On systematic mortality risk and risk-minimization with survivor swaps

Mikkel Dahl
Nordea Markets
Strandgade 3
DK-0900 Copenhagen C
Denmark
mikkel.dahl@nordea.com

Martin Melchior
PFA Pension
Sundkrogsgade 4
DK-2100 Copenhagen Ø
Denmark
mml@pfa.dk

Thomas Møller
PFA Pension
Sundkrogsgade 4
DK-2100 Copenhagen Ø
Denmark
thm@pfa.dk

Abstract. A new market for so-called mortality derivatives is now appearing with survivor swaps (also called mortality swaps), longevity bonds and other specialized solutions. The development of these new financial instruments is triggered by the increased focus on the systematic mortality risk inherent in life insurance contracts, and their main focus is thus to allow the life insurance companies to hedge their systematic mortality risk. At the same time this new class of financial contracts is interesting from an investor's point of view since they increase the possibility for an investor to diversify the investment portfolio. The systematic mortality risk stems from the uncertainty related to the future development of the mortality intensities. Mathematically this uncertainty is described by modeling the underlying mortality intensities via stochastic processes. We consider two different portfolios of insured lives, where the underlying mortality intensities are correlated, and study the combined financial and mortality risk inherent in a portfolio of general life insurance contracts. In order to hedge this risk we allow for investments in survivor swaps and derive risk-minimizing strategies in markets where such contracts are available. The strategies are evaluated numerically.

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1 Introduction

Life and pension insurance companies typically use deterministic mortality intensities when determining premiums and reserves. These mortality intensities are often assessed via historical data from the portfolios in the company. Given the estimated mortality intensities the companies have traditionally been explicitly concerned with the unsystematic mortality risk, which is the risk associated with the randomness of deaths in an insurance portfolio with known mortality. According to the law of large numbers, the unsystematic mortality risk is diversifiable, which means that it is eliminated if the portfolio is sufficiently large. However, the mortality is not deterministic, so the companies are exposed to systematic mortality risk as well. The systematic mortality risk, which refers to risk associated with changes in the underlying mortality intensity (or the mortality table), is fundamentally different than the unsystematic mortality risk. It is not diversifiable, and thus it is not eliminated when the size of the portfolio is increased. In practice, the life and pension insurance companies deal with the systematic mortality risk according to the so-called prudent man principle, i.e. they set the mortality intensities to the safe side. This will, hopefully, create a surplus which is then redistributed to the insured using the so-called contribution principle.

During the last years, the average lifetime has increased dramatically. This fact has to some extent been neglected, since focus has been on the financial risk, which is much easier to observe and handle. It is now clear that the mortality tables used 20 years ago for the pricing of life annuities were in some cases not sufficiently conservative, since whole life annuities are being paid out for several years longer than expected. The current challenge for the life insurance companies is therefore to control the combined financial and insurance risk inherent in life insurance contracts.

There are essentially two ways of addressing mortality risk: One can link or adapt benefits to the current mortality, or one can invest in financial assets, which are correlated with the mortality or number of survivors. Linking benefits to the mortality has been called mortality-linked contracts, see e.g. Dahl (2004). In the present paper, we focus on the possibility of hedging mortality risk by investing in mortality-linked derivatives. The key difference between financial risk and mortality risk, is that the financial market provides a vast number of financial assets which the companies can use to hedge the financial risk, whereas this is not the case with mortality risk. There is, however, much focus on the subject, and some financial assets linked to the mortality have been issued. In this paper, we consider the so-called survivor swap. With this contract, we swap (exchange) a fixed rate of survivors for the actual number of survivors. Hedging with survivor swaps has also been considered by Lin and Cox (2005). Blake, Cairns and Dowd (2006) have studied other types of mortality-linked derivatives, such as longevity bonds, mortality swaps and mortality swaptions.

In the literature, various models for the stochastic mortality have been proposed, see e.g. Marocco and Pitacco (1998), Milevsky and Promislow (2001), Dahl (2004), Cairns, Blake and Dowd (2004), Biffis and Millossovich (2004), Miltersen and Persson (2006) and references therein. In this paper, we consider a model with stochastic interest rates and stochastic mortality intensities. The model is inspired by the one proposed in Dahl (2004) and Dahl and Møller (2006) and uses the so-called CIR-processes known from the financial literature for the modeling of mortality intensities. We study a model with two mortality intensities and two underlying random processes. Both mortality intensities may be driven by both underlying processes. The first mortality intensity represents the mortality of the insurance portfolio, and the second intensity represents the mortality of a population.

We consider different financial markets, which contain a zero coupon bond and possibly one or more survivor swaps, and study the possibilities of hedging in these markets. In all the markets we have more sources of risk (financial risk and mortality risks) than financial assets, so we apply theory from incomplete markets. More precisely, we use the criterion of risk-minimization introduced by Föllmer and Sondermann (1986) for contingent claims and extended to payment processes by Møller (2001) to determine risk-minimizing strategies. The strategies illustrate how the combined insurance and financial risk can be hedged partly with bonds and survivor swaps. This extends the work of Dahl and Møller (2006).

The paper is organized as follows: Section 2.1 introduces the basic financial market, and Sections 2.2 and 2.3 review the mortality theory of Dahl and Møller (2006) within a two-dimensional model with two portfolios, where the underlying mortalities are correlated. Section 2.4 considers the combined model. In Section 3, we introduce survivor swaps in the financial market and define their price processes. Section 4 introduces a portfolio of general life insurance contracts and the market value. In Section 5, we give a brief review of the theory of risk-minimization and apply these results for determining risk-minimizing strategies in the various financial markets. Finally, the strategies are compared numerically in Section 6. Proofs of some technical results are presented in the Appendix.

2 The model

In this section we introduce the combined model for the financial risk and the insurance risk. The model is inspired by the one of Dahl and Møller (2006). Let T be a fixed finite time horizon and $(\Omega, \mathcal{F}, P)$ a probability space equipped with a filtration $\mathbb{F} = (\mathcal{F}(t))_{0\leq t\leq T}$, which contains all available information. We define $\mathbb{F}$ as the natural filtration generated by three independent standard Brownian motions $W^{\mu} = (W_1^{\mu}, W_2^{\mu})$ and W^r and a 2-dimensional counting process $N(x) = (N_1(x), N_2(x))$. The process $N(x)$ is used to keep track of the number of deaths in two portfolios, whereas W^{μ} drives the mortality intensities and W^r determines the interest rate. In addition we consider sub-filtrations $\mathbb{G}$, $\mathbb{I}$ and $\mathbb{H}$ generated by W^r, W^{μ} and $N(x)$, respectively. We assume that W^r and $(W^{\mu}, N(x))$ are stochastically independent. For an extension with dependence between the financial market and the mortality intensities, see Miltersen and Persson (2006).

2.1 The financial market

In this section, we introduce the financial market, which exists of two traded assets: A savings account and a zero coupon bond with maturity T with price processes B and $P(\cdot, T)$, respectively. The short rate is determined by a so-called Vasiček model, i.e. the short rate dynamics under P are

$$dr(t) = (\gamma^r - \delta^r r(t))dt + \sigma^r dW^r(t),$$

with $r(0) = r_0$. Here, γ^r, δ^r and σ^r are constants, and W^r is a standard Brownian motion. As in Dahl and Møller (2006), we assume that

$$P(t,T) = E^{Q^r}\left[e^{-\int_t^T r(u)du} \middle| \mathcal{F}(t)\right],$$

where the measure Q^r is defined via $\frac{dQ^r}{dP} = \widehat{\Lambda}(T)$, with $d\widehat{\Lambda}(t) = \widehat{\Lambda}(t)h^r(t)dW^r(t)$, $\widehat{\Lambda}(0) = 1$, and where

$$h^r(t) = -\left(\frac{\widetilde{c}}{\sigma^r} + \frac{cr(t)}{\sigma^r}\right). \tag{2.1}$$

Here, c and $\widetilde{c}$ are constants. It follows by Girsanov's theorem that the dynamics of r under Q^r are given by

$$dr(t) = \left(\gamma^{r,Q} - \delta^{r,Q}r(t)\right)dt + \sigma^r dW^{r,Q}(t), \tag{2.2}$$

with $r(0) = r_0$, where $W^{r,Q}$ is a standard Brownian motion under Q^r, and where

$$\begin{aligned}\gamma^{r,Q} &= \gamma^r - \widetilde{c},\\ \delta^{r,Q} &= \delta^r + c.\end{aligned}$$

The model is affine under Q^r, and it is well-known that

$$P(t,T) = e^{A^r(t,T) - B^r(t,T)r(t)},$$

where $A^r(t,T)$ and $B^r(t,T)$ are given by

$$\begin{aligned}B^r(t,T) &= \frac{1}{\delta^{r,Q}}\left(1 - e^{-\delta^{r,Q}(T-t)}\right),\\ A^r(t,T) &= \frac{(B^r(t,T) - T + t)(\gamma^{r,Q}\delta^{r,Q} - \frac{1}{2}(\sigma^r)^2)}{(\delta^{r,Q})^2} - \frac{(\sigma^r)^2(B^r(t,T))^2}{4\delta^{r,Q}}.\end{aligned}$$

The dynamics under P of the zero coupon bond price process are

$$dP(t,T) = \left(r(t) - h^r(t,r(t))\sigma^p(t,r(t))\right)P(t,T)dt + \sigma^p(t,r(t))P(t,T)dW^r(t),$$

where

$$\sigma^p(t,r(t)) = -\sigma^r B^r(t,T).$$

The dynamics under Q^r of the price processes are

$$\begin{aligned}dB(t) &= r(t)B(t)dt, \quad B(0) = 1,\\ dP(t,T) &= r(t)P(t,T)dt + \sigma^p(t,r(t))P(t,T)dW^{r,Q}(t).\end{aligned} \tag{2.3}$$

We note that Q^r is the unique equivalent martingale measure for the financial model.

2.2 The mortality intensities

We consider two portfolios and introduce two different mortalities, which may be correlated. Inspired by the model of Dahl and Møller (2006), we define for each portfolio the processes

$$d\zeta_j(x,t) = (\gamma_j(x,t) - \delta_j(x,t)\zeta_j(x,t))dt + \sqrt{\zeta_j(x,t)}\sigma_j(x,t)dW^\mu(t), \tag{2.4}$$

$\zeta_j(x,0) = 1$, $j = 1,2$. The process ζ_1 is related to an insurance portfolio, whereas ζ_2 is related to a larger population. Each of the two groups are assumed to exist of individuals of equal age x. Here, σ_j is a two-dimensional row vector and W^μ is a two-dimensional standard Brownian motion.

The mortality intensity processes are given by $\mu_j(x,t) = \mu_j^\circ(x+t)\zeta_j(x,t)$, where μ_j° are the initial mortality intensities at time 0, and the survival probabilities are defined by

$$\mathcal{S}_j(x,t,T) = E^P\left[e^{-\int_t^T \mu_j(x,\tau)d\tau}\,\middle|\,\mathcal{F}(t)\right] = E^P\left[e^{-\int_t^T \mu_j^\circ(x+\tau)\,\zeta_j(x,\tau)d\tau}\,\middle|\,\mathcal{F}(t)\right]. \tag{2.5}$$

For more details, see Dahl (2004) and Dahl and Møller (2006). In the case where $\sigma_1^\mu(x,t) = \big(\sigma_{1,1},\ 0\big)$ and $\sigma_2^\mu(x,t) = \big(0,\ \sigma_{2,2}\big)$, the two mortality intensities are independent. If instead we take $\sigma_1^\mu(x,t) = \big(\sigma_{1,1}, \sigma_{1,2}\big)$, the two mortalities are no longer independent.

It follows by Itô's formula that, for $j = 1,2$, the dynamics of the mortality intensities are

$$d\mu_j(x,t) = (\gamma_j^\mu(x,t) - \delta_j^\mu(x,t)\mu_j(x,t))dt + \sqrt{\mu_j(x,t)}\sigma_j^\mu(x,t)dW^\mu(t), \tag{2.6}$$

where

$$\gamma_j^\mu(x,t) = \mu_j^\circ(x+t)\gamma_j(x,t), \tag{2.7}$$

$$\delta_j^\mu(x,t) = \delta_j(x,t) - \frac{\frac{d}{dt}\mu_j^\circ(x+t)}{\mu_j^\circ(x+t)}, \tag{2.8}$$

$$\sigma_j^\mu(x,t) = \sqrt{\mu_j^\circ(x+t)}\sigma_j(x,t). \tag{2.9}$$

In order to ensure that the mortality intensities are strictly positive, we assume that

$$2\gamma_j^\mu(x,t) \geq \big(\sigma_j(x,t)\big)\big(\sigma_j(x,t)\big)^{\mathrm{tr}}, \tag{2.10}$$

where a^{tr} denotes the vector a transposed.

2.3 The lifetimes in the portfolios

This section describes the lifetimes in each of the two portfolios. For simplicity, we assume that the portfolios consist of n_j, $j = 1,2$, lives, all aged x years at time 0. Furthermore, we assume, that the lives in portfolio 1 are different from the lives in portfolio 2, i.e. the two portfolios consist of disjoint lives. We adopt the natural assumption that the lifetimes in a portfolio are mutually independent and identically distributed conditional on the mortality intensities. The remaining lifetimes at time 0 are described by a sequence of non-negative random variables $T_{j,1}, \ldots, T_{j,n_j}$, $j = 1,2$. The probability of a single individual surviving to time t, given the information on the mortality intensity until time t, is given by

$$P(T_{j,1} > t|\mathcal{I}(t)) = e^{-\int_0^t \mu_j(x,s)ds}, \qquad j = 1,2.$$

All considered insurance contracts have expiration time T, so the mortality processes are only modeled on the interval $[0,T]$. We therefore introduce *censored lifetimes* given by $T_{j,i}^* = T_{j,i} \wedge T$, $j = 1,2$, $i = 1,\ldots,n_j$. In each portfolio the censored lifetimes are now i.i.d. given $\mathcal{I}(T)$. We emphasize, that even though the mortality intensities for the two portfolios may be correlated we have conditional independence between the lives in the two portfolios.

The number of deaths at time $t \in [0,T]$ in portfolio j is described by the counting process $N_j(x) = (N_j(x,t))_{t\in[0,T]}$, where

$$N_j(x,t) = \sum_{i=1}^{n_j} 1_{\{T_{j,i}\leq t\}}, \tag{2.11}$$

for $j = 1, 2$. The stochastic intensity process $\lambda_j(x) = (\lambda_j(x,t))_{t\in[0,T]}$ related to $N_j(x)$ is given informally by

$$\lambda_j(x,t)dt = E^P[dN_j(x,t)|\mathcal{H}(t-) \vee \mathcal{I}(t)] = (n_j - N_j(x,t-))\mu_j(x,t)dt, \tag{2.12}$$

$j = 1, 2$. Hence the transition rates are simply the mortality intensity multiplied by the number of survivors just before time t.

