2007) and Genest et al. (2009). However, empirical results, presented in Section 5, indicate the validity of this approach, described as follows.

(1) Given the data $\boldsymbol{U}_i$, $i \in \{1, \dots, n\}$, build the vectors of componentwise scaled ranks $\boldsymbol{U}_{(i)} = (U_{i(1)}, \dots, U_{i(d)})^T$, where $U_{i(j)} := (\sum_{k=1}^n \mathbb{1}_{\{U_{kj} \le U_{ij}\}})/(n+1)$, $i \in \{1, \dots, n\}$, $j \in \{1, \dots, d\}$. Based on $\boldsymbol{U}_{(i)}$, $i \in \{1, \dots, n\}$, estimate the unknown parameter ϑ by $\hat{\vartheta}$.

(2) Based on $\boldsymbol{U}_{(i)}$, $i \in \{1, \dots, n\}$, compute the first $d-1$ components U'_{ij}, $i \in \{1, \dots, n\}$, $j \in \{1, \dots, d-1\}$, of the transformation T as in Equation (4) and $Y_i := \sum_{j=1}^{d-1} (\Phi^{-1}(U'_{ij}))^2$, $i \in \{1, \dots, n\}$. Then compute the Anderson-Darling test statistic $A_n := -n - \frac{1}{n} \sum_{i=1}^n (2i-1)[\log(F_{\chi^2_{d-1}}(Y_i)) + \log(1 - F_{\chi^2_{d-1}}(Y_i))]$.

(3) Choose the number N of bootstrap repetitions. For each $k \in \{1, \dots, N\}$ do:

(3.1) Generate the random sample $\boldsymbol{U}_{i,k}^* = (U_{i1,k}^*, \dots, U_{id,k}^*)^T \sim C_{\hat{\vartheta}}$, $i \in \{1, \dots, n\}$, and compute the vectors of componentwise scaled ranks $\boldsymbol{U}_{(i),k}^*$, $i \in \{1, \dots, n\}$. Based on $\boldsymbol{U}_{(i),k}^*$, $i \in \{1, \dots, n\}$, estimate the unknown parameter ϑ by $\hat{\vartheta}_k^*$.

(3.2) Based on $\boldsymbol{U}_{(i),k}^*$, $i \in \{1, \dots, n\}$, compute the first $d-1$ components $U_{ij,k}'^*$, $i \in \{1, \dots, n\}$, $j \in \{1, \dots, d-1\}$, of the transformation T as in Equation (4) and $Y_{i,k}^* := \sum_{j=1}^{d-1} (\Phi^{-1}(U_{ij,k}'^*))^2$, $i \in \{1, \dots, n\}$. Then compute the Anderson-Darling test statistic $A_{n,k}^* := -n - \frac{1}{n} \sum_{i=1}^n (2i-1)[\log(F_{\chi_{d-1}^2}(Y_{i,k}^*)) + \log(1 - F_{\chi_{d-1}^2}(Y_{i,k}^*))]$.

(4) An approximate p-value for the test (T_{d-1}, N_{d-1}, AD) is given by $\frac{1}{N} \sum_{k=1}^N \mathbb{1}_{\{A_{n,k}^* > A_n\}}$.

5 Results

In this section we present the results from the large-scale simulation study conducted for the nine different goodness-of-fit approaches listed in (5) and described in Section 4. To reduce the rather huge amount of test results, we are restricted to report only a limited amount of such, summarizing the characteristics we found in the simulation study.

As an overall effect, we find that all goodness-of-fit tests show larger power against the investigated alternatives if the sample size n increases from $n = 150$ to $n = 500$, as well as if the dependence increases from $\tau = 0.25$ to $\tau = 0.5$. This is supported by intuition, as it is usually easier to distinguish between data sets if the number of samples is larger or the dependence is stronger. We therefore only report results for the case $n = 150$ and $\tau = 0.25$. One of the main results we found is that the empirical power against all investigated alternatives increases if the dimension increases. However, so does runtime. Further, the obtained results indicate that ten dimensions represent an intermediate state between five and twenty dimensions according to the empirical power against the investigated alternatives and runtime. We therefore report the results only for the cases $d = 5$ and $d = 20$, and address the case $d = 10$ only when needed.

