# An Arbitrage-Free Family of Longevity Bonds*

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**Abstract**

Stochastic mortality modeling has recently attracted considerable interest in academia as well as among practitioners. Different methods have been proposed on how to define and model the involved quantities, in particular for the stochastic force of mortality. Most of these approaches use the credit risk modeling framework in order to define stochastic mortalities. We show how stochastic mortality can be define via an exogenously given longevity bond market.

Besides providing the necessary definitions and propositions, we will give insights how continuous stochastic mortality models generally could be categorized, and assess under which conditions the longevity bond market is arbitrage-free. Furthermore, we discuss one modeling approach, namely the forward force of mortality modeling framework in more detail, and show how it can be applied to price advanced mortality derivatives.

Keywords: stochastic mortality, longevity bond, bond market, forward force of mortality, HJM-framework.

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# 1 Introduction

Stochastic mortality modeling has attracted considerable interest in academia as well as among practitioners. This is mainly due to the insight, that the evolution of mortality trends over time is not predictable, but that mortality rather evolves in a stochastic fashion (see for example Currie et al. (2004) or Macdonald et al. (1998)). Turner (2005) even concludes that there is no real basis for making mathematical precise estimates of confidence ranges for *longevity risk*. Of course, this presents a considerable risk factor to annuity providers and corporates with big pension schemes, since their underlying estimates may well be wrong and, thus, their reserves may not be sufficient to cover their future liabilities.

The classical way to deal with this type of risk is reinsurance. However, reinsurers' appetite for this longevity risk is limited, since it is a systematic type of risk and cannot be diversified away. Thus, new methods have been proposed on how to manage this risk. One of the most prominent alternatives is *securitization*, i.e. "isolating the cash flows that are linked to longevity risk and repackaging them into cash flows that are traded in capital markets" (see Cowley and Cummins (2005)). Several instruments have been proposed by the academic literature, such as longevity or survivor bonds (see Blake and Burrows (2001)) in various forms (see, e.g., Blake et al. (2006) or Lin and Cox (2005)) or survivor swaps (see Dowd et al. (2006)).

However, so far there have only been few issues. There have been some bonds dependent on the mortality of a certain population (SWISS Re Vita I and Vita II[1]) or asset backed type securities indemnifying closed books of life insurance policies (Swiss Re Queensgate, Alps II), however, these securities are dealing with catastrophic mortality risk rather than longevity risk. So far, no instrument has been issued which is suitable for hedging longevity risk, although in 2004, there has been an announcement for the issue of a longevity bond by the European Investment Bank (EIB) together with BNP Paribas and Partner Re. There is no accordance why the issue was not successful, but as a fact, the demand was rather limited. Possible reasons could be that the payoff structure was inadequate or the price was percepted as too high. Besides these public deals, there supposably was a number of over-the-counter deals, however we are not aware of any details.

All in all, this could lead to the conclusion that mortality modeling is an academic problem rather than a practical one, since there is no market, and maybe not even the demand for a mortality derivatives market. But there is another possible answer why longevity bonds or similar instruments have not been successfully offered yet, namely that there are no standard methodologies for modeling and valuating mortality derivatives. For example the credit risk derivative market, i.e. the market of structured products on defaultable bonds, took off after the technology to estimate credit risk had reached a level adequate to sustain a market (see Dowd et al. (2006)). This is also indicated by a *market price of mortality puzzle*: most authors and practitioners seem to agree that there is or should be a risk premium for systematic mortality risk, however, there is no accordance of its form or size.

Thus, the search for adequate methodologies for mortality modeling and, maybe even more importantly, for pricing mortality derivatives, may not be void after all, but trigger the market to be born and, eventually, to grow. Several methods have been proposed on how to price mortality derivatives (see, e.g., Cairns et al. (2005a), Cairns et al. (2005b), Dahl (2004), Lin and Cox (2005), or Milevsky et al. (2005)), however there have only been few basic contributions on the classification and categorization of continuous stochastic mortality models for pricing mortality

[1] Short term bonds offering a certain spread over LIBOR (about 90-190 bps) but not paying back the full principal in case of an extreme event (e.g. an epidemic similar to the Spanish Flue in 1918).

derivatives.