2.4 Change of measure

Inspired by Dahl and Møller (2006), we consider martingale measures, with a likelihood process Λ on the form

$$d\Lambda(t) = \Lambda(t-)\Big(h^r(t)dW^r(t) + h^\mu(t)dW^\mu(t) + g(t)dM(x,t)\Big), \tag{2.13}$$

with $\Lambda(0) = 1$. We assume that $E^P[\Lambda(T)] = 1$ and define an equivalent martingale measure Q by $\frac{dQ}{dP} = \Lambda(T)$. Here, h^μ and g are two-dimensional processes. For simplicity, we require that g is deterministic, continuously differentiable and $g_j > -1$, $j = 1, 2$. The process h^r is defined in (2.1). The other terms in (2.13) are related to the mortality. We take the Girsanov kernels on the special form

$$\begin{aligned} h_1^\mu(t,\mu_1,\mu_2) &= \sigma_{1,2}^\mu(x,t)\left(\frac{\beta_2^\mu(x,t)\sqrt{\mu_2(x,t)}}{\bar\sigma^\mu(x,t)} - \frac{\widetilde\beta_2^\mu(x,t)}{\bar\sigma^\mu(x,t)\sqrt{\mu_2(x,t)}}\right) \\ &- \ \sigma_{2,2}^\mu(x,t)\left(\frac{\beta_1^\mu(x,t)\sqrt{\mu_1(x,t)}}{\bar\sigma^\mu(x,t)} - \frac{\widetilde\beta_1^\mu(x,t)}{\bar\sigma^\mu(x,t)\sqrt{\mu_1(x,t)}}\right), \end{aligned} \tag{2.14}$$

$$\begin{aligned} h_2^\mu(t,\mu_1,\mu_2) &= \sigma_{2,1}^\mu(x,t)\left(\frac{\beta_1^\mu(x,t)\sqrt{\mu_1(x,t)}}{\bar\sigma^\mu(x,t)} - \frac{\widetilde\beta_1^\mu(x,t)}{\bar\sigma^\mu(x,t)\sqrt{\mu_1(x,t)}}\right) \\ &- \ \sigma_{1,1}^\mu(x,t)\left(\frac{\beta_2^\mu(x,t)\sqrt{\mu_2(x,t)}}{\bar\sigma^\mu(x,t)} - \frac{\widetilde\beta_2^\mu(x,t)}{\bar\sigma^\mu(x,t)\sqrt{\mu_2(x,t)}}\right), \end{aligned} \tag{2.15}$$

where $\bar\sigma^\mu(x,t) = \sigma_{1,2}^\mu(x,t)\sigma_{2,1}^\mu(x,t) - \sigma_{1,1}^\mu(x,t)\sigma_{2,2}^\mu(x,t)$ and β_j^μ and $\widetilde\beta_j^\mu$ are continuous functions. This ensures that the mortality intensities follow CIR models under Q. Moreover, the restrictions on g and h^μ ensure that the stochastic independence between the financial market and the insurance elements is preserved under Q. Indeed, straight-forward calculations show that

$$d\mu_j(x,t) = \Big(\gamma_j^{\mu,Q}(x,t) - \delta_j^{\mu,Q}(x,t)\mu_j(x,t)\Big)dt + \sqrt{\mu_j(x,t)}\sigma_j^\mu(x,t)dW^{\mu,Q}(t),$$

where $W^{\mu,Q}$ is a 2-dimensional standard Brownian motion under Q, and where

$$\begin{aligned} \gamma_j^{\mu,Q}(x,t) &= \gamma_j^\mu(x,t) - \widetilde\beta_j^\mu(x,t), \\ \delta_j^{\mu,Q}(x,t) &= \delta_j^\mu(x,t) - \beta_j^\mu(x,t). \end{aligned}$$

We emphasize that β_j^μ and $\widetilde\beta_j^\mu$ must fulfill certain conditions in order to prevent zero-valued mortality intensities, see Dahl and Møller (2006).

Finally, we define the Q-martingales M_j^Q by $dM_j^Q(x,t) = dN_j(x,t) - \lambda_j^Q(x,t)dt$, where $\lambda_j^Q(x,t) = (n_j - N_j(x,t-))(1+g_j(t))\mu_j(x,t)$, $j = 1, 2$. We can interpret the quantities $\mu_j^Q(x,t) = (1+g_j(t))\mu_j(x,t)$ as the mortality intensities under Q.

2.4.1 Survival probabilities under Q

We define the survival probabilities under Q by

$$\mathcal{S}_j^Q(x,t,T) = E^Q\left[e^{-\int_t^T \mu_j^Q(x,\tau)d\tau}\middle|\mathcal{F}(t)\right],$$

and the associated Q-martingales by

$$\mathcal{S}_j^{Q,M}(x,t,T) = E^Q\left[e^{-\int_0^T \mu_j^Q(x,\tau)d\tau}\middle|\mathcal{F}(t)\right] = e^{-\int_0^t \mu_j^Q(x,\tau)d\tau}\mathcal{S}_j^Q(x,t,T).$$

A closer inspection of the Q-mortality intensities μ_j^Q reveals that the dynamics are on the form

$$d\mu_j^Q(x,t) = \left(\gamma_j^{\mu,Q,g}(x,t) - \delta_j^{\mu,Q,g}(x,t)\mu_j^Q(x,t)\right)dt + \sqrt{\mu_j^Q(x,t)}\sigma_j^{\mu,Q,g}(x,t)dW^{\mu,Q}(t),$$

where

$$\begin{aligned}
\gamma_j^{\mu,Q,g}(x,t) &= (1+g_j(t))\gamma_j^{\mu,Q}(x,t),\\
\delta_j^{\mu,Q,g}(x,t) &= \delta_j^{\mu,Q}(x,t) - \frac{\frac{d}{dt}g_j(t)}{1+g_j(t)},\\
\sigma_j^{\mu,Q,g}(x,t) &= \sqrt{1+g_j(t)}\sigma_j^{\mu}(x,t).
\end{aligned}$$

In this case, the diffusion term is two-dimensional, so in order to use the affine theory from Section 2.1, we first rewrite $\mu_j^Q(x,t)$. It is well-known that

$$\sigma_{j,1}^{\mu,Q,g}(x,t)dW_1^{\mu,Q}(t) + \sigma_{j,2}^{\mu,Q,g}(x,t)dW_2^{\mu,Q}(t) = \widetilde{\sigma}_j^{\mu,Q,g}(x,t)d\widetilde{W}_j^{\mu,Q}(t),$$

where $\widetilde{W}_j^{\mu,Q}$ are standard Brownian motions, and

$$\widetilde{\sigma}_j^{\mu,Q,g}(x,t) = \sqrt{(\sigma_{j,1}^{\mu,Q,g}(x,t))^2 + (\sigma_{j,2}^{\mu,Q,g}(x,t))^2}.$$

The mortality intensities can now be written on a form, where the diffusion term is one-dimensional, i.e.

$$d\mu_j^Q(x,t) = \left(\gamma_j^{\mu,Q,g}(x,t) - \delta_j^{\mu,Q,g}(x,t)\mu_j^Q(x,t)\right)dt + \sqrt{\mu_j^Q(x,t)}\widetilde{\sigma}_j^{\mu,Q,g}(x,t)d\widetilde{W}_j^{\mu,Q}(t),$$

such that the drift and squared diffusion terms for $\mu_j^Q(x,t)$ are affine in $\mu_j^Q(x,t)$. This is similar to the situation in Dahl and Møller (2006), and the results obtained there now show that the Q-survival probabilities $\mathcal{S}_j^Q(x,t,T)$, $j=1,2$, are given by

$$\mathcal{S}_j^Q(x,t,T) = e^{A_j^{\mu,Q}(x,t,T) - B_j^{\mu,Q}(x,t,T)\mu_j^Q(x,t)},$$

where $A_j^{\mu,Q}$ and $B_j^{\mu,Q}$ are determined from

$$\begin{aligned}
\frac{\partial}{\partial t}B_j^{\mu,Q}(x,t,T) &= \delta_j^{\mu,Q,g}(x,t)B_j^{\mu,Q}(x,t,T) + \frac{1}{2}(\widetilde{\sigma}_j^{\mu,Q,g}(x,t))^2(B_j^{\mu,Q}(x,t,T))^2 - 1,\\
B_j^{\mu,Q}(x,T,T) &= 0,\\
\frac{\partial}{\partial t}A_j^{\mu,Q}(x,t,T) &= \gamma_j^{\mu,Q,g}(x,t)B_j^{\mu,Q}(x,t,T),\\
A_j^{\mu,Q}(x,T,T) &= 0.
\end{aligned}$$

The forward mortality intensities under Q is for $j=1,2$ given by

$$f^{\mu_j,Q}(x,t,T) = -\frac{\partial}{\partial T}\log\mathcal{S}_j^Q(x,t,T) = \mu_j^Q(x,t)\frac{\partial}{\partial T}B_j^{\mu,Q}(x,t,T) - \frac{\partial}{\partial T}A_j^{\mu,Q}(x,t,T),$$

see Dahl and Møller (2006).

3 Survivor swaps

Inspired by interest rate swaps, so-called *survivor swaps* have been introduced, where one can exchange a fixed number of survivors with the actual number of survivors in a portfolio. The portfolio could for example be the insured lives in an insurance portfolio or the lives of a certain age in some population.

3.1 The payment process associated with a survivor swap

When a buyer and a seller agree to enter a survivor swap, they essentially agree on some fixed survival probability, which is here determined at time 0 and given by

$$ {}_t\widetilde{p}_x = e^{-\int_0^t \widetilde{\mu}^0(x,\tau)d\tau}. $$

The intensity $\widetilde{\mu}^0$ determines the fixed payments during the period of the contract.

A survivor swap on each of the two portfolios can now be described by payment processes A_j^{swap}, $j = 1, 2$, with dynamics

$$ dA_j^{\text{swap}}(x,t) = (n_j - N_j(x,t))dt - n_j \, {}_t\widetilde{p}_x \, dt, \tag{3.1} $$

and $A_j^{\text{swap}}(x,0) = 0$. The payment rate in (3.1) is the difference between the actual number $(n_j - N_j(x,t))$ of survivors in portfolio j at time t and the expected number $n_j \, {}_t\widetilde{p}_x$, which is calculated at time 0 by using the survival probability ${}_t\widetilde{p}_x$. Thus, the swap leads to a continuous payment if the actual number of survivors exceeds the predetermined level of survivors. If on the other hand the predetermined level of survivors exceeds the actual number of survivors, the payment rate (3.1) is negative, and the buyer has to pay the difference to the seller of the contract.

3.2 Market values

In the remaining of the paper we consider a fixed but arbitrary measure Q from the class of equivalent martingale measures introduced in section 2.4.

Let the discounted payments $A^{*,\,\text{swap}}$ from the survivor swap be defined by

$$ dA_j^{*,\,\text{swap}}(x,t) = e^{-\int_0^t r(u)du} dA_j^{\text{swap}}(x,t), $$

and $A^{*,\,\text{swap}}(x,0) = 0$. For $j = 1, 2$ we now introduce the process $Z_j^{*,Q}$ given by

$$ Z_j^{*,Q}(x,t) = E^Q\left[\int_0^T e^{-\int_0^\tau r(u)du} dA_j^{\text{swap}}(x,\tau) \middle| \mathcal{F}(t) \right]. \tag{3.2} $$

Hence $Z_j^{*,Q}(x,t)$ is the conditional expected value at time t of discounted payments from the survivor swap on portfolio j. In this paper we adopt the terminology of Föllmer and Sondermann (1986) and refer to a process on a form similar to (3.2) as an intrinsic value process. The asterisk $*$ in $Z_j^{*,Q}(x,t)$ and $A^{*,\,\text{swap}}$ indicates that we are working with discounted values. We will use this notation in the rest of the paper. It follows that

$$ \begin{aligned} Z_j^{*,Q}(x,t) &= A_j^{*,\,\text{swap}}(x,t) + e^{-\int_0^t r(u)du} E^Q\left[\int_t^T e^{-\int_t^\tau r(u)du} dA_j^{\text{swap}}(x,\tau) \middle| \mathcal{F}(t) \right] \\ &= A_j^{*,\,\text{swap}}(x,t) + \widetilde{Z}_j^{*,Q}(x,t), \end{aligned} \tag{3.3} $$

where we have introduced the notation

$$\widetilde{Z}_j^{*,Q}(x,t) = e^{-\int_0^t r(u)du} E^Q \left[\int_t^T e^{-\int_t^\tau r(u)du} dA_j^{\mathrm{swap}}(x,\tau) \middle| \mathcal{F}(t) \right].$$

Here, $\widetilde{Z}_j^{*,Q}(t)$ is the discounted *market value* of the future payments and represents the discounted expected value of future payments given the current information. Using (3.1) and the independence between the financial market and the insured lives, we get that

$$\widetilde{Z}_j^{*,Q}(x,t) = (n_j - N_j(x,t)) \int_t^T P^*(t,\tau)\mathcal{S}^Q(x,t,\tau)d\tau - n_j\, {}_t\widetilde{p}_x \int_t^T P^*(t,\tau)\, {}_{\tau-t}\widetilde{p}_{x+t} d\tau,$$

where $P^*(t,\tau)$ is the discounted price of a zero coupon bond. The first term is the discounted market value of the variable payments, and the second term is the discounted market value of the fixed payments.

We assume below that assets with discounted price processes (3.2) can be traded dynamically in the financial market. These assets may now be used for hedging the combined insurance and financial risk inherent in the insurance portfolio. We note that the discounted price processes are martingales under the chosen measure Q, such that Q is also a martingale measure in the extended markets, where the survivor swaps can be traded dynamically.

3.3 A stochastic representation of survivor swaps

In this section, we derive a stochastic representation of (3.3), which provides insight regarding the different types of risks associated with a survivor swap. Furthermore, it is useful for determining risk-minimizing strategies in the situation where the survivor swaps can be traded dynamically.

In the remaining of the paper we work under the following assumption.

Assumption 3.1 $\widetilde{Z}_j^{*,Q} \in C^{1,2,2}$, *i.e.* $\widetilde{Z}_j^{*,Q}$ *is continuously differentiable with respect to* t *and twice continuously differentiable with respect to* r *and* μ.