Let us first discuss the methods that show a comparably weak performance in the conducted simulation study. We start with the results that are based on the test statistics $S_{n,d-1}^B$ or $S_{n,d}^B$ to reduce the dimension. Although keeping the error probability of the first kind, the goodness-of-fit tests $(T_{d-1}, S_{n,d-1}^B)$, $(T_d, S_{n,d}^B)$, and $(R, S_{n,d}^B)$ show a comparably weak performance against the investigated alternatives, at least in our test setting as described in Section 4.2. For example, for $n = 150$, $d = 5$, and $\tau = 0.25$, the method $(R, S_{n,d}^B)$ leads to an empirical power of 3.4% for testing Clayton's copula when the simulated copula is Ali-Mikhail-Haq's, 2.0% for testing the Gaussian copula on Ali-Mikhail-Haq copula data, 4.1% for testing Ali-Mikhail-Haq's copula on data from Frank's copula, and 3.4% for testing Gumbel's copula on data from Joe's copula. Similarly for $n = 500$, $d \in \{10, 20\}$, $\tau = 0.5$, and the methods $(T_{d-1}, S_{n,d-1}^B)$ and $(T_d, S_{n,d}^B)$. We therefore do not further report on the methods involving $S_{n,d-1}^B$ or $S_{n,d}^B$. The method (T_d, K_Π, AD) also shows a rather weak performance for both investigated sample sizes n and all three investigated dimensions d, however, note that the empirical power against the investigated alternative families of Clayton, Frank, Gumbel, and Joe significantly increases for $\tau = 0.5$ in this case. The goodness-of-fit testing approach (K_C, χ^2) faced problems uniformly over all dimensions $d \in \{5, 10, 20\}$ and dependencies $\tau \in \{0.25, 0.5\}$, where the empirical error probability of the first kind is around 10%. Only for $n = 500$, uniformly over all d and τ, the expected level of approximately 5% is obtained.

Now consider the goodness-of-fit testing approaches (T_{d-1}, N_{d-1}, AD), (K_C, AD), and (T_d, N_d, AD). Recall that (T_{d-1}, N_{d-1}, AD) is based on the first $d-1$ components of the transformation T addressed in Equation (4), (K_C, AD) applies only the last component of T, and (T_d, N_d, AD) applies the whole transformation T in d dimensions, where all three approaches use the Anderson-Darling test to test H_0^*. The test results for the three goodness-of-fit tests with $n = 150$, $\tau = 0.25$, and $d \in \{5, 20\}$ are reported in Tables 1, 2, and 3, respectively. As mentioned above, we test and report three different parameters, the known and correct parameter ϑ which is reported in the middle of the relevant block, and the parameters ϑ_- and ϑ_+ which are reported above, respectively below, the true parameter. Note that, for the true parameter, all three approaches keep the error probability of the first kind at approximately 5% as expected. However, we see significant deviations thereof when ϑ_- or ϑ_+ is used. This indicates that a bootstrap procedure is needed when the parameter involved is estimated. As all procedures have these effects in common, we first compare them without conducting a bootstrap. First consider the low-dimensional case $d = 5$. On average, method (K_C, AD) shows the largest empirical power against the investigated Archimedean alternatives and method (T_{d-1}, N_{d-1}, AD) the largest against the Gaussian and the t_4 copula. For (T_d, N_d, AD), even if the transformation T with all d components is applied, there surprisingly seems to be no gain in power, which can be inferred from Table 3. Now let us turn to the large-dimensional case $d = 20$. Again on average, method (K_C, AD) shows the largest empirical power against the Archimedean alternatives and method (T_{d-1}, N_{d-1}, AD) has the largest empirical power against the Gaussian and the t_4 the copula. However, the differences in power vanish. This is also supported by the case $d = 10$ which we do not report.

Table 4 shows the empirical power of the method (R, N_d, AD). In comparison to the goodness-of-fit approach (T_{d-1}, N_{d-1}, AD), both methods seem to have advantages over each other. On average, for $d = 5$, the empirical power for the H_0 copula of Clayton is larger for (R, N_d, AD) than for (T_{d-1}, N_{d-1}, AD); the other way round for Gumbel's copula; for the family of Frank, if the true copula is Ali-Mikhail-Haq's or Clayton's copula, then (R, N_d, AD) shows larger power, and if the true copula is Gumbel's, Joe's, or the Gaussian copula, then (T_{d-1}, N_{d-1}, AD) shows larger power. For the large-dimensional case $d = 20$ the differences among the methods are again much smaller.