Lemma 3.2 *A survivor swap on portfolio* j *with fixed survival probability* ${}_t\widetilde{p}_x$ *admits the representation*

$$Z_j^{*,Q}(x,t) = Z_j^{*,Q}(x,0) + \int_0^t \nu_j^{Z,Q}(\tau)dM_j^Q(x,\tau) + \int_0^t \eta_j^{Z,Q}(\tau)dW^{r,Q}(\tau) + \int_0^t \rho_j^{Z,Q}(\tau)dW^{\mu,Q}(\tau),$$

where $\rho_j^{Z,Q} = (\rho_{j,1}^{Z,Q}, \rho_{j,2}^{Z,Q})$, *and*

$$\nu_j^{Z,Q}(t) = -\int_t^T P^*(t,\tau)\mathcal{S}_j^Q(x,t,\tau)d\tau, \tag{3.4}$$

$$\begin{aligned} \eta_j^{Z,Q}(t) &= -(n_j - N_j(x,t-))\sigma^r \int_t^T B^r(t,\tau)P^*(t,\tau)\mathcal{S}_j^Q(x,t,\tau)d\tau \\ &\quad + n_j\, {}_t\widetilde{p}_x\, \sigma^r \int_t^T B^r(t,\tau)P^*(t,\tau)\, {}_{\tau-t}\widetilde{p}_{x+t}d\tau, \end{aligned} \tag{3.5}$$

$$\begin{aligned} \rho_{j,i}^{Z,Q}(t) &= -\sigma_{j,i}^{\mu}(x,t)\sqrt{\mu_j(x,t)}(n_j - N_j(x,t-))(1+g_j(t)) \\ &\quad \times \int_t^T B_j^{\mu,Q}(t,\tau)P^*(t,\tau)\mathcal{S}_j^Q(x,t,\tau)d\tau, \quad i = 1,2. \end{aligned} \tag{3.6}$$

Before proving the lemma, we comment briefly on the result above. There are essentially three types of risk associated with the value of the survivor swap. First we have the unsystematic mortality risk driven by the martingale M_j^Q, which is the risk associated with a death in the portfolio underlying the swap. Second, we have the interest rate risk related to changes in the underlying process $W^{r,Q}$ driving the interest rate. Finally we have the systematic mortality risk driven by the underlying processes $W^{\mu,Q}$. Since $W^{\mu,Q}$ is two-dimensional, we have both systematic mortality risk generated by $W_1^{\mu,Q}$ and systematic mortality risk generated by $W_2^{\mu,Q}$.

Proof of Lemma 3.2: Using Itô's formula on the martingale $Z_j^{*,Q}$ defined in (3.3) and the dynamics for $A_j^{\text{swap}}(x,t)$ from (3.1), we get that

$$\begin{aligned} dZ_j^{*,Q}(x,t) &= dA_j^{*,\text{ swap}}(x,t) + d\widetilde{Z}_j^{*,Q}(x,t) \\ &= \psi(t)dt + d\widetilde{Z}_j^{*,Q}(x,t). \end{aligned} \tag{3.7}$$

Here, ψ is some process, whose exact form we do not need to know, since the drift of a martingale is zero. We use the processes ψ and ψ_1 as some buffers for all quantities that concerns the drift. Using (3.7) and Itô's formula on $\widetilde{Z}_j^{*,Q}$, we get that

$$\begin{aligned} dZ_j^{*,Q}(x,t) &= \psi_1(t)dt + \sigma^r \frac{\partial}{\partial r}\widetilde{Z}_j^{*,Q}(x,t-)dW^{r,Q}(t) \\ &\quad + \sqrt{\mu_j(x,t)}\frac{\partial}{\partial \mu_j}\widetilde{Z}_j^{*,Q}(x,t-)\sigma_j^{\mu}(x,t)dW^{\mu,Q}(t) \\ &\quad - dN_j(x,t)\int_t^T P^*(t,\tau)\mathcal{S}_j^Q(x,t,\tau)d\tau \\ &= \sigma^r \frac{\partial}{\partial r}\widetilde{Z}_j^{*,Q}(x,t-)dW^{r,Q}(t) + \sqrt{\mu_j(x,t)}\frac{\partial}{\partial \mu_j}\widetilde{Z}_j^{*,Q}(x,t-)\sigma_j^{\mu}(x,t)dW^{\mu,Q}(t) \\ &\quad - dM_j^Q(x,t)\int_t^T P^*(t,\tau)\mathcal{S}_j^Q(x,t,\tau)d\tau. \end{aligned}$$

In the second equality, we have rewritten $dZ_j^{*,Q}$ such that we get a term with respect to the dynamics of the martingale M_j^Q. We notice, that the drift term disappears in the last equation since $Z_j^{*,Q}$ is a martingale. Now, recalling that

$$P(t,T) = e^{A^r(t,T) - B^r(t,T)r(t)},$$

we see that

$$\begin{aligned} \frac{\partial}{\partial r}\widetilde{Z}_j^{*,Q}(x,t-) &= -(n_j - N_j(x,t-))\int_t^T B^r(t,\tau)P^*(t,\tau)\mathcal{S}_j^Q(x,t,\tau)d\tau \\ &\quad + n_j\, {}_t\widetilde{p}_x \int_t^T B^r(t,\tau)P^*(t,\tau)\, {}_{\tau-t}\widetilde{p}_{x+t}d\tau. \end{aligned} \tag{3.8}$$

Furthermore, we recall that

$$\mathcal{S}_j^Q(x,t,T) = e^{A_j^{\mu,Q}(t,T) - B_j^{\mu,Q}(x,t,T)\mu_j^Q(x,t)} = e^{A_j^{\mu,Q}(t,T) - B_j^{\mu,Q}(x,t,T)\mu_j(x,t)(1+g_j(t))},$$

and that ${}_\tau\widetilde{p}_x$ is deterministic at time t. Hence, we get

$$\frac{\partial}{\partial \mu_j}\widetilde{Z}_j^{*,Q}(x,t-) = -(n_j - N_j(x,t-))(1+g_j(t))\int_t^T B_j^{\mu,Q}(t,\tau)P^*(t,\tau)\mathcal{S}_j^Q(x,t,\tau)d\tau. \tag{3.9}$$

Collecting (3.7)-(3.9) and remembering that $Z_j^{*,Q}$ is a martingale, we get

$$dZ_j^{*,Q}(x,t) = \nu_j^{Z,Q}(t)dM_j^Q(x,t) + \eta_j^{Z,Q}(t)dW^{r,Q}(t) + \rho_j^{Z,Q}(t)dW^{\mu,Q}(t),$$

where $\nu_j^{Z,Q}$, $\eta_j^{Z,Q}$ and $\rho_j^{Z,Q}$ are given in (3.4), (3.5) and (3.6), respectively. This proves the lemma.

□

4 Insurance contracts

An insurance contract specifies a payment process with premiums paid by the policy-holders and benefits paid by the insurance company. We consider a portfolio of fairly general life insurance contracts. Each contract allows for a single premium paid at time 0, continuous premiums, a lump sum payment upon retirement, a single payment upon death and life annuity payments. The payment process generated by the portfolio of insurance contracts is formally the net payments to the policy-holders, which means that premiums are negative and benefits are positive.

4.1 The payment process

The payment process is described by

$$\begin{aligned} dA(t) &= -n_1\pi^s(0)d1_{\{t\geq 0\}} - \pi^c(t)(n_1 - N_1(x,t))1_{\{0\leq t<\overline{T}\}}dt + a^d(t)dN_1(x,t) \\ &\quad + (n_1 - N_1(x,\overline{T}))a^r(\overline{T})d1_{\{t\geq \overline{T}\}} + a^p(t)(n_1 - N_1(x,t))1_{\{\overline{T}\leq t\leq T\}}dt. \end{aligned} \tag{4.1}$$

Here, n_1 is the number of people in the insurance portfolio, and $N_1(x,t)$ is the number of deaths in the insurance portfolio during $(0,t]$. The term of the contract is T and the time of retirement is $\overline{T} \leq T$. The first term in (4.1) is the single premium π^s paid by all n_1 policy-holders upon signing the contracts, and the second term is continuous premiums π^c paid by the current $(n_1 - N_1(x,t))$ survivors until retirement. The third term is payments a^d in case of death, and the fourth term is the lump sum payment a^r paid to the remaining policy-holders alive at time $\overline{T}$. The last term is life annuity payments a^p to the remaining policy-holders alive in the period from retirement until the end of the insurance period. We assume that π^c, a^d and a^p are piecewise continuous functions.

4.2 Market reserves

The intrinsic value process associated with payment process A is defined by

$$V^{*,Q}(t) = E^Q\left[\int_{0-}^T dA^*(\tau)\middle|\mathcal{F}(t)\right] = E^Q\left[\int_{0-}^T e^{-\int_0^\tau r(u)du}dA(\tau)\middle|\mathcal{F}(t)\right], \tag{4.2}$$

for $0 \leq t \leq T$. Using that A and r are adapted processes, we see that

$$\begin{aligned} V^{*,Q}(t) &= \int_{0-}^t e^{-\int_0^\tau r(u)du}dA(\tau) + E^Q\left[\int_t^T e^{-\int_0^\tau r(u)du}dA(\tau)\middle|\mathcal{F}(t)\right] \\ &= A^*(t) + \widetilde{V}^{*,Q}(t). \end{aligned} \tag{4.3}$$

The process $\widetilde{V}^{*,Q}(t)$ is referred to as the discounted *market reserve*. It represents the discounted conditional expected value of future payments calculated at time t. As in Dahl and Møller (2006), we formulate the following proposition.

Proposition 4.1 *The discounted market reserve $\widetilde{V}^{*,Q}(t)$ is given by*

$$\widetilde{V}^{*,Q}(t) = (n_1 - N_1(x,t))\widetilde{V}_p^{*,Q}(t),$$

where $\widetilde{V}_p^{*,Q}$ is the discounted market reserve for one policy-holder and is determined by

$$\begin{aligned}\widetilde{V}_p^{*,Q}(t) =& \int_t^T P^*(t,\tau)\mathcal{S}_1^Q(x,t,\tau)\Big(a^d(\tau)f^{\mu_1,Q}(x,t,\tau) - \pi^c(\tau)1_{\{0\le\tau<\overline{T}\}} + a^p(\tau)1_{\{\overline{T}\le\tau\le T\}}\Big)d\tau \\ &+ P^*(t,\overline{T})\mathcal{S}_1^Q(x,t,\overline{T})a^r(\overline{T})1_{\{t<\overline{T}\}}.\end{aligned}$$

As in Dahl and Møller (2006), we give a short comment on the result above: The market reserve (for one policy-holder alive at time t) is a function of the current level for the short rate $r(t)$ and the insurance portfolio's mortality intensity $\mu_1(x,t)$. The market reserve is on the same form as usual reserves, but it now involves the price $P(t,\tau)$ of a zero coupon bond instead of the usual discount factor and the stochastic survival probability $\mathcal{S}_1^Q(x,t,\tau)$ for the portfolio instead of the usual deterministic mortality intensity. Furthermore we note that the deterministic mortality intensity is replaced by the Q-forward mortality intensity $f^{\mu_1,Q}(x,t,\tau)$ in the term involving the payment a^d upon a death. Finally, we emphasize that the market reserve depends on the choice of measure Q. For a proof of Proposition 4.1, we refer to Dahl and Møller (2006).

In addition to Assumption 3.1 we assume the following holds for the remaining of the paper.

Assumption 4.2 $\widetilde{V}_p^{*,Q} \in C^{1,2,2}$, *i.e.* $\widetilde{V}_p^{*,Q}$ *is continuously differentiable with respect to t and twice continuously differentiable with respect to r and μ.*

4.3 A stochastic representation of the insurance contract

Similarly to Lemma 3.2 we have the following stochastic representation for the intrinsic value process of the insurance payment process.

Lemma 4.3 *The intrinsic value process (4.2) associated with the payment process (4.1) admits the representation*

$$V^{*,Q}(t) = V^{*,Q}(0) + \int_0^t \nu^{V,Q}(\tau)dM_1^Q(x,\tau) + \int_0^t \eta^{V,Q}(\tau)dW^{r,Q}(\tau) + \int_0^t \rho^{V,Q}(\tau)dW^{\mu,Q}(\tau), \tag{4.4}$$

where

$$\nu^{V,Q}(t) = B(t)^{-1}a^d(t) - \widetilde{V}_p^{*,Q}(t), \tag{4.5}$$

$$\begin{aligned}\eta^{V,Q}(t) = -\sigma^r(n_1 - N_1(x,t-))\Bigg(&\int_t^T B^r(t,\tau)P^*(t,\tau)\mathcal{S}_1^Q(x,t,\tau) \\ &\times \Big(a^d(\tau)f^{\mu_1,Q}(x,t,\tau) - \pi^c(\tau)1_{\{0\le\tau\le\overline{T}\}} + a^p(\tau)1_{\{\overline{T}\le\tau\le T\}}\Big)d\tau \\ &+ B^r(t,\overline{T})P^*(t,\overline{T})\mathcal{S}_1^Q(x,t,\overline{T})a^r(\overline{T})1_{\{t<\overline{T}\}}\Bigg),\end{aligned} \tag{4.6}$$

and where

$$\rho_j^{V,Q}(t) = -\sigma_{1,j}^{\mu}(x,t)\sqrt{\mu_1(x,t)}(n_1 - N_1(x,t-))(1+g_1(t))\Bigg(\int_t^T P^*(t,\tau)B_1^{\mu,Q}(x,t,\tau)\mathcal{S}_1^Q(x,t,\tau)$$
$$\times\left(a^d(\tau)\left(f^{\mu_1,Q}(x,t,\tau) - \frac{\frac{\partial}{\partial\tau}B_1^{\mu,Q}(x,t,\tau)}{B_1^{\mu,Q}(x,t,\tau)}\right) - \pi^c(\tau)1_{\{0\le\tau\le\overline{T}\}} + a^p(\tau)1_{\{\overline{T}\le\tau\le T\}}\right)d\tau$$
$$+ P^*(t,\overline{T})B_1^{\mu,Q}(x,t,\overline{T})\mathcal{S}_1^Q(x,t,\overline{T})a^r(\overline{T})1_{\{t<\overline{T}\}}\Bigg), \quad j=1,2. \tag{4.7}$$

Before proving the lemma, we briefly explain the result. We note that the representation of the intrinsic value process for the insurance payment process has the same form as the representation of the intrinsic value process for a survivor swap in Lemma 3.2. Thus, the value process for the insurance payment process A consists of three terms relating to the unsystematic mortality risk, the interest rate risk and the systematic mortality risk, respectively.

Proof of Lemma 4.3: The proof is similar to the one for Lemma 3.2. Recall from (4.3) that the Q-martingale $V^{*,Q}$ can be written as

$$V^{*,Q}(t) = A^*(t) + \widetilde{V}^{*,Q}(t),$$

where $\widetilde{V}^{*,Q}$ is the discounted market value of the payment process given by

$$\widetilde{V}^{*,Q}(t) = (n_1 - N_1(x,t))\int_t^T P^*(t,\tau)\mathcal{S}_1^Q(x,t,\tau)$$
$$\times\left(a^d(\tau)f^{\mu_1,Q}(x,t,\tau) - \pi^c(\tau)1_{\{0\le\tau\le\overline{T}\}} + a^p(\tau)1_{\{\overline{T}\le\tau\le T\}}\right)d\tau$$
$$+ (n_1 - N_1(x,t))P^*(t,\overline{T})\mathcal{S}_1^Q(x,t,\overline{T})a^r(\overline{T})1_{\{t<\overline{T}\}},$$

see Proposition 4.1. The dynamics of $V^{*,Q}$ are given by

$$\begin{aligned} dV^{*,Q}(t) &= dA^*(t) + d\widetilde{V}^{*,Q}(t) \\ &= \psi^V(t)dt + B(t)^{-1}a^d(t)dM_1^Q(x,t) + d\widetilde{V}^{*,Q}(t), \end{aligned} \tag{4.8}$$

where ψ^V is some process. The exact form of this process is not important since the value process is a Q-martingale and therefore has no drift. In (4.8), we have added and subtracted the quantity $B(t)^{-1}a^d(t)\lambda_1^Q(x,t)dt$ in order to obtain a term involving $dM^Q(t)$.

In order to determine $dV^{*,Q}$, we now need to find the dynamics of the discounted market value $d\widetilde{V}^{*,Q}(t)$. This is again obtained via Itô's formula:

$$\begin{aligned} d\widetilde{V}^{*,Q}(t) &= \psi_1^V(t)dt + \sigma^r\frac{\partial}{\partial r}\widetilde{V}^{*,Q}(t-)dW^{r,Q}(t) + \sqrt{\mu_1(x,t)}\frac{\partial}{\partial\mu_1}\widetilde{V}^{*,Q}(t-)\sigma_1^{\mu}(x,t)dW^{\mu,Q}(t) \\ &\quad - \widetilde{V}_p^{*,Q}(t)dN_1(x,t) \\ &= \psi_2^V(t)dt + \sigma^r\frac{\partial}{\partial r}\widetilde{V}^{*,Q}(t-)dW^{r,Q}(t) + \sqrt{\mu_1(x,t)}\frac{\partial}{\partial\mu_1}\widetilde{V}^{*,Q}(t-)\sigma_1^{\mu}(x,t)dW^{\mu,Q}(t) \\ &\quad - \widetilde{V}_p^{*,Q}(t)dM_1^Q(x,t). \end{aligned} \tag{4.9}$$

Again ψ_1^V and ψ_2^V are some processes, whose exact form we do not need to know. In (4.9) we used the same idea as in (4.8) in order to get an expression that involves M_1^Q. We can now use (4.9) in (4.8) to obtain

$$dV^{*,Q}(t) = \Big(\psi^V(t) + \psi_2^V(t)\Big)dt + \Big(B(t)^{-1}a^d(t) - \widetilde{V}_p^{*,Q}(t)\Big)dM_1^Q(x,t)$$
$$+ \sigma^r \frac{\partial}{\partial r}\widetilde{V}^{*,Q}(t-)dW^{r,Q}(t) + (\sigma_1^\mu(x,t))^{\mathrm{tr}}\sqrt{\mu_1(x,t)}\frac{\partial}{\partial \mu_1}\widetilde{V}^{*,Q}(t-)dW^{\mu,Q}(t).$$

Using the techniques from Dahl and Møller (2006), it now follows that

$$dV^{*,Q}(t) = \nu^{V,Q}(t)dM_1^Q(x,t) + \eta^{V,Q}(t)dW^{r,Q}(t) + \rho^{V,Q}(t)dW^{\mu,Q}(t), \tag{4.10}$$

where $\nu^{V,Q}$ and $\eta^{V,Q}$ are given by (4.5) and (4.6) and $\rho^{V,Q} = (\rho_1^{V,Q}, \rho_2^{V,Q})$ are given in (4.7). Hence we have proved the lemma.

□

5 Risk-minimizing strategies

5.1 Motivation

When an insurance company signs a life insurance contract with a policy-holder, the company is exposed to both financial and mortality risk. Typically this combined risk cannot be hedged perfectly. A way to handle the risk is to use the criterion of *risk-minimization* suggested by Föllmer and Sondermann (1986); see also Schweizer (2001) for a survey. In the following, we first give a brief introduction to the criterion of risk-minimization. We then determine risk-minimizing strategies in different financial markets. The first market is the one studied in Dahl and Møller (2006), which consists of a savings account and a zero coupon bond. The other markets in addition contain survivor swaps.

5.2 Introduction to risk-minimization

Consider a financial market with a savings account B and a risky asset with discounted price process X. Here, the Q-martingale X may be a vector process. A strategy is a process $\varphi = (\xi, \eta)$ satisfying certain integrability conditions, where ξ is the number of risky assets held, and η is the discounted deposit in the savings account. The discounted value at time t associated with the strategy is $V^*(t,\varphi) = \xi(t)X(t) + \eta(t)$. An investment strategy with $V^*(T,\varphi) = 0$ is called 0-admissible. The *cost process* at time t is given by

$$C(t,\varphi) = V^*(t,\varphi) - \int_0^t \xi(u)dX(u) + A^*(t).$$

Thus, the accumulated costs until time t are the discounted value of the investment portfolio, $V^*(t,\varphi)$, reduced by discounted trading gains and added discounted net payments to the policy-holders.

If a strategy φ minimizes the so-called *risk process* $R(\cdot,\varphi)$ defined by

$$R(t,\varphi) = E^Q\left[\Big(C(T,\varphi) - C(t,\varphi)\Big)^2\Big|\mathcal{F}(t)\right],$$

it is called *risk-minimizing*. The strategy can be determined from the *Galtchouk-Kunita-Watanabe* decomposition given by

$$V^{*,Q}(t) = E^Q\left[A^*(T)|\mathcal{F}(t)\right] = V^{*,Q}(0) + \int_0^t \xi^Q(u)dX(u) + L^Q(t),$$

where ξ^Q is a predictable process, and L^Q is a zero-mean Q-martingale which is orthogonal to X. There exists a unique 0-admissible risk-minimizing strategy $\varphi^* = (\xi^*, \eta^*)$, given by

$$\varphi^*(t) = (\xi^*(t), \eta^*(t)) = (\xi^Q(t), V^{*,Q}(t) - \xi^Q(t)X(t) - A^*(t)),$$

see Møller (2001, Theorem 2.1). The intrinsic risk process, which measures the minimum obtainable risk, is given by

$$R(t, \varphi^*) = E^Q \left[(L^Q(T) - L^Q(t))^2 \middle| \mathcal{F}(t) \right]. \tag{5.1}$$

Typically, this quantity has to be evaluated numerically by simulation.

5.3 Risk-minimization in a bond market

Consider the market introduced in Section 2.1 with a savings account and a zero coupon bond. This market was also studied in Dahl and Møller (2006). They showed that the risk-minimizing strategy is given by

$$\varphi_B^*(t) = (\xi_B^*(t), \eta_B^*(t)) = (\xi_B^Q(t),\ \widetilde{V}^{*,Q}(t) - \xi_B^Q(t)P^*(t,T)),$$

where

$$\xi_B^Q(t) = \frac{\eta^{V,Q}(t)}{-\sigma^r B^r(t,T)P^*(t,T)}. \tag{5.2}$$

The unhedgeable risk is determined by the process L^Q with

$$dL^Q(t) = \nu^{V,Q}(\tau)dM_1^Q(x,\tau) + \rho_1^{V,Q}(\tau)dW_1^{\mu,Q}(\tau) + \rho_2^{V,Q}(\tau)dW_2^{\mu,Q}(\tau). \tag{5.3}$$

By inserting (5.3) in (5.1), we see that the intrinsic risk process is given by

$$R(t, \varphi_B^*) = E^Q \left[\int_t^T \left(\left(\nu^{V,Q}(\tau)\right)^2 \lambda_1^Q(x,\tau) + \sum_{j=1}^{2} \left(\rho_j^{V,Q}(\tau)\right)^2 \right) d\tau \middle| \mathcal{F}(t) \right].$$

We evaluate the initial intrinsic risk $R(0, \varphi_B^*)$ numerically in Section 6.

5.4 Risk-minimization with an insurance portfolio survivor swap

Now consider the market (B, P, Z_1), which in addition to the savings account B and the zero coupon bond P, includes a survivor swap on the insurance portfolio. From Lemma 3.2 we have the following stochastic representation for intrinsic value process of the survivor swap

$$dZ_1^{*,Q}(x,t) = \nu_1^{Z,Q}(t)dM_1^Q(x,t) + \eta_1^{Z,Q}(t)dW^{r,Q}(t) + \rho_1^{Z,Q}(t)dW^{\mu,Q}(t).$$

Thus, we assume that we can trade a survivor swap on the insurance portfolio dynamically. It would indeed be more more realistic to work with less frequent trading of the survivor swap. However, the current model with continuous time trading of the survivor swap still gives an idea of how the insurance company could reduce risk in the extended markets. The case with discrete time trading of the survivor swap is postponed to future research.

In order to determine the risk-minimizing strategy, it is useful to introduce certain processes $Y_{1,1}^{*,Q}$ and $Y_{1,2}^{*,Q}$ which are orthogonal to the traded assets. These processes will help us construct a zero-mean Q-martingale, which is orthogonal with the zero coupon bond and the survivor swap. To simplify notation, we define $\chi_{i,j}^{Z,Q}(t) = 1_{\{\rho_{i,j}^{Z,Q}(t) \neq 0\}}$ for $i, j \in \{1, 2\}$.

Let the zero-mean martingales $Y_{1,j}^{*,Q}$, $j = 1, 2$, be given by

$$dY_{1,j}^{*,Q}(t) = \chi_{1,j}^{Z,Q}(t) \left(dM_1^Q(x,t) - \kappa_{1,j}^Q(t) dW_j^{\mu,Q}(t) \right), \tag{5.4}$$

with $Y_{1,j}^{*,Q}(0) = 0$. For $\rho_{1,j}^{Z,Q}(t) \neq 0$, we furthermore define $\kappa_{1,j}^Q(t)$ by

$$\kappa_{1,j}^Q(t) = \frac{\nu_1^{Z,Q}(t)\lambda_1^Q(x,t)}{\rho_{1,j}^{Z,Q}(t)}. \tag{5.5}$$

This construction ensures that $Y_{1,1}^{*,Q}$ and $Y_{1,2}^{*,Q}$ are indeed orthogonal to the discounted zero coupon price process $P^*(\cdot, T)$ and the discounted price process $Z_1^{*,Q}$ associated with the survivor swap. In the situation where $\rho_{1,j}^{Z,Q}(t) = 0$, we see that $dY_{1,j}^{*,Q}(t) = 0$. In the special case without systematic mortality risk, we have that $\rho_{1,1}^{Z,Q}(t) = \rho_{1,2}^{Z,Q}(t) = 0$ for all t, such that $Y_{1,1}^*$ and $Y_{1,2}^*$ are constant and equal to 0. In fact, this implies that the insurance payment process A is attainable and thus it can be hedged perfectly. The risk-minimizing strategy in this case was essentially obtained in Møller (1998, Section 5).

Proposition 5.1 *The Galtchouk-Kunita-Watanabe decomposition of $V^{*,Q}$ in the market (B, P, Z_1) is given by*

$$V^{*,Q}(t) = V^{*,Q}(0) + \int_0^t \xi_1^Q(\tau) dP^*(\tau, T) + \int_0^t \vartheta_1^Q(\tau) dZ_1^{*,Q}(x,\tau) + L_1^Q(t), \tag{5.6}$$

where $V^{*,Q}(0) = -n_1\pi^s(0) + n_1\widetilde{V}_p^Q(0)$ and

$$\begin{aligned} L_1^Q(t) = &\int_0^t \left(\nu^{V,Q}(\tau) - \vartheta_1^Q(\tau)\nu_1^{Z,Q}(\tau) \right) dM_1^Q(x,\tau) \\ &+ \sum_{j=1}^2 \int_0^t \left(\rho_j^{V,Q}(\tau) - \vartheta_1^Q(\tau)\rho_{1,j}^{Z,Q}(\tau) \right) dW_j^{\mu,Q}(\tau), \end{aligned}$$

with

$$\xi_1^Q(t) = \frac{\eta^{V,Q}(t) - \vartheta_1^Q(t)\eta_1^{Z,Q}(t)}{-\sigma^r B^r(t,T) P^*(t,T)}, \tag{5.7}$$

$$\vartheta_1^Q(t) = \frac{\nu^{V,Q}(t) + \rho_1^{V,Q}(t)(\kappa_{1,1}^Q(t))^{-1}\chi_{1,1}^{Z,Q}(t) + \rho_2^{V,Q}(t)(\kappa_{1,2}^Q(t))^{-1}\chi_{1,2}^{Z,Q}(t)}{\nu_1^{Z,Q}(t) + \rho_{1,1}^{Z,Q}(t)(\kappa_{1,1}^Q(t))^{-1}\chi_{1,1}^{Z,Q}(t) + \rho_{1,2}^{Z,Q}(t)(\kappa_{1,2}^Q(t))^{-1}\chi_{1,2}^{Z,Q}(t)}. \tag{5.8}$$

The proof is postponed to the Appendix. The optimal number of zero-coupon bonds determined by ξ_1^Q is the optimal number ξ_B^Q from the bond market adjusted by a term originating from the interest rate risk inherent in the survivor swap. The optimal number of survivor swaps can be interpreted as a ratio between risk-weighted averages of the mortality risk associated with the intrinsic value process of the insurance payment process and the price process of the survivor swap, respectively. The unhedged risk L_1^Q consists of both unsystematic and systematic mortality risk. The unsystematic mortality risk is driven by the compensated counting process M_1^Q, and the systematic mortality risk is driven by the 2-dimensional Brownian motion $W^{\mu,Q}$. The unsystematic mortality risk is the standard discounted sum at risk $\nu^{V,Q}$ reduced by the discounted sum at risk related to the investment in survivor swaps. Similarly, the unhedged systematic mortality risk is the original systematic risk, reduced by the risk from the survivor swaps.

From the general theory of risk-minimization in Section 5.2, we get the unique 0-admissible risk-minimizing strategy for the payment process (4.1)

$$\varphi_1^*(t) = (\xi_1^*(t), \vartheta_1^*(t), \eta_1^*(t)) = (\xi_1^Q(t),\, \vartheta_1^Q(t),\, \widetilde{V}^{*,Q}(t) - \xi_1^Q(t)P^*(t,T) - \vartheta_1^Q Z_1^{*,Q}(t)),$$

where ξ_1^Q and ϑ_1^Q are given in Proposition 5.1. From (5.1) we get the intrinsic risk process

$$\begin{aligned} R(t,\varphi_1^*) &= E^Q\Bigg[\Bigg(\int_t^T \Big(\nu^{V,Q}(\tau) - \vartheta_1^Q(\tau)\nu_1^{Z,Q}(\tau)\Big)dM_1^Q(x,\tau) \\ &\qquad + \sum_{j=1}^2 \Big(\rho_j^{V,Q}(\tau) - \vartheta_1^Q(\tau)\rho_{1,j}^{Z,Q}(\tau)\Big)dW_j^{\mu,Q}(\tau)\Bigg)^2\Bigg|\mathcal{F}(t)\Bigg] \\ &= E^Q\Bigg[\Bigg(\int_t^T \Big(\nu^{V,Q}(\tau) - \vartheta_1^Q(\tau)\nu_1^{Z,Q}(\tau)\Big)^2\lambda_1^Q(x,\tau) \\ &\qquad + \sum_{j=1}^2 \Big(\rho_j^{V,Q}(\tau) - \vartheta_1^Q(\tau)\rho_{1,j}^{Z,Q}(\tau)\Big)^2\Bigg)d\tau\Bigg|\mathcal{F}(t)\Bigg]. \end{aligned}$$

Here we have used, that M_1^Q, $W_1^{\mu,Q}$ and $W_2^{\mu,Q}$ are mutually independent and the fact that $d\langle M_1^Q(x,t)\rangle = \lambda_1^Q(t)dt$ and $d\langle W_j^{\mu,Q}(t)\rangle = dt$.