Another aspect, especially in large-scale simulation studies is computational efficiency. In going from the low- to the large-dimensional case we faced several problems during our computations. As indicated by the symbol $*$ in Table 4, the approach (R, N_d, AD) shows difficulties in testing the H_0 copula of Ali-Mikhail-Haq for $d = 20$. Although applying Maple with `Digits` set to 15, all of the $N = 1\,000$ goodness-of-fit tests indicated numerical problems and so no test result was obtained. Another problem appeared partly for the Gaussian and the t_4 copula. Although we only used $N = 100$ replications for testing these two H_0 copulas, we did not obtain a result within seven days of runtime,

	True copula, $d = 5$							True copula, $d = 20$						
H_0	A	C	F	G	J	Ga	t_4	A	C	F	G	J	Ga	t_4
	16.4	29.4	11.5	37.1	48.2	6.1	91.3	28.0	62.6	10.6	66.7	81.8	47.9	100.0
A	**5.1**	8.0	5.5	8.5	13.4	11.7	77.9	**4.2**	20.5	17.3	18.9	76.4	82.8	100.0
	20.0	9.9	29.1	10.1	11.2	50.6	66.4	35.1	8.2	76.5	45.6	98.3	99.6	100.0
	8.0	20.6	8.0	21.5	32.6	6.4	90.7	21.9	42.1	87.8	92.9	100.0	97.2	100.0
C	14.1	**4.7**	25.9	8.8	20.9	38.8	79.6	80.9	**5.0**	99.7	99.5	100.0	100.0	100.0
	51.0	22.9	71.3	44.7	59.6	84.8	78.8	99.5	49.3	100.0	100.0	100.0	100.0	100.0
	15.2	26.2	13.6	41.9	57.0	7.3	91.5	31.1	59.2	25.7	89.7	94.2	74.8	100.0
F	6.6	9.0	**4.3**	12.4	21.4	8.4	77.8	5.8	29.4	**4.9**	44.4	65.8	83.4	100.0
	16.2	16.0	14.9	4.0	5.3	34.0	66.7	22.4	33.1	27.6	6.1	26.4	96.5	100.0
	16.5	32.2	9.6	23.3	40.4	14.0	95.2	50.1	87.6	22.6	55.5	85.7	99.8	100.0
G	34.0	43.7	25.5	**2.9**	6.6	40.4	90.4	87.5	98.5	78.7	**4.3**	16.6	99.9	100.0
	74.7	76.2	68.6	23.5	10.8	80.6	90.8	99.8	99.9	99.8	61.7	27.4	100.0	100.0
	51.0	70.0	28.6	13.7	26.6	45.0	97.3	98.4	100.0	86.1	23.5	73.6	100.0	100.0
J	83.7	90.3	69.0	11.6	**4.3**	80.7	98.1	100.0	100.0	99.5	35.4	**4.9**	100.0	100.0
	98.5	99.0	96.1	61.9	33.0	97.5	99.1	100.0	100.0	100.0	98.4	78.1	100.0	100.0

Table 1 Empirical power in % for testing ϑ_-, ϑ, and ϑ_+ for (T_{d-1}, N_{d-1}, AD) based on $N = 1\,000$ replications with $n = 150$, $\tau = 0.25$, and $d = 5$ (left), respectively $d = 20$ (right).

	True copula, $d = 5$							True copula, $d = 20$						
H_0	A	C	F	G	J	Ga	t_4	A	C	F	G	J	Ga	t_4
	7.2	14.9	19.1	57.9	96.7	7.1	12.0	9.7	55.0	75.5	98.9	100.0	31.2	49.9
A	**3.4**	8.8	21.7	69.8	98.3	12.0	17.8	**5.4**	35.2	83.3	99.5	100.0	45.8	54.3
	5.2	7.8	30.2	80.2	98.9	25.2	29.5	7.4	21.7	86.0	99.9	100.0	63.0	62.9
	15.6	9.5	59.4	84.5	99.7	23.2	20.4	38.1	9.8	99.2	100.0	100.0	57.5	32.1
C	10.8	**5.1**	54.0	86.5	99.7	28.0	24.7	29.8	**5.9**	98.6	100.0	100.0	70.4	45.8
	13.0	5.6	54.5	88.2	99.6	38.6	36.4	30.6	7.3	98.6	100.0	100.0	81.7	63.9
	22.6	54.7	7.4	22.1	61.4	7.7	18.4	99.5	100.0	8.5	70.8	99.9	86.8	98.6
F	17.4	52.5	**5.5**	21.4	64.5	8.6	21.3	98.3	100.0	**4.9**	73.5	100.0	88.0	98.8
	15.4	54.2	5.8	24.5	70.2	13.2	31.2	95.7	100.0	4.5	79.1	100.0	90.2	99.3
	51.0	70.5	21.4	8.0	13.0	13.9	19.7	99.0	99.6	66.0	8.4	66.9	57.7	82.8
G	37.4	63.3	12.9	**5.1**	12.1	9.5	17.0	96.7	98.9	38.3	**5.6**	74.7	42.4	76.7
	29.7	61.1	9.2	5.7	14.8	8.3	16.8	89.8	98.4	23.7	6.0	83.8	38.1	76.1
	96.1	98.4	56.0	19.0	7.7	59.2	65.3	100.0	100.0	100.0	98.6	12.9	100.0	100.0
J	94.0	98.7	40.1	12.1	**5.2**	55.3	65.2	100.0	100.0	100.0	94.2	**5.6**	100.0	100.0
	93.7	98.8	30.6	10.2	5.2	58.3	70.5	100.0	100.0	100.0	88.4	7.4	100.0	100.0