5.5 Risk-minimization with a population survivor swap

As an alternative to the market (B,P,Z_1) considered in the previous section, we now study the market (B,P,Z_2). Hence, we allow for investments in a survivor swap on the population instead of the insurance portfolio.

From Lemma 3.2, we have the following representation for the intrinsic value process of the population survivor swap

$$dZ_2^{*,Q}(t) = \nu_2^{Z,Q}(t)dM_2^Q(x,t) + \eta_2^{Z,Q}(t)dW^{r,Q}(t) + \rho_2^{Z,Q}(t)dW^{\mu,Q}(t).$$

Here, we note that the unsystematic mortality risk is driven by the random source M_2^Q, which means that we introduce a new random source compared to the market (B,P,Z_1) from Section 5.4 above. Consequently, we now need three zero-mean martingales $Y_{2,j}^{*,Q}$, $j = 1,2,3$, in order to span all risk, since we have five random sources in the market, and only two risky assets to hedge the risk.

Let the zero-mean martingales $Y_{2,j}^{*,Q}$, $j = 1,2,3$, be given by

$$\begin{aligned} dY_{2,j}^{*,Q}(t) &= \chi_{2,j}^{Z,Q}(t)\left(dM_2^Q(x,t) - \kappa_{2,j}^Q(t)dW_j^{\mu,Q}(t)\right),\ j = 1,2, \\ dY_{2,3}^{*,Q}(t) &= dM_1^Q(x,t), \end{aligned}$$

where $Y_{2,j}^{*,Q}(0) = 0$, $j = 1,2,3$, and for $\rho_{2,j}^{Z,Q}(t) \neq 0$, $j = 1,2$, we define $\kappa_{2,j}^Q(t)$ by

$$\kappa_{2,j}^Q(t) = \frac{\nu_2^{Z,Q}(t)\lambda_2^Q(x,t)}{\rho_{2,j}^{Z,Q}(t)}.$$

Then $Y_{2,j}^{*,Q}$ are orthogonal to $P^*(\cdot,T)$ and $Z_2^{*,Q}$. We note that even in the case without systematic mortality risk, the insurance payment process A is not attainable in the market (B,P,Z_2), since the unsystematic mortality risk cannot be eliminated. This is in contrast to the situation in the previous market (B,P,Z_1).

Proposition 5.2 *The Galtchouk-Kunita-Watanabe decomposition of $V^{*,Q}$ in the market (B,P,Z_2) is given by*

$$V^{*,Q}(t) = V^{*,Q}(0) + \int_0^t \xi_2^Q(\tau)dP^*(\tau,T) + \int_0^t \vartheta_2^Q(\tau)dZ_2^{*,Q}(x,\tau) + L_2^Q(t),$$

where

$$\begin{aligned} V^{*,Q}(0) &= -n_1\pi^s(0) + n_1\widetilde{V}_p^{*,Q}(0), \\ L_2^Q(t) &= \int_0^t \nu^{V,Q}(\tau)dM_1^Q(x,\tau) - \int_0^t \vartheta_2^Q(\tau)\nu_2^{Z,Q}(\tau)dM_2^Q(x,\tau) \\ &\quad + \sum_{j=1}^2 \int_0^t \left(\rho_j^{V,Q}(\tau) - \vartheta_2^Q(\tau)\rho_{2,j}^{Z,Q}\right)dW_j^{\mu,Q}(\tau), \end{aligned} \tag{5.9}$$

and

$$\xi_2^Q(t) = \frac{\eta^{V,Q}(t) - \vartheta_2^Q(t)\eta_2^{Z,Q}(t)}{-\sigma^r B^r(t,T)P^*(t,T)}, \tag{5.10}$$

$$\vartheta_2^Q(t) = \frac{\rho_1^{V,Q}(t)(\kappa_{2,1}^Q(t))^{-1}\chi_{2,1}^{Z,Q}(t) + \rho_2^{V,Q}(t)(\kappa_{2,2}^Q(t))^{-1}\chi_{2,2}^{Z,Q}(t)}{\nu_2^{Z,Q}(t) + \rho_{2,1}^{Z,Q}(t)(\kappa_{2,1}^Q(t))^{-1}\chi_{2,1}^{Z,Q}(t) + \rho_{2,2}^{Z,Q}(t)(\kappa_{2,2}^Q(t))^{-1}\chi_{2,2}^{Z,Q}(t)}. \tag{5.11}$$

The interpretation of the Galtchouk-Kunita-Watanabe decomposition obtained in Proposition 5.2 is essentially identical to that of Proposition 5.1. However in this case the unhedgeable unsystematic mortality risk consists of two terms. A term which stems from the insurance portfolio driven by M_1^Q and a term driven by M_2^Q originating from investments in survivor swaps.

The unique 0-admissible risk-minimizing strategy for the payment process (4.1) is given by

$$\varphi_2^*(t) = (\xi_2^*(t), \vartheta_2^*(t), \eta_2^*(t)) = (\xi_2^Q(t),\, \vartheta_2^Q(t),\, \widetilde{V}^{*,Q}(t) - \xi_2^Q(t)P^*(t,T) - \vartheta_2^Q Z_2^{*,Q}(t)),$$

where ξ_2^Q and ϑ_2^Q are given in (5.10) and (5.11). The intrinsic risk process $R(\cdot,\varphi_2^*)$ can be determined as in the previous section. This leads to

$$\begin{aligned} R(t,\varphi_2^*) = E^Q\Bigg[\Bigg(\int_t^T & \left(\nu^{V,Q}(\tau)\right)^2\lambda_1^Q(x,\tau) + \left(\vartheta_2^Q(\tau)\nu_2^{Z,Q}(\tau)\right)^2\lambda_2^Q(x,\tau) \\ & + \sum_{j=1}^2 \left(\rho_j^{V,Q}(\tau) - \vartheta_2^Q(\tau)\rho_{2,j}^{Z,Q}(\tau)\right)^2\Bigg)d\tau\Bigg|\mathcal{F}(t)\Bigg]. \end{aligned}$$

5.6 Risk-minimization with survivor swaps on both portfolios

Now consider the situation where the market includes both survivor swaps. In this case the considered market (B, P, Z_1, Z_2) includes three risky assets, whereas we have five random sources in the market: Two random sources driving the unsystematic mortality risks, two random sources driving the systematic mortality risks and one random source driving the interest rate risk. Thus, the market is still incomplete.

Proposition 5.3 *The Galtchouk-Kunita-Watanabe decomposition of $V^{*,Q}$ in the market (B, P, Z_1, Z_2) is given by*

$$V^{*,Q}(t) = V^{*,Q}(0) + \int_0^t \xi_3^Q(\tau)dP^*(\tau,T) + \sum_{j=1}^{2} \int_0^t \vartheta_{3,j}^Q(\tau)dZ_j^{*,Q}(x,\tau) + L_3^Q(t),$$

where

$$\vartheta_{3,1}^Q(t) = \frac{\nu^{V,Q}(t) + \frac{\rho_1^{V,Q}(t)}{\kappa_{1,1}^Q(t)}\chi_{1,1}^{Z,Q}(t) + \frac{\rho_2^{V,Q}(t)}{\kappa_{1,2}^Q(t)}\chi_{1,2}^{Z,Q}(t) - \left(\frac{\rho_{2,1}^{Z,Q}(t)}{\kappa_{1,1}^Q(t)}\chi_{1,1}^{Z,Q}(t) + \frac{\rho_{2,2}^{Z,Q}(t)}{\kappa_{1,2}^Q(t)}\chi_{1,2}^{Z,Q}(t)\right)\vartheta_2^Q(t)}{\nu_1^{Z,Q}(t) + \frac{\rho_{1,1}^{Z,Q}(t)}{\kappa_{1,1}^Q(t)}\chi_{1,1}^{Z,Q}(t) + \frac{\rho_{1,2}^{Z,Q}(t)}{\kappa_{1,2}^Q(t)}\chi_{1,2}^{Z,Q}(t) - \left(\frac{\rho_{2,1}^{Z,Q}(t)}{\kappa_{1,1}^Q(t)}\chi_{1,1}^{Z,Q}(t) + \frac{\rho_{2,2}^{Z,Q}(t)}{\kappa_{1,2}^Q(t)}\chi_{1,2}^{Z,Q}(t)\right)\widetilde{\vartheta}_2^Q(t)},$$

$$\vartheta_{3,2}^Q(t) = \frac{\frac{\rho_1^{V,Q}(t)}{\kappa_{2,1}^Q(t)}\chi_{2,1}^{Z,Q}(t) + \frac{\rho_2^{V,Q}(t)}{\kappa_{2,2}^Q(t)}\chi_{2,2}^{Z,Q}(t) - \vartheta_{3,1}^Q(t)\left(\frac{\rho_{1,1}^{Z,Q}(t)}{\kappa_{2,1}^Q(t)}\chi_{2,1}^{Z,Q}(t) + \frac{\rho_{1,2}^{Z,Q}(t)}{\kappa_{2,2}^Q(t)}\chi_{2,2}^{Z,Q}(t)\right)}{\nu_2^{Z,Q}(t) + \frac{\rho_{2,1}^{Z,Q}(t)}{\kappa_{2,1}^Q(t)}\chi_{2,1}^{Z,Q}(t) + \frac{\rho_{2,2}^{Z,Q}(t)}{\kappa_{2,2}^Q(t)}\chi_{2,2}^{Z,Q}(t)},$$

$$\xi_3^Q(t) = \frac{\eta^{V,Q}(t) - \vartheta_{3,1}^Q(t)\eta_1^Z(t) - \vartheta_{3,2}^Q(t)\eta_2^Z(t)}{-\sigma^r B^r(t,T)P^*(t,T))},$$

and where

$$\vartheta_2^Q(t) = \frac{\rho_1^{V,Q}(t)(\kappa_{2,1}^Q(t))^{-1}\chi_{2,1}^{Z,Q}(t) + \rho_2^{V,Q}(t)(\kappa_{2,2}^Q(t))^{-1}\chi_{2,2}^{Z,Q}(t)}{\nu_2^{Z,Q}(t) + \rho_{2,1}^{Z,Q}(t)(\kappa_{2,1}^Q(t))^{-1}\chi_{2,1}^{Z,Q}(t) + \rho_{2,2}^{Z,Q}(t)(\kappa_{2,2}^Q(t))^{-1}\chi_{2,2}^{Z,Q}(t)},$$

$$\widetilde{\vartheta}_2^Q(t) = \frac{\rho_{1,1}^{Z,Q}(t)(\kappa_{2,1}^Q(t))^{-1}\chi_{2,1}^{Z,Q}(t) + \rho_{1,2}^{Z,Q}(t)(\kappa_{2,2}^Q(t))^{-1}\chi_{2,2}^{Z,Q}(t)}{\nu_2^{Z,Q}(t) + \rho_{2,1}^{Z,Q}(t)(\kappa_{2,1}^Q(t))^{-1}\chi_{2,1}^{Z,Q}(t) + \rho_{2,2}^{Z,Q}(t)(\kappa_{2,2}^Q(t))^{-1}\chi_{2,2}^{Z,Q}(t)}.$$

Furthermore

$$\begin{aligned} V^{*,Q}(0) &= -n_1\pi^s(0) + n_1\widetilde{V}_p^{*,Q}(0), \\ L_3(t) &= \sum_{j=1}^{2}\int_0^t \left(\rho_j^{V,Q}(\tau) - \vartheta_{3,1}^Q(\tau)\rho_{1,j}^{Z,Q}(\tau) - \vartheta_{3,2}^Q(\tau)\rho_{2,j}^{Z,Q}(\tau)\right)dW_j^{\mu,Q}(\tau) \\ &\quad + \int_0^t \left(\nu^{V,Q}(\tau) - \vartheta_{3,1}^Q(\tau)\nu_1^{Z,Q}(\tau)\right)dM_1^Q(x,\tau) - \int_0^t \vartheta_{3,2}^Q(\tau)\nu_2^{Z,Q}(\tau)dM_2^Q(x,\tau). \end{aligned}$$

□

The proof of the proposition, which is postponed to the Appendix, is carried out using the same techniques as in the proof of Proposition 5.1.

We observe that the Galtchouk-Kunita-Watanabe decomposition essentially is of the same form as in the cases with only one survivor swap. However the coefficients are far more complex here than in the previous markets.

Here the unique 0-admissible risk-minimizing strategy for the payment process (4.1) is given by

$$\begin{aligned}\varphi_3^*(t) &= (\xi_3^*(t), \vartheta_{3,1}^*(t), \vartheta_{3,2}^*(t), \eta_3^*(t)) \\ &= (\xi_3^Q(t),\, \vartheta_{3,1}^Q(t),\, \vartheta_{3,2}^Q(t),\, \widetilde{V}^{*,Q}(t) - \xi_3^Q(t)P^*(t,T) - \vartheta_{3,1}^Q Z_1^{*,Q}(t) - \vartheta_{3,2}^Q Z_2^{*,Q}(t)),\end{aligned}$$

where ξ_3^Q, $\vartheta_{3,1}^Q$ and $\vartheta_{3,2}^Q$ are given in the above proposition. From (5.1) we get the intrinsic risk process

$$\begin{aligned}R(t,\varphi_3^*) = E^Q\Bigg[\int_t^T \Bigg(&\Big(\nu^{V,Q}(\tau) - \vartheta_{3,1}^Q(\tau)\nu_1^{Z,Q}(\tau)\Big)^2 \lambda_1^Q(x,\tau) + \Big(\vartheta_{3,2}^Q(\tau)\nu_2^{Z,Q}(\tau)\Big)^2 \lambda_2^Q(\tau) \\ &+ \sum_{j=1}^2 \Big(\rho_j^{V,Q}(\tau) - \vartheta_{3,1}^Q(\tau)\rho_{1,j}^{Z,Q}(\tau) - \vartheta_{3,2}^Q(\tau)\rho_{2,j}^{Z,Q}(\tau)\Big)^2\Bigg) d\tau \Bigg| \mathcal{F}(t)\Bigg].\end{aligned}$$

6 Numerical examples

In this section, we study the risk-minimizing strategies and their efficiency. We first focus on investments in the survivor swaps in the relevant markets. We then study investments in the zero-coupon bond, which is included in all the markets. Finally, we compare the efficiency of the various investment strategies.

6.1 Setup

Unless stated otherwise we consider an insurance portfolio with 100 policy-holders aged 30, who pay a continuous premium of $\pi^c(t) = 0.2$ during $[0, \overline{T}]$, where $\overline{T} = 30$. In case of a death at time t the insurance company pays a lump sum of $a^d(t) = 5 \cdot 1_{\{0 \le t < \overline{T}\}}$. Hence the death benefit is paid out upon a death before retirement only. At the age of retirement, which is 60 years, a lump sum of $a^r(\overline{T}) = 3$ is paid to all survivors. Finally, the contract contains a 30 year life annuity starting at age 60 with a rate of $a^p(t) = 1$, which implies that $T = 60$. The population portfolio exists of 1000 lives aged 30.