Table 2 Empirical power in % for testing ϑ_-, ϑ, and ϑ_+ for (K_C, AD) based on $N = 1\,000$ replications with $n = 150$, $\tau = 0.25$, and $d = 5$ (left), respectively $d = 20$ (right).

	True copula, $d = 5$							True copula, $d = 20$						
H_0	A	C	F	G	J	Ga	t_4	A	C	F	G	J	Ga	t_4
	10.5	16.9	7.3	14.7	15.2	7.2	91.4	19.9	41.2	7.3	54.6	63.1	46.8	100.0
A	**5.5**	6.9	7.1	9.8	14.7	10.1	83.4	**4.5**	14.1	22.1	43.6	73.0	81.0	100.0
	14.0	7.2	33.1	28.9	50.3	38.4	75.6	30.9	10.5	81.0	79.4	98.3	99.1	100.0
	8.5	11.4	25.3	31.6	56.3	17.6	92.5	36.9	36.1	96.0	97.5	100.0	98.3	100.0
C	19.4	**6.0**	52.7	53.6	79.0	47.1	89.0	84.8	**5.8**	99.8	100.0	100.0	100.0	100.0
	47.9	14.6	81.5	82.5	95.6	81.8	90.8	99.5	43.6	100.0	100.0	100.0	100.0	100.0
	8.5	11.1	8.6	17.5	22.1	6.1	90.0	7.2	26.4	18.0	70.3	86.0	57.7	100.0
F	4.7	6.3	**4.1**	8.5	8.4	6.2	82.0	7.4	20.5	**4.4**	33.9	52.0	77.6	100.0
	11.0	9.5	8.9	6.7	5.6	19.6	71.9	35.5	43.5	22.9	19.4	31.3	93.9	100.0
	10.6	10.6	6.6	12.2	24.7	10.0	89.4	35.6	52.7	20.2	44.4	85.0	97.8	100.0
G	24.4	25.6	18.7	**4.9**	6.7	26.3	84.8	84.5	89.2	74.5	**4.3**	19.4	99.7	100.0
	55.2	53.3	48.6	14.9	6.5	60.0	85.7	99.7	99.7	99.4	52.6	18.0	100.0	100.0
	47.9	55.8	28.4	6.4	14.4	40.6	93.5	99.9	100.0	91.4	8.6	62.2	100.0	100.0
J	81.9	83.7	61.4	15.1	**4.8**	73.4	95.0	100.0	100.0	99.9	66.2	**4.3**	100.0	100.0
	97.2	97.3	90.2	50.7	19.7	94.1	97.8	100.0	100.0	100.0	99.5	69.6	100.0	100.0

Table 3 Empirical power in % for testing ϑ_-, ϑ, and ϑ_+ for (T_d, N_d, AD) based on $N = 1\,000$ replications with $n = 150$, $\tau = 0.25$, and $d = 5$ (left), respectively $d = 20$ (right).

	True copula, $d = 5$							True copula, $d = 20$						
H_0	A	C	F	G	J	Ga	t_4	A	C	F	G	J	Ga	t_4
	10.7	18.1	7.2	12.4	9.8	5.1	86.7	$*$	$*$	$*$	$*$	$*$	$*$	$*$
A	**4.7**	7.1	6.6	8.0	13.2	13.3	75.9	$*$	$*$	$*$	$*$	$*$	$*$	$*$
	15.5	7.1	31.8	33.9	49.9	48.0	69.3	$*$	$*$	$*$	$*$	$*$	$*$	$*$
	7.8	12.2	20.4	27.7	47.4	14.7	85.9	50.4	38.0	98.0	99.3	100.0	98.6	100.0
C	17.9	**5.2**	49.3	54.9	77.6	48.5	83.9	87.1	**5.7**	99.9	100.0	100.0	100.0	100.0
	49.8	14.9	81.8	84.0	95.3	86.1	89.5	99.4	45.4	100.0	100.0	100.0	100.0	100.0
	8.4	13.3	9.4	19.9	23.9	5.3	90.9	9.4	39.8	17.1	77.6	92.9	47.7	100.0
F	5.3	6.6	**4.2**	7.7	10.2	9.0	81.5	20.2	36.0	**4.6**	33.1	63.9	80.0	100.0
	12.6	11.5	9.6	6.2	5.5	28.6	71.2	55.3	60.7	22.8	8.6	24.0	96.3	100.0
	9.7	10.1	6.8	12.0	23.3	8.6	87.0	34.6	33.6	15.8	44.0	83.4	78.9	100.0
G	25.2	21.2	20.7	**5.0**	6.5	28.4	81.7	85.1	80.9	74.9	**4.7**	20.7	97.2	100.0
	56.4	50.0	48.1	15.2	6.4	62.5	81.2	99.8	99.4	99.7	51.5	16.0	100.0	100.0
	55.4	54.8	33.8	7.9	14.5	41.6	94.6	100.0	99.8	98.7	22.3	52.8	100.0	100.0
J	83.3	81.7	64.3	16.0	**5.2**	73.1	94.9	100.0	100.0	100.0	76.6	**3.9**	100.0	100.0
	98.2	97.9	91.2	49.2	19.4	94.4	97.3	100.0	100.0	100.0	99.7	65.0	100.0	100.0
	20.0	30.0	12.0	26.0	21.0	10.0	96.0	–	–	–	–	–	–	100.0
Ga	12.0	10.0	12.0	11.0	12.0	**1.0**	94.0	–	–	–	–	–	–	100.0
	15.0	11.0	26.0	9.0	11.0	11.0	87.0	–	–	–	–	–	–	100.0
	39.0	31.0	53.0	27.0	29.0	61.0	14.0	100.0	100.0	100.0	–	–	–	–
t_4	65.0	48.0	69.0	54.0	52.0	86.0	**5.0**	100.0	100.0	100.0	–	–	–	–
	86.0	70.0	91.0	77.0	71.0	96.0	10.0	100.0	100.0	100.0	–	–	–	–

Table 4 Empirical power in % for testing ϑ_-, ϑ, and ϑ_+ for (R, N_d, AD) based on $N = 1\,000$ replications ($N = 100$ for the Gaussian and the t_4 copula) with $n = 150$, $\tau = 0.25$, and $d = 5$ (left), respectively $d = 20$ (right).

indicated by the symbol − in Table 4. Concerning the testing approaches (K_C, AD) and (T_d, N_d, AD) involving the evaluation of the probability integral transformation $K_C(t)$, we even had to choose a rather large precision of 64 for `Digits` in order to obtain reliable testing results. This has an effect on the computational time for these testing methods. Tables 5 and 6 present the runtimes in minutes for simulating the empirical power of a goodness-of-fit procedure for testing one parameter based on $N = 1\,000$ replications with $n = 150$ and $\tau = 0.25$ for the methods (T_{d-1}, N_{d-1}, AD) and (K_C, AD), respectively. The results are obtained by taking the average of the corresponding runtimes for the three parameters ϑ_-, ϑ, and ϑ_+ as presented in Tables 1 and 2. Similarly for Table 7, where the entries for the Gaussian and the t_4 H_0 copulas are based on $N = 100$ runs. Note that the runtimes for the goodness-of-fit approach (T_d, N_d, AD) were slightly larger than those for (K_C, AD) and much larger than those for (T_{d-1}, N_{d-1}, AD), as expected, and are therefore omitted. As the results indicate, for $d = 5$, the goodness-of-fit approach (T_{d-1}, N_{d-1}, AD) is the fastest, then (K_C, AD), and finally (R, N_d, AD), but all in all the differences in runtimes for these approaches seem to be acceptable small. For the large-dimensional case $d = 20$, again our proposed method (T_{d-1}, N_{d-1}, AD) is the fastest, however, the differences in runtimes between this and the other methods is much larger than for the case $d = 5$. Note that the small runtimes for the approach (K_C, AD) and the H_0 copula of Clayton stem from the fact that for Clayton's family, the function K_C can be efficiently evaluated, as mentioned in Section 4.2. For all other H_0 copulas, evaluation of K_C takes quite different and rather large amounts of runtime, due to the derivatives involved.