The initial mortality intensities μ_j° are taken on the Gompertz-Makeham form

$$\mu_j^\circ(x+t) = a_j + b_j(c_j)^{x+t},$$

with parameters given in Table 1. The parameters for portfolio 1 are taken from Dahl and Møller (2006). For portfolio 2, we have modified the parameters slightly, such that there is a minor difference between the two initial mortalities. The parameters for the

Portfolio (j)	a	b	c
1	0.0001340	0.0000353	1.1020000
2	0.0001360	0.0000350	1.1030000

Table 1: *Gompertz-Makeham parameters.*

dynamics for the mortality intensities are also inspired by Dahl and Møller (2006) and can

Portfolio (j)	$\mu_j(x,0)$	$\gamma_j(x,t)$	$\delta_j(x,t)$	$\sigma_{j,1}(x,t)$	$\sigma_{j,2}(x,t)$
1	$\mu_1^0(x)$	0.0001800	0.0080	0.006	0.018
2	$\mu_2^0(x)$	0.0001805	0.0081	0.000	0.019

Table 2: *Parameters for mortality intensities.*

be found in Table 2. Here, however, the mortality intensities are driven by two underlying Brownian motions. The parameters for the financial market are listed in Table 3. Using these parameters in the Vasiček model, we get a short rate model with Q-mean reversion level $\gamma^{r,Q}/\delta^{r,Q} = 0.055$. The speed of mean reversion is determined by $\delta^{r,Q}$. Figure 1

$r(0)$	$\gamma^{r,Q}$	$\delta^{r,Q}$	σ^r
0.03	0.011	0.2	0.01

Table 3: *Parameters for the financial market.*

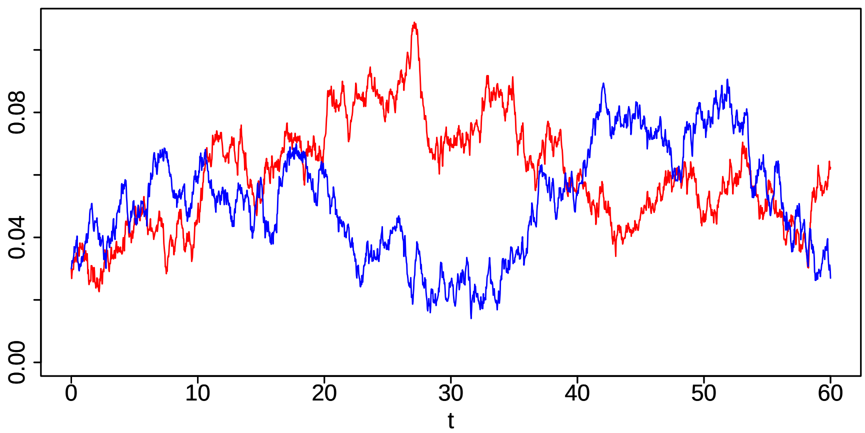

Figure 1: *The short rate in two different stochastic scenarios (red and blue lines).*

shows the development of the short rate in two stochastic scenarios (red and blue lines). We observe that the level of the short rate is similar in the two scenarios until time 20. In the time interval from 20 to 40, the short rate in the blue (red) scenario is relatively low (high), whereas the situation is reversed from time 40 to 55. In the last 5 years the short rates again lie at the same level. The corresponding intrinsic value processes for the insurance payment process in these two scenarios are plotted in Figure 2. Comparing Figures 1 and 2 we see that a low (high) interest rate has a positive (negative) impact on the intrinsic value process for the insurance payment process.

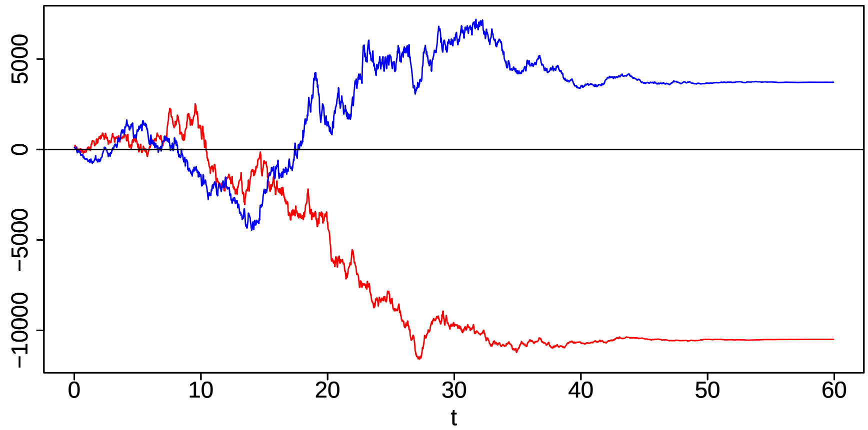

Figure 2: *Intrinsic value processes for the insurance contract the red and blue scenario.*

6.2 Risk sensitivity of the insurance contract

Figure 3 shows the outcomes of the so-called risk sensitivities $\nu^{V,Q}$, $\eta^{V,Q}$ and $\rho^{V,Q}$ in the red and blue scenario. We observe that due the relatively large number of insured lives

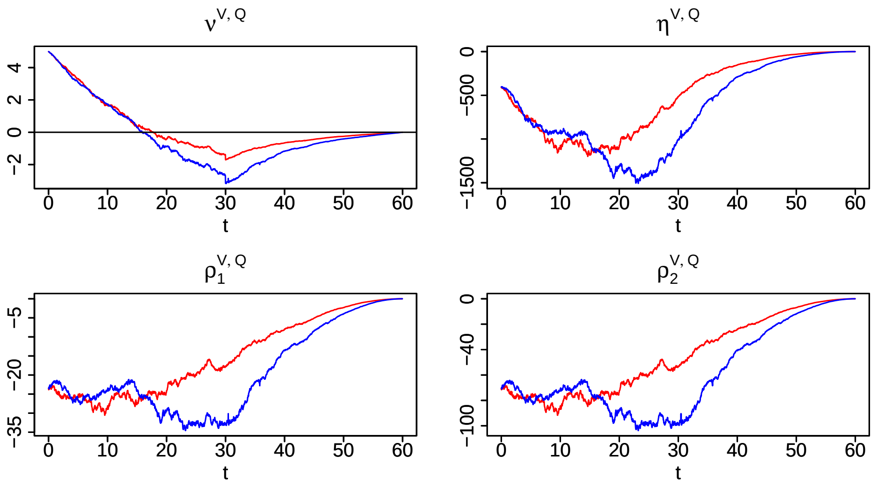

Figure 3: *Risk sensitivities for the insurance contract in the red and blue scenario.*

in the insurance portfolio, the magnitude of the sensitivity $\nu^{V,Q}$ to unsystematic mortality risk is insignificant compared to the other sensitivities. Initially $\nu^{V,Q}$ is positive reflecting that the discounted death benefit is larger than the discounted individual market reserve. From time 0 to the time of retirement, $\nu^{V,Q}$ decreases due to the premiums paid. After approximately 15 years $\nu^{V,Q}$ becomes negative. At the time of retirement we observe a negative jump in $\nu^{V,Q}$ since the death benefit is larger than the sum at retirement. At the age of retirement, only payments from the life annuity remains, and here $\nu^{V,Q}$ attains its minimal value. Hereafter it increases steadily towards the value 0 at the expiry of the contract. We see that $\nu^{V,Q}$ attain lower values in the blue scenario than in the red scenario. This difference is most prominent between times 20 and 40, where we also observe considerable differences for the short rates.

The sensitivity to interest rate risk $\eta^{V,Q}$ is the most significant risk due to the large interest rate volatility. As noted already this fact is especially evident if we focus on the value process in Figure 2 in the period from 20 to 35 years. The differences in the sensitivities to interest rate risk in the two scenarios can be explained by the development in the value processes.

Finally, we recall from Table 2 that the dependence of μ_1 on $W_2^{\mu,Q}$ is three times that of the dependence on $W_1^{\mu,Q}$. This relationship essentially carries through to the risk sensitivities $\rho_1^{V,Q}$ and $\rho_2^{V,Q}$. Once again the differences in the sensitivities to systematic mortality risk in the two scenarios can be explained by the development of the value processes.

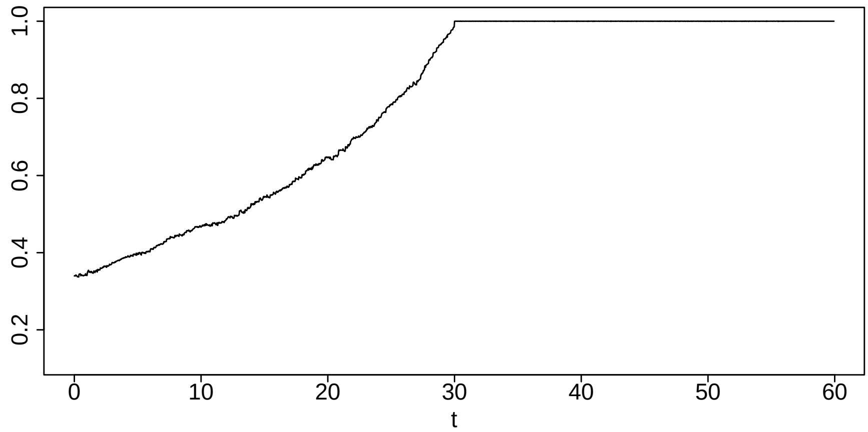

Figure 4: *Number of survivor swaps on the insurance portfolio held at time t in the (B, P, Z_1) market (in the red scenario).*

6.3 Investment strategies

The risk-minimizing strategy minimizes the total risk, which consists of unsystematic and systematic mortality risk and interest rate risk. Here we first focus on investments in survivor swaps and then we turn to investments in bonds. In this section we diverge from the standard setup and consider an insurance portfolio of 10,000 individuals and a population of 100,000 individuals.

In the (B, P, Z_1) market, the survivor swap can be used to hedge both unsystematic mortality risk and systematic mortality risk generated by $W_1^{\mu,Q}$ and $W_2^{\mu,Q}$. The large number of insured implies that the unsystematic mortality risk is insignificant compared to the systematic mortality risks. Until the age of retirement the investments in the survivor swap is therefore essentially the ratio between a weighted average of the sensitivities to systematic mortality risk from the insurance contract and the sensitivities to systematic mortality risk from the survivor swap. At the age of retirement, the only remaining payments from the insurance contract is a life annuity, which has the exact same sensitivity to mortality risk as a survivor swap on the insurance portfolio. Hence, from the time of retirement the mortality risk can be hedged perfectly by holding exactly one survivor swap. The investment strategy is seen in Figure 4.

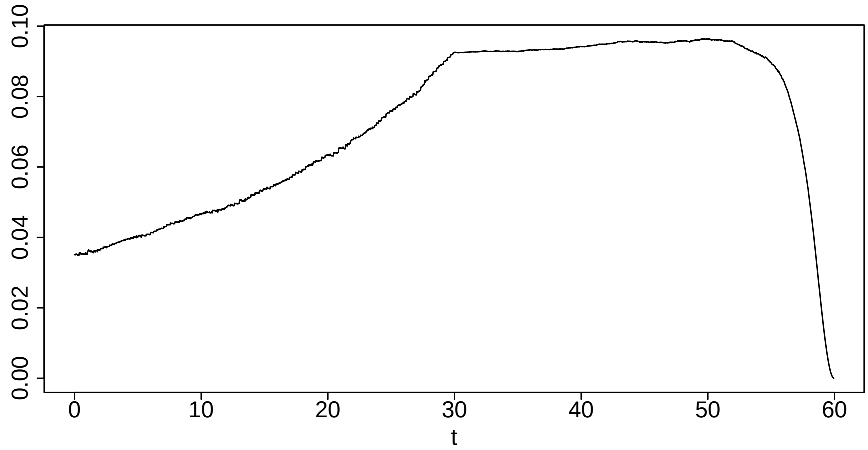

Figure 5: *Number of survivor swaps on population held at time t in the* (B, P, Z_2) *market.*

In the (B, P, Z_2) market, the survivor swap can be used to hedge systematic mortality risk generated by $W_2^{\mu,Q}$ only. Furthermore, the survivor swap introduces an additional source of unsystematic mortality risk, M_2^Q. Since the unsystematic mortality risk is insignificant compared to the $W_2^{\mu,Q}$-systematic mortality risk, the investments in the survivor swap is essentially the ratio between a weighted average of the sensitivity to $W_2^{\mu,Q}$-systematic mortality risk from the insurance portfolio and the sensitivity to $W_2^{\mu,Q}$-systematic mortality risk from the survivor swap. The investment strategy is seen in Figure 5. Comparing Figures 4 and 5 we observe a large similarity between the shapes of ϑ_1 and ϑ_2. However, we notice, that the investment in the (B, P, Z_2) market tends to zero at the end of the insurance period. This can be explained by the fact that the sensitivity to systematic mortality risk from the insurance contract converges to 0, whereas the swap's sensitivity to unsystematic mortality risk does not converge to 0. Since the number of individuals in the population is 10 times the number in the insurance portfolio the magnitude of ϑ_1 is essentially 10 times the magnitude of ϑ_2.

In the (B, P, Z_1, Z_2) market, the survivor swaps are used to hedge both unsystematic mortality risk related to the insurance portfolio and the systematic mortality risk. Furthermore, the survivor swap on the population introduces the additional unsystematic mortality risk. First we recall that, from the time of retirement, a survivor swap on the portfolio eliminates all mortality risk, which in turn means that no investments in the survivor swap on the population is needed after the time of retirement. Prior to the time of retirement we use both survivor swaps to hedge the mortality risks. From the strategies in Figure 6, we observe that, upon correcting for the larger number of individuals in the population, the total number of swaps is slightly higher than the number of swaps held in the (B, P, Z_1) market.