	True copula, $d = 5$							True copula, $d = 20$						
H_0	A	C	F	G	J	Ga	t_4	A	C	F	G	J	Ga	t_4
A	0.7	0.7	0.7	0.7	1.4	0.7	0.7	3.5	3.5	3.5	3.5	4.2	3.5	3.5
C	0.7	0.7	0.7	0.7	1.4	0.7	0.7	3.5	3.5	3.5	3.5	4.2	3.5	3.5
F	0.7	0.7	0.7	0.7	1.4	0.7	0.7	3.5	3.5	3.5	3.5	4.2	3.5	3.5
G	0.7	0.7	0.7	0.7	1.4	0.7	0.7	3.5	3.5	3.5	3.5	4.2	3.5	3.5
J	0.7	0.7	0.7	0.7	1.4	0.7	0.7	3.5	3.5	3.5	3.5	4.2	3.5	3.5

Table 5 Runtimes in minutes for (T_{d-1}, N_{d-1}, AD) based on $N = 1\,000$ replications with $n = 150$, $\tau = 0.25$, and $d = 5$ (left), respectively $d = 20$ (right).

Due to the small runtimes for our proposed goodness-of-fit approach (T_{d-1}, N_{d-1}, AD) it is possible to conduct a bootstrap procedure, see Section 4.2, for accessing the empirical power of this goodness-of-fit approach. As an estimation procedure, we use the averaged pairwise Kendall's tau estimator as described in Section 4.2. The results for the bootstrap procedure are reported in Table 8. Note that for the case $d = 5$ the empirical error probability of the first kind is around or below 5% in most cases. However, for Clayton's and Gumbel's copula the levels are rather large, being 6.0% for Clayton's and 7.4% for Gumbel's copula. When the dimension is raised to $d = 20$ both levels drop. In the latter case we get 5.3% for Gumbel's copula, which seems to be a reasonable level. In the

	True copula, $d = 5$							True copula, $d = 20$						
H_0	A	C	F	G	J	Ga	t_4	A	C	F	G	J	Ga	t_4
A	2.0	1.9	1.9	1.9	2.6	1.8	1.8	58.2	61.8	60.5	62.6	65.2	64.3	62.7
C	0.9	0.8	0.8	0.8	1.5	0.8	0.8	3.7	3.7	3.7	3.8	4.4	3.7	3.8
F	3.0	3.0	2.9	2.9	3.6	3.0	2.9	326.3	329.3	338.1	338.3	335.3	338.1	332.8
G	3.0	3.1	3.0	3.1	3.7	3.0	3.0	98.8	99.6	100.3	105.9	101.3	102.6	99.5
J	3.1	3.1	3.0	3.0	3.8	3.0	3.0	40.4	42.9	45.3	42.5	44.6	40.1	41.9

Table 6 Runtimes in minutes for (K_C, AD) based on $N = 1\,000$ replications with $n = 150$, $\tau = 0.25$, and $d = 5$ (left), respectively $d = 20$ (right).

	True copula, $d = 5$							True copula, $d = 20$						
H_0	A	C	F	G	J	Ga	t_4	A	C	F	G	J	Ga	t_4
A	4.0	3.8	3.5	3.6	4.2	3.7	3.7	31.2	31.5	31.7	32.2	32.7	30.8	30.4
C	6.1	5.7	5.7	5.6	6.1	5.5	5.6	24.9	24.4	24.7	24.5	25.0	24.4	24.2
F	4.6	4.6	4.6	4.5	5.1	4.3	4.5	44.7	44.8	44.6	47.1	47.8	46.9	46.0
G	10.2	10.1	10.5	10.5	10.9	10.1	10.3	367.7	361.7	367.5	366.5	365.7	367.0	358.8
J	9.8	10.1	10.0	10.0	10.6	9.9	10.0	335.3	327.0	331.1	335.9	329.0	335.3	325.7
Ga	798	1238	1226	1244	1206	1221	476	–	–	–	–	–	–	1076
t_4	1442	1430	1448	1439	1464	1440	1389	1134	615	533	–	–	–	–

Table 7 Runtimes in minutes for (R, N_d, AD) based on $N = 1\,000$ replications ($N = 100$ for the Gaussian and the t_4 copula) with $n = 150$, $\tau = 0.25$, and $d = 5$ (left), respectively $d = 20$ (right).

Clayton case the level is now 4.1% so there is again a small deviation, however, this time on the conservative side. All other levels for the case $d = 20$ are around or below 5%, except for Frank's family, where we get 6.3%.

As in the tests where no bootstrap procedure is used, the power against wrong alternatives increases when the dimension increases, uniformly over all cases. In comparison to (T_{d-1}, N_{d-1}, AD) without the bootstrap, the power increases in most cases when $d = 5$. Contrary, for $d = 20$ there is a loss of power for the larger part of alternatives. The case $d = 10$, which we do not report here, can again be seen as an intermediate state regarding this fact, some family combinations show a decrease, some an increase in empirical power. As before, the empirical power is especially large, when the simulated copula comes from the Gaussian or t_4 family. All in all, the bootstrap seems to work, although there is no theoretical proof for the convergence of the procedure.

	True copula, $d = 5$							True copula, $d = 20$						
H_0	A	C	F	G	J	Ga	t_4	A	C	F	G	J	Ga	t_4
A	**4.0**	11.3	5.8	7.7	21.0	5.6	93.2	**4.3**	32.7	12.8	55.0	87.9	82.8	100.0
C	6.6	**6.0**	18.9	25.2	52.4	13.1	95.6	18.8	**4.1**	79.9	91.2	99.2	84.1	100.0
F	8.7	26.4	**4.6**	3.8	4.6	10.7	94.3	16.6	72.2	**6.3**	29.1	70.0	98.8	100.0
G	46.0	70.0	27.1	**7.4**	3.0	45.2	99.0	69.9	96.7	25.3	**5.3**	14.9	100.0	100.0
J	89.6	94.2	69.6	16.3	**4.6**	85.1	100.0	99.6	99.8	89.1	18.0	**4.7**	100.0	100.0

Table 8 Empirical power in % for (T_{d-1}, N_{d-1}, AD) applying a bootstrap procedure with $M = 1\,000$ bootstrap replications based on $N = 1\,000$ replications with $n = 150$, $\tau = 0.25$, and $d = 5$ (left), respectively $d = 20$ (right).

6 Conclusion

A goodness-of-fit test for Archimedean copulas was presented, especially suited to large dimensions. The proposed test is based on a transformation T known for generating random variates and can therefore be viewed as an analogon to Rosenblatt's transformation, which is also used for sampling, known as the conditional distribution method. The suggested goodness-of-fit test proceeds in two steps. In a first step, the first $d - 1$ components of T are applied, which build a simple and easy-to-use transformation from d to $d - 1$ dimensions. This complements one of the known goodness-of-fit tests using only the d-th component of T requiring the knowledge of the generator derivatives. In a second step, the $d - 1$ components are mapped to a one-dimensional setting, where the hypothesis is tested. This second step is common to many goodness-of-fit approaches and hence all known goodness-of-fit tests in a one dimensional setting can be applied.

The power of the proposed testing approach was investigated in a large-scale simulation study, in comparison to other known goodness-of-fit tests. In this study, goodness-of-fit tests even in dimension $d = 20$ were investigated for the first time. In such large dimensions the computational effort for applying commonly known testing procedures turned out to be quite demanding according to precision and runtime. However, the proposed approach is easily applied in any dimension and its evaluation requires only small numerical precision. Due to the small runtime, the proposed testing method could also be investigated with a bootstrap procedure based on the averaged pairwise Kendall's tau estimator, again showing good performance in large dimensions.

All in all, investigating large-dimensional problems involving copulas is still an open field of research and more efficient methods in dealing with large dimensions are needed.

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References

Anderson, T. W. and Darling, D. A. (1952), “Asymptotic theory of certain goodness-of-fit criteria based on stochastic processes”, Annals of Mathematical Statistics, 23, 193–212.

Anderson, T. W. and Darling, D. A. (1954), “A test of goodness of fit”, Journal of the American Statistical Association, 49, 765–769.

Barbe, P., Genest, C., Ghoudi, K., and Rémillard, B. (1996), “On Kendall's Process”, Journal of Multivariate Analysis, 58, 197–229.