In the four markets considered in Sections 5.3-5.6 all interest rate risk can be hedged. When we invest in survivor swap(s), we introduce additional interest rate risk, and the investments in the bond eliminate the interest rate risk from the insurance payment process corrected for the interest sensitivity from the survivor swap(s). Thus, the investment

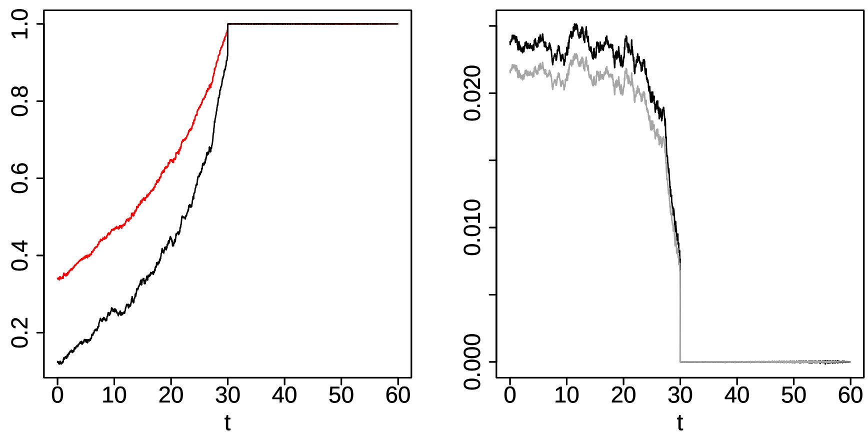

Figure 6: *The left plot shows the number of survivor swaps on the insurance portfolio held at time t in the (B, P, Z_1, Z_2) market (black line). The red line is the number of survivor swaps on the insurance portfolio held at time t in the (B, P, Z_1) market. The right plot shows the number of survivor swaps on the population held at time t in the (B, P, Z_1, Z_2) market (black line). The grey line is the difference between the investments in the survivor swap on the insurance portfolio from the (B, P, Z_1) market and the (B, P, Z_1, Z_2) market scaled by a factor 10.*

strategies for the zero-coupon bond will differ slightly, depending on the investments in survivor swap(s). The investment strategies are plotted in Figure 7.

6.4 Efficiency of the strategies

The efficiency of the strategies is assessed by calculating the initial intrinsic risks $R(0, \cdot)$. We use Monte Carlo simulation with 1,000 simulations to calculate $R(0, \cdot)$ in different markets with portfolios of different sizes. In order to quantify the total risk inherent in the insurance contract we consider the initial intrinsic risk in the market including only the risk free asset B. In this case the intrinsic risk process for the insurance contract is given by

$$R(t, \varphi_V^*) = E^Q \left[\int_t^T \left(\left(\nu^{V,Q}(\tau) \right)^2 \lambda_1^Q(x, \tau) \, + \, \left(\eta^{V,Q}(\tau) \right)^2 + \sum_{j=1}^{2} \left(\rho_j^{V,Q}(\tau) \right)^2 \right) d\tau \middle| \mathcal{F}(t) \right].$$

The results of the Monte Carlo simulations are collected Table 4. First of all, we observe that introducing a zero coupon bond eliminates approximately 80% of the risk, so the interest rate is without doubt the most significant source of risk. Investigating the markets without survivor swaps we observe how the unsystematic mortality risk is diversified when the size of the insurance portfolio is increased. The initial intrinsic risk in the markets without survivor swaps is obviously unaffected by the size of the population, since the insurance payment process is independent of the unsystematic mortality risk of the population. Introducing survivor swaps allow us to hedge some of the mortality risk in addition to the interest rate risk. We observe that more mortality risk can be eliminated in the case with a survivor swap on the portfolio than with the survivor swap on the population. This is due to the fact that the systematic mortality risk more closely resembles

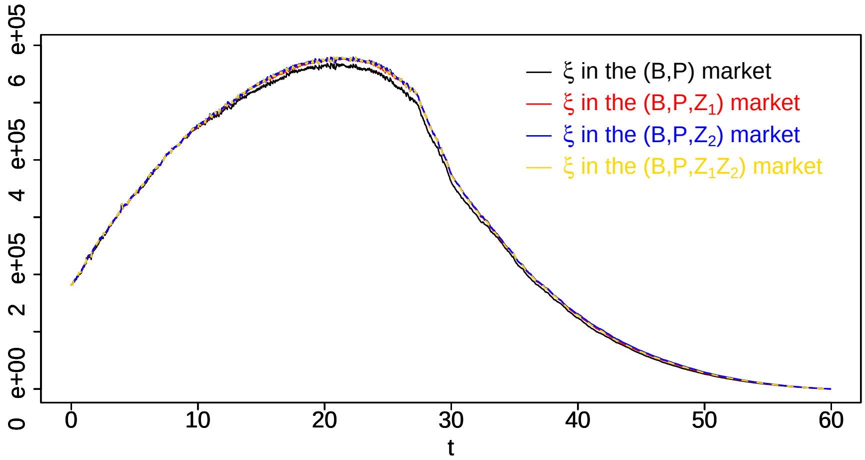

Figure 7: *Number of zero coupon bonds held at time t.*

n_1	n_2	$\frac{\sqrt{R(0,\varphi_V^*)}}{n_1}$	$\frac{\sqrt{R(0,\varphi_B^*)}}{n_1}$	$\frac{\sqrt{R(0,\varphi_1^*)}}{n_1}$	$\frac{\sqrt{R(0,\varphi_2^*)}}{n_1}$	$\frac{\sqrt{R(0,\varphi^*)}}{n_1}$
100	1,000	0.632	0.111	0.048	0.103	0.033
100	10,000	0.633	0.111	0.048	0.100	0.020
1,000	10,000	0.628	0.062	0.032	0.037	0.018
1,000	100,000	0.628	0.062	0.032	0.035	0.015
10,000	100,000	0.622	0.055	0.013	0.020	0.010

Table 4: *Intrinsic risks calculated at time 0.*

the one associated with the insurance payment process and that it allows for hedging the unsystematic mortality risk. In contrast, the survivor swap on the population introduces a new source of unsystematic mortality risk. If we allow for investments in both survivor swaps we obviously obtain an even smaller initial intrinsic risk. Furthermore, we see that the difference between the two strategies with only one survivor swap is largest when the number of individuals underlying the survivor swaps are low, since the unsystematic mortality risks become more important in these cases.

When we have a very large number of lives in both portfolios, we observe that the markets with survivor swaps hedge almost the same amount of risk. It might therefore be better to invest in only one survivor swap instead of both survivor swaps, since one might introduce too much credit risk and administration costs compared to the small advantage of the (B, P, Z_1, Z_2) market.

A Appendix

A.1 Proof of Proposition 5.1

From Lemma 4.3 we have the following dynamics of $V^{*,Q}$:

$$dV^{*,Q}(t) = \nu^{V,Q}(t)dM_1^Q(x,t) + \eta^{V,Q}(t)dW^{r,Q}(t) + \rho^{V,Q}(t)dW^{\mu,Q}(t). \tag{A.1}$$

Furthermore we know from Section 5.2 that the Galtchouk-Kunita-Watanabe decomposition has the form

$$dV^{*,Q}(t) = \xi_1^Q(t)dP^*(t,T) + \vartheta_1^Q(t)dZ_1^{*,Q}(x,t) + dL_1^Q(t), \tag{A.2}$$

where ξ_1^Q and ϑ_1^Q are predictable processes, and L_1^Q is a zero-mean Q-martingale orthogonal to P^* and $Z^{*,Q}$. Since $Y_{1,1}^{*,Q}$ and $Y_{1,2}^{*,Q}$ are orthogonal to $P^*(\cdot,T)$ and $Z_1^{*,Q}$ and the four processes together span the space of Q-martingales driven by the underlying stochastic processes $W^{r,Q}$, $W^{\mu,Q}$ and M_1^Q, we may take L_1^Q of the form

$$dL_1^Q(t) = \phi_1^Q(t)dY_1^{*,Q}(t), \tag{A.3}$$

where ϕ_1^Q is a two-dimensional predictable process, and where

$$dY_{1,j}^{*,Q}(t) = \chi_{1,j}^{Z,Q}(t)\Big(dM_1^Q(x,t) - \kappa_{1,j}^Q(t)dW_j^{\mu,Q}(t)\Big),$$

$j = 1, 2$, are the zero-mean Q-martingales introduced in (5.4). Hence L_1^Q is indeed a zero-mean martingale and orthogonal to P^* and $Z_1^{*,Q}$ as required.

Now, recall that the discounted zero coupon bond price dynamics under Q are given by

$$dP^*(t,T) = -\sigma^r B^r(t,T)P^*(t,T)dW^{r,Q}(t). \tag{A.4}$$

Inserting the representation of the survivor swap in Lemma 3.2 together with (A.3) and (A.4) in (A.2), we rewrite the Galtchouk-Kunita-Watanabe decomposition in terms of the underlying stochastic processes. This leads to the representation

$$\begin{aligned} dV^{*,Q}(t) = {} & \Big(\vartheta_1^Q(t)\nu_1^{Z,Q}(t) + \phi_{1,1}^Q(t)\chi_{1,1}^{Z,Q}(t) + \phi_{1,2}^Q(t)\chi_{1,2}^{Z,Q}(t)\Big)dM_1^Q(x,t) \\ & + \Big(\vartheta_1^Q(t)\eta_1^{Z,Q}(t) - \xi_1^Q(t)\sigma^r B^r(t,T)P^*(t,T)\Big)dW^{r,Q}(t) \\ & + \sum_{j=1}^{2}\Big(\vartheta_1^Q(t)\rho_{1,j}^{Z,Q}(t) - \phi_{1,j}^Q(t)\chi_{1,j}^{Z,Q}(t)\kappa_{1,j}^Q(t)\Big)dW_j^{\mu,Q}(t), \end{aligned} \tag{A.5}$$

which involves the unknown processes $(\xi_1^Q, \vartheta_1^Q, \phi_1^Q)$. We now determine these processes by simply equating the terms in the two decompositions (A.1) and (A.5). Hence, we need to solve the following linear system with four equations and the four unknowns $(\xi_1^Q, \vartheta_1^Q, \phi_{1,1}^Q, \phi_{1,2}^Q)$:

$$\begin{aligned} \nu^{V,Q}(t) &= \vartheta_1^Q(t)\nu_1^{Z,Q}(t) + \phi_{1,1}^Q(t)\chi_{1,1}^{Z,Q}(t) + \phi_{1,2}^Q(t)\chi_{1,2}^{Z,Q}(t), \\ \eta^{V,Q}(t) &= \vartheta_1^Q(t)\eta_1^{Z,Q}(t) - \xi_1^Q(t)\sigma^r B^r(t,T)P^*(t,T), \\ \rho_1^{V,Q}(t) &= \vartheta_1^Q(t)\rho_{1,1}^{Z,Q}(t) - \phi_{1,1}^Q\chi_{1,1}^{Z,Q}(t)\kappa_{1,1}^Q(t), \\ \rho_2^{V,Q}(t) &= \vartheta_1^Q(t)\rho_{1,2}^{Z,Q}(t) - \phi_{1,2}^Q\chi_{1,2}^{Z,Q}(t)\kappa_{1,2}^Q(t). \end{aligned}$$

Here, the second equation determines $\xi_1^Q(t)$ in terms of $\vartheta_1^Q(t)$, and this gives (5.7). From the last two equations, we determine $\phi_{1,1}^Q$ and $\phi_{1,2}^Q$ in terms of ϑ_1^Q provided that $\rho_{1,1}^{Z,Q}$ and $\rho_{1,2}^{Z,Q}$ are not equal to 0, such that we can devide by $\kappa_{1,1}^Q$ and $\kappa_{1,2}^Q$. In this case, we get

$$\phi_{1,j}^Q(t)\chi_{1,j}^{Z,Q}(t) = \frac{\vartheta_1^Q(t)\rho_{1,j}^{Z,Q}(t) - \rho_j^{V,Q}(t)}{\kappa_{1,j}^Q(t)}\chi_{1,j}^{Z,Q}(t),$$

$j = 1,2$. By inserting thse expressions in the first line, we get the equation

$$\begin{aligned}\nu^{V,Q}(t) &= \vartheta_1^Q(t)\Big(\nu_1^{V,Q}(t) + \frac{\rho_{1,1}^{Z,Q}(t)}{\kappa_{1,1}^Q(t)}\chi_{1,1}^{Z,Q}(t) + \frac{\rho_{1,2}^{Z,Q}(t)}{\kappa_{1,2}^Q(t)}\chi_{1,2}^{Z,Q}(t)\Big)\\ &\quad - \frac{\rho_1^{V,Q}(t)}{\kappa_{1,1}^Q(t)}\chi_{1,1}^{Z,Q}(t) - \frac{\rho_2^{V,Q}(t)}{\kappa_{1,2}^Q(t)}\chi_{1,2}^{Z,Q}(t).\end{aligned}$$

This now gives (5.8). The expression expression for L_1^Q follows e.g. by inserting φ_1^Q in (A.3) and writing $Y_1^{*,Q}$ in terms of the underlying Q-martingales.

$\square$

A.2 Proof of Proposition 5.3

Define processes $Y_{3,1}^*$ and $Y_{3,2}^*$ by

$$\begin{aligned}dY_{3,1}^*(t) &= \chi_{1,1}^{Z,Q}(t)\Big(dM_1^Q(x,t) + \alpha_1(t)dM_2^Q(x,t) - \kappa_{1,1}^Q(t)dW_1^{\mu,Q}(t)\Big),\\ dY_{3,2}^*(t) &= \chi_{1,2}^{Z,Q}(t)\Big(dM_1^Q(x,t) + \alpha_2(t)dM_2^Q(x,t) - \kappa_{1,2}^Q(t)dW_2^{\mu,Q}(t)\Big),\end{aligned}$$

and $Y_{3,j}^*(0) = 0$, $j = 1,2$. Clearly, these processes are orthogonal to $P^*(\cdot,T)$ and Z_1^*. Now we determine α_1 and α_2 such that $Y_{3,j}^*$ and Z_2^* are orthogonal. Since

$$d\langle Y_{3,j}^*, Z_2^*\rangle(t) = \chi_{1,j}^{Z,Q}(t)\Big(\alpha_j(t)\nu_2^{Z,Q}(t)\lambda_2^Q(t) - \kappa_{1,j}^Q(t)\rho_{2,j}^{Z,Q}(t)\Big)dt, \tag{A.6}$$

we get that

$$\alpha_j(t) = \frac{\kappa_{1,j}^Q(t)\rho_{2,j}^{Z,Q}(t)}{\nu_2^{Z,Q}(t)\lambda_2^Q(t)} = \frac{\nu_1^{Z,Q}(t)\lambda_1^Q(t)\rho_{2,j}^{Z,Q}(t)}{\nu_2^{Z,Q}(t)\lambda_2^Q(t)\rho_{1,j}^{Z,Q}(t)}, \tag{A.7}$$