Berg, D. (2009), “Copula goodness-of-fit testing: an overview and power comparison”, The European Journal of Finance, `http://www.informaworld.com/10.1080/13518470802697428` (2009-03-25).

Berg, D. and Bakken, H. (2006), “Copula Goodness-of-fit Tests: A Comparative Study”, `http://www.danielberg.no/publications/CopulaGOF.pdf` (2008-11-01).

Berg, D. and Bakken, H. (2007), “A copula goodness-of-fit approach based on the conditional probability integral transformation”, `http://www.danielberg.no/publications/Btest.pdf` (2008-11-01).

Breymann, W., Dias, A., and Embrechts, P. (2003), “Dependence structures for multivariate high-frequency data in finance”, Quantitative Finance, 3, 1–14.

bwGRiD, `http://www.bw-grid.de/` (2009-03-25).

Cherubini, U., Luciano, E., and Vecchiato, W. (2004), “Copula Methods in Finance”, Wiley.

Devroye, L. (1986), “Non-Uniform Random Variate Generation”, Springer.

Dobrić, J. and Schmid, F. (2007), “A goodness of fit test for copulas based on Rosenblatt's transformation”, Computational Statistics & Data Analysis, 51, 4633–4642.

Embrechts, P., Lindskog, F., and McNeil, A. J. (2001), “Modelling Dependence with Copulas and Applications to Risk Management”, `http://www.risklab.ch/ftp/papers/DependenceWithCopulas.pdf` (2008-11-01).

Fermanian, J.-D. (2005), “Goodness of fit tests for copulas”, Journal of Multivariate Analysis, 95, 1, 119–152.

Genest, C. and Boies, J.-C. (2003), “Detecting dependence with Kendall plots”, The American Statistician, 57, 275–284.

Genest, C. and Rivest, L.-P. (1993), "Statistical Inference Procedures for Bivariate Archimedean Copulas", Journal of the American Statistical Association, 88, 423, 1034–1043.

Genest, C., Quessy, J. F., and Rémillard, B. (2006), "Goodness-of-fit procedures for copula models based on the probability integral transformation", Scandinavian Journal of Statistics, 33, 337–366.

Genest, C., Rémillard, B., and Beaudoin, D. (2009), "Goodness-of-fit tests for copulas: A review and a power study", Insurance: Mathematics and Economics, 44.

GSL, `http://www.gnu.org/software/gsl/` (2008-11-16).

Maplesoft, `http://www.maplesoft.com/` (2009-03-28).

Marsaglia, G. and Marsaglia, J. C. W. (2004), "Evaluating the Anderson Darling Distribution", Journal of Statistical Software, 9, 2, 1–5, `http://www.jstatsoft.org/v09/i02/paper` (2009-03-25).

McNeil, A. J. and Nešlehová, J. (2009), "Multivariate Archimedean copulas, d-monotone functions and l_1-norm symmetric distributions", The Annals of Statistics, in press.

McNeil, A. J., Frey, R., and Embrechts, P. (2005), "Quantitative Risk Management: Concepts, Techniques, and Tools", Princeton University Press.

NAG, `http://www.nag.co.uk/` (2008-11-16).

Nelsen, R. B. (2007), "An Introduction to Copulas", Springer.

Rao, C. R. (2001), "Linear Statistical Inference and its Applications", Wiley-Interscience.

Rosenblatt, M. (1952), "Remarks on a Multivariate Transformation", The Annals of Mathematical Statistics, 23, 3, 470–472.

Savu, C. and Trede, M. (2008), "Goodness-of-fit tests for parametric families of Archimedean copulas", Quantitative Finance, 8, 2, 109–116.

Schmitz, V. (2003), "Copulas and Stochastic Processes", Rheinisch-Westfälische Technische Hochschule Aachen, `http://darwin.bth.rwth-aachen.de/opus3/volltexte/2004/935/pdf/Schmitz_Volker.pdf` (2009-02-25).

Sklar, A. (1996), "Random variables, distribution functions, and copulas—a personal look backward and forward", Distributions with Fixed Marginals and Related Topics, 28, 1–14.

Wagner, R. (2003), "Mersenne Twister Random number Generator", `http://www-personal.umich.edu/~wagnerr/MersenneTwister.html` (2008-08-10).

Wu, F., Valdez, E. A., and Sherris, M. (2006), "Simulating Exchangeable Multivariate Archimedean Copulas and its Applications", Communications in Statistics - Simulation and Computation, 36, 5, 1019–1034.