$j = 1,2$, which are well-defined for $\rho_{1,j}^{Z,Q}(t) \neq 0$. We need to rewrite the decomposition

$$dV^{*,Q}(t) = \nu^{V,Q}(t)dM_1^Q(x,t) + \eta^{V,Q}(t)dW^{r,Q}(t) + \rho^{V,Q}(t)dW^{\mu,Q}(t) \tag{A.8}$$

on the form

$$dV^{*,Q}(t) = \xi_3^Q(t)dP^*(t,T) + \vartheta_{3,1}^Q(t)dZ_1^{*,Q}(x,t) + \vartheta_{3,2}^Q(t)dZ_2^{*,Q}(x,t) + dL_3^Q(t), \tag{A.9}$$

where L_3^Q is a Q-martingale, which is orthogonal to the traded processes $P^*(\cdot,T)$, Z_1^* and Z_2^*. Here, we take L_3^Q on the form:

$$dL_3^Q(t) = \phi_{3,1}^Q(t)dY_{3,1}^*(t) + \phi_{3,2}^Q(t)dY_{3,2}^*(t).$$

We need to determine the processes $(\xi_3^Q, \vartheta_{3,1}^Q, \vartheta_{3,2}^Q, \phi_{3,1}^Q, \phi_{3,2}^Q)$. We insert the dynamics for the traded processes in (A.9) and collect terms to obtain that

$$\begin{aligned}
dV^*(t) &= \xi_3^Q(t)\Big(-\sigma^r B^r(t,T)P^*(t,T)\Big)dW^{r,Q}(t) \\
&\quad + \sum_{j=1}^{2} \vartheta_{3,j}^Q(t)\Big(\nu_j^{Z,Q}(t)dM_j^Q(x,t) + \eta_j^{Z,Q}(t)dW^{r,Q}(t) + \sum_{i=1}^{2}\rho_{j,i}^{Z,Q}(t)dW_i^{\mu,Q}(t)\Big) \\
&\quad + \sum_{j=1}^{2} \phi_{3,j}^Q(t)\chi_{1,j}^{Z,Q}(t)\Big(dM_1^Q(x,t) + \alpha_j(t)dM_2^Q(x,t) - \kappa_{1,j}^Q dW_j^{\mu,Q}(t)\Big).
\end{aligned}$$

We now collect the terms from the driving process and rewrite dV^* on the form

$$\begin{aligned}
dV^*(t) &= \Big(\xi_3^Q(t)(-\sigma^r B^r(t,T)P^*(t,T)) + \vartheta_{3,1}^Q(t)\eta_1^Z(t) + \vartheta_{3,2}^Q(t)\eta_2^Z(t)\Big)dW^r(t) \\
&\quad + \Big(\vartheta_{3,1}^Q(t)\nu_1^{Z,Q}(t) + \phi_{3,1}^Q(t)\chi_{1,1}^{Z,Q}(t) + \phi_{3,2}^Q(t)\chi_{1,2}^{Z,Q}(t)\Big)dM_1^Q(x,t) \\
&\quad + \Big(\vartheta_{3,2}^Q(t)\nu_2^{Z,Q}(t) + \phi_{3,1}^Q(t)\alpha_1(t)\chi_{1,1}^{Z,Q}(t) + \phi_{3,2}^Q(t)\alpha_2(t)\chi_{1,2}^{Z,Q}(t)\Big)dM_2^Q(x,t) \\
&\quad + \sum_{j=1}^{2}\Big(\vartheta_{3,1}^Q(t)\rho_{1,j}^{Z,Q}(t) + \vartheta_{3,2}^Q(t)\rho_{2,j}^{Z,Q}(t) - \phi_{3,1}^Q(t)\chi_{1,j}^{Z,Q}(t)\kappa_{1,j}^Q(t)\Big)dW_j^{\mu,Q}(t).
\end{aligned}$$

We now identify the processes with the ones appearing in the original decomposition (A.8) to obtain the system

$$\eta^{V,Q}(t) = \xi_3^Q(t)(-\sigma^r B^r(t,T)P^*(t,T)) + \vartheta_{3,1}^Q(t)\eta_1^Z(t) + \vartheta_{3,2}^Q(t)\eta_2^Z(t), \tag{A.10}$$

$$\nu^{V,Q}(t) = \vartheta_{3,1}^Q(t)\nu_1^{Z,Q}(t) + \phi_{3,1}^Q(t)\chi_{1,1}^{Z,Q}(t) + \phi_{3,2}^Q(t)\chi_{1,2}^{Z,Q}(t), \tag{A.11}$$

$$0 = \vartheta_{3,2}^Q(t)\nu_2^{Z,Q}(t) + \phi_{3,1}^Q(t)\alpha_1(t)\chi_{1,1}^{Z,Q}(t) + \phi_{3,2}^Q(t)\alpha_2(t)\chi_{1,2}^{Z,Q}(t), \tag{A.12}$$

$$\rho_1^{V,Q}(t) = \vartheta_{3,1}^Q(t)\rho_{1,1}^{Z,Q}(t) + \vartheta_{3,2}^Q(t)\rho_{2,1}^{Z,Q}(t) - \phi_{3,1}^Q(t)\chi_{1,1}^{Z,Q}(t)\kappa_{1,1}^Q(t), \tag{A.13}$$

$$\rho_2^{V,Q}(t) = \vartheta_{3,1}^Q(t)\rho_{1,2}^{Z,Q}(t) + \vartheta_{3,2}^Q(t)\rho_{2,2}^{Z,Q}(t) - \phi_{3,2}^Q(t)\chi_{1,2}^{Z,Q}(t)\kappa_{1,2}^Q(t). \tag{A.14}$$

Using (A.10), we first see that

$$\xi_3^Q(t) = \frac{\eta^{V,Q}(t) - \vartheta_{3,1}^Q(t)\eta_1^Z(t) - \vartheta_{3,2}^Q(t)\eta_2^Z(t)}{-\sigma^r B^r(t,T)P^*(t,T))}. \tag{A.15}$$

Next, we determine the parameters $\phi_{3,1}$ and $\phi_{3,2}$ in terms of $\vartheta_{3,1}^Q(t)$ and $\vartheta_{3,2}^Q(t)$ from (A.13) and (A.14). This leads to

$$\begin{aligned}
\phi_{3,1}^Q(t) &= \frac{1}{\kappa_{1,1}^Q(t)}\Big(\vartheta_{3,1}^Q(t)\rho_{1,1}^{Z,Q}(t) + \vartheta_{3,2}^Q(t)\rho_{2,1}^{Z,Q}(t) - \rho_1^{V,Q}(t)\Big) \\
&= \vartheta_{3,1}^Q(t)\frac{\rho_{1,1}^{Z,Q}(t)}{\kappa_{1,1}^Q(t)} + \vartheta_{3,2}^Q(t)\frac{\rho_{2,1}^{Z,Q}(t)}{\kappa_{1,1}^Q(t)} - \frac{\rho_1^{V,Q}(t)}{\kappa_{1,1}^Q(t)},
\end{aligned}$$

for $\rho_{1,1}^{Z,Q}(t) \neq 0$, and

$$\phi_{3,2}^Q(t) = \vartheta_{3,1}^Q(t)\frac{\rho_{1,2}^{Z,Q}(t)}{\kappa_{1,2}^Q(t)} + \vartheta_{3,2}^Q(t)\frac{\rho_{2,2}^{Z,Q}(t)}{\kappa_{1,2}^Q(t)} - \frac{\rho_2^{V,Q}(t)}{\kappa_{1,2}^Q(t)},$$

for $\rho_{1,2}^{Z,Q}(t) \neq 0$. Now, these terms may be inserted in equations (A.11) and (A.12). In this way, we get the two equations

$$
\begin{aligned}
&\nu^{V,Q}(t) + \frac{\rho_1^{V,Q}(t)}{\kappa_{1,1}^Q(t)}\chi_{1,1}^{Z,Q}(t) + \frac{\rho_2^{V,Q}(t)}{\kappa_{1,2}^Q(t)}\chi_{1,2}^{Z,Q}(t) \\
&= \vartheta_{3,1}^Q(t)\Big(\nu_1^{Z,Q}(t) + \frac{\rho_{1,1}^{Z,Q}(t)}{\kappa_{1,1}^Q(t)}\chi_{1,1}^{Z,Q}(t) + \frac{\rho_{1,2}^{Z,Q}(t)}{\kappa_{1,2}^Q(t)}\chi_{1,2}^{Z,Q}(t)\Big) \\
&\quad + \vartheta_{3,2}^Q(t)\Big(\frac{\rho_{2,1}^{Z,Q}(t)}{\kappa_{1,1}^Q(t)}\chi_{1,1}^{Z,Q}(t) + \frac{\rho_{2,2}^{Z,Q}(t)}{\kappa_{1,2}^Q(t)}\chi_{1,2}^{Z,Q}(t)\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\alpha_1(t)\rho_1^{V,Q}(t)}{\kappa_{1,1}^Q(t)}\chi_{1,1}^{Z,Q}(t) + \frac{\alpha_2(t)\rho_2^{V,Q}(t)}{\kappa_{1,2}^Q(t)}\chi_{1,2}^{Z,Q}(t) \\
&= \vartheta_{3,1}^Q(t)\Big(\frac{\alpha_1(t)\rho_{1,1}^{Z,Q}(t)}{\kappa_{1,1}^Q(t)}\chi_{1,1}^{Z,Q}(t) + \frac{\alpha_2(t)\rho_{1,2}^{Z,Q}(t)}{\kappa_{1,2}^Q(t)}\chi_{1,2}^{Z,Q}(t)\Big) \\
&\quad + \vartheta_{3,2}^Q(t)\Big(\nu_2^{Z,Q}(t) + \frac{\alpha_1(t)\rho_{2,1}^{Z,Q}(t)}{\kappa_{1,1}^Q(t)}\chi_{1,1}^{Z,Q}(t) + \frac{\alpha_2(t)\rho_{2,2}^{Z,Q}(t)}{\kappa_{1,2}^Q(t)}\chi_{1,2}^{Z,Q}(t)\Big).
\end{aligned}
$$

Here, we furthermore note that $\alpha_1(t)/\kappa_{1,1}^Q(t) = 1/\kappa_{2,1}^Q(t)$ and $\alpha_2(t)/\kappa_{1,2}^Q(t) = 1/\kappa_{2,2}^Q(t)$. By solving these equations, we get

$$
\vartheta_{3,1}^Q(t) = \frac{\nu^{V,Q}(t) + \frac{\rho_1^{V,Q}(t)}{\kappa_{1,1}^Q(t)}\chi_{1,1}^{Z,Q}(t) + \frac{\rho_2^{V,Q}(t)}{\kappa_{1,2}^Q(t)}\chi_{1,2}^{Z,Q}(t) - \left(\frac{\rho_{2,1}^{Z,Q}(t)}{\kappa_{1,1}^Q(t)}\chi_{1,1}^{Z,Q}(t) + \frac{\rho_{2,2}^{Z,Q}(t)}{\kappa_{1,2}^Q(t)}\chi_{1,2}^{Z,Q}(t)\right)\vartheta_2^Q(t)}{\nu_1^{Z,Q}(t) + \frac{\rho_{1,1}^{Z,Q}(t)}{\kappa_{1,1}^Q(t)}\chi_{1,1}^{Z,Q}(t) + \frac{\rho_{1,2}^{Z,Q}(t)}{\kappa_{1,2}^Q(t)}\chi_{1,2}^{Z,Q}(t) - \left(\frac{\rho_{2,1}^{Z,Q}(t)}{\kappa_{1,1}^Q(t)}\chi_{1,1}^{Z,Q}(t) + \frac{\rho_{2,2}^{Z,Q}(t)}{\kappa_{1,2}^Q(t)}\chi_{1,2}^{Z,Q}(t)\right)\widetilde{\vartheta}_2^Q(t)},
$$

$$
\vartheta_{3,2}^Q(t) = \frac{\frac{\rho_1^{V,Q}(t)}{\kappa_{2,1}^Q(t)}\chi_{2,1}^{Z,Q}(t) + \frac{\rho_2^{V,Q}(t)}{\kappa_{2,2}^Q(t)}\chi_{2,2}^{Z,Q}(t) - \vartheta_{3,1}^Q(t)\left(\frac{\rho_{1,1}^{Z,Q}(t)}{\kappa_{2,1}^Q(t)}\chi_{2,1}^{Z,Q}(t) + \frac{\rho_{1,2}^{Z,Q}(t)}{\kappa_{2,2}^Q(t)}\chi_{2,2}^{Z,Q}(t)\right)}{\nu_2^{Z,Q}(t) + \frac{\rho_{2,1}^{Z,Q}(t)}{\kappa_{2,1}^Q(t)}\chi_{2,1}^{Z,Q}(t) + \frac{\rho_{2,2}^{Z,Q}(t)}{\kappa_{2,2}^Q(t)}\chi_{2,2}^{Z,Q}(t)},
$$

where

$$
\begin{aligned}
\vartheta_2^Q(t) &= \frac{\rho_1^{V,Q}(t)(\kappa_{2,1}^Q(t))^{-1}\chi_{2,1}^{Z,Q}(t) + \rho_2^{V,Q}(t)(\kappa_{2,2}^Q(t))^{-1}\chi_{2,2}^{Z,Q}(t)}{\nu_2^{Z,Q}(t) + \rho_{2,1}^{Z,Q}(t)(\kappa_{2,1}^Q(t))^{-1}\chi_{2,1}^{Z,Q}(t) + \rho_{2,2}^{Z,Q}(t)(\kappa_{2,2}^Q(t))^{-1}\chi_{2,2}^{Z,Q}(t)}, \\
\widetilde{\vartheta}_2^Q(t) &= \frac{\rho_{1,1}^{Z,Q}(t)(\kappa_{2,1}^Q(t))^{-1}\chi_{2,1}^{Z,Q}(t) + \rho_{1,2}^{Z,Q}(t)(\kappa_{2,2}^Q(t))^{-1}\chi_{2,2}^{Z,Q}(t)}{\nu_2^{Z,Q}(t) + \rho_{2,1}^{Z,Q}(t)(\kappa_{2,1}^Q(t))^{-1}\chi_{2,1}^{Z,Q}(t) + \rho_{2,2}^{Z,Q}(t)(\kappa_{2,2}^Q(t))^{-1}\chi_{2,2}^{Z,Q}(t)}.
\end{aligned}
$$

We complete the proof by rewriting the orthogonal term. First, using the definition of $Y_{3,j}^*$, we see that

$$
\begin{aligned}
dL_3(t) &= \phi_{3,1}^Q(t)dY_{3,1}^*(t) + \phi_{3,2}^Q(t)dY_{3,2}^*(t) \\
&= -\sum_{j=1}^{2}\phi_{3,j}^Q(t)\chi_{1,j}^{Z,Q}(t)\kappa_{1,j}^Q(t)dW_j^{\mu,Q}(t) + \Big(\sum_{j=1}^{2}\phi_{3,j}^Q(t)\chi_{1,j}^{Z,Q}(t)\Big)dM_1^Q(x,t) \\
&\quad + \Big(\sum_{j=1}^{2}\phi_{3,j}^Q(t)\chi_{1,j}^{Z,Q}(t)\alpha_j(t)\Big)dM_2^Q(x,t).
\end{aligned}
$$

Using (A.10)–(A.14), we finally see that

$$
\begin{aligned}
dL_3(t) = &\sum_{j=1}^{2} (\rho_j^{V,Q}(t) - \vartheta_{3,1}^{Q}(t)\rho_{1,j}^{Z}(t) - \vartheta_{3,2}^{Q}(t)\rho_{2,j}^{Z}(t)) dW_j^{\mu,Q}(t) \\
&+ (\nu^{V,Q}(t) - \vartheta_{3,1}^{Q}(t)\nu_1^{Z,Q}(t)) dM_1^{Q}(x,t) - \vartheta_{3,2}^{Q}(t)\nu_2^{Z,Q}(t) dM_2^{Q}(x,t).
\end{aligned}
$$

This completes the proof.

□

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