Cairns et al. (2005b) offer a pricing framework for the securitization of mortality risk. They define the $(T, x_0)$-Bond, i.e. a security paying out the financial equivalent of the proportion of survivors of a cohort of $x_0$-year olds at time 0 to time $T$, as their basic instrument and derive the definitions for quantities such as for example the spot force of mortality or the forward force of mortality using these basic longevity bonds. Furthermore, they discuss the consequences of these definitions on mortality modeling. In particular, they propose four types of continuous mortality models: models of the spot force of mortality, forward mortality models, a positive mortality framework, and mortality market models. The latter can be interpreted as the mortality modeling equivalent to the well-known LIBOR market models from interest rate modeling (see, e.g., Bingham and Kiesel (2003)).

However, in this framework, Cairns et al. (2005b) assume that the dynamics of the term structure of mortality and the dynamics term structure of interest are independent. Even though the death of an individual policy holder can be considered independent of the development of the financial markets, there is evidence that the force of mortality in the future is to a large extent dependent on the development of the economy in general, and thereby also of the financial markets (cf. Miltersen and Persson (2005)). Furthermore, even if the development of future mortality and of the financial markets was uncorrelated under the physical probability measure, there still might be some dependency under the pricing measure. For example, it is a widely accepted fact that the current increasing attention to stochastic mortality modeling is mainly due to the low interest rates. When rates are low, the value of insurers' liabilities increase and, thus, insurers are generally more concerned about their future liabilities (see Dowd et al. (2006)). Also, annuity contracts often include mortality dependent options such as Guaranteed Annuity Options (GAO) or Guaranteed Minimum Income Benefits (GMIB) which may be *in the money* and therefore exercised more often in a low interest environment, inducing a stronger affection of insurers to longevity risk when rates are low.

Therefore, we believe that this independence assumption of the term structure of mortality and the term structure of interest rates is at least questionable. The current paper offers a pricing framework without this assumption and generalizes the approach of Cairns et al. (2005b) in various other ways. For example, we also allow for multiple inception dates of the longevity bonds.

The remainder is organized as follows. Section 2 introduces the basic definitions and properties. In Section 3, we show how survival probabilities can be defined similar to interest rates in fixed income modeling. Here, we follow Bjørk (1999) and carry over his results for the fixed income market to the mortality derivatives market. Furthermore, we explain how our results can be employed for mortality modeling. Although in Section 3 we limit our considerations to a single inception date of longevity bonds, namely 0, in Section 4, we explain how the definitions can be generalized toward multiple inception dates, and we provide conditions that need to be satisfied in order for the market to be arbitrage-free. In Section 5, one of the approaches proposed in 3, namely the modeling of the forward force of mortality, is considered in more detail, since this framework comprises the other modeling approaches. We recover a restriction on the drift term of the models as originally found by Miltersen and Persson (2005), which basically presents the well-known Heath-Jarrow-Morton drift condition for mortality modeling. In 6, we show how the modeling framework can be employed for pricing more complex mortality derivatives. Finally, Section 7 summarizes the main results and gives an outlook on future research.

## 2 Basic Definitions

Following and generalizing the ideas of Cairns et al. (2005b), we define a $(\tau, T, x_\tau)$-Bond as a financial security paying out ${}_{T-\tau}p_{x_\tau}^{(T)}$ at time $T$. Note that there seems to be a redundancy in the notation, since we have $\tau$ and $x_\tau$ in the definition. However, in general $x_\tau$ could be any age, or, in other words, the bond could be defined for any cohort of $x_\tau$ year old at time $\tau$. For the sake of simplicity, we will generally only consider one cohort, i.e. we fix a cohort of $x_0$ year olds at time 0 for our considerations. In particular, this leads to

$$x_\tau = x_0 + \tau. \tag{1}$$

We will come back to the general case in the outlook on future research (Section 7). In the above definition, ${}_{T-\tau}p_{x_\tau}^{(T)}$ denotes the (realized) proportion of the population still alive at time $\tau$ and $x_\tau$ years old at time $\tau$, who are still alive at time $T$, i.e. who survived $(T - \tau)$ periods from time $\tau$ to time $T$. More general, ${}_\mu p_{x_{\nu-\mu}}^{(\nu)}$ for $\nu \geq \mu$ denotes the proportion of *the* population alive at time $(\nu - \mu)$ and $x_{\nu-\mu}$ years old at time $(\nu - \mu)$ who survive until time $\nu$, i.e. the realized survival probability from time $(\nu - \mu)$ to time $\nu$ of $x_{\nu-\mu}$ year olds at time $(\nu - \mu)$.

We denote the time $t$ value of a $(\tau, T, x_\tau)$-Bond by $\Pi_t(\tau, T, x_\tau)$, where $t \leq T$. The above is summarized by the following definitions:

**Definition 2.1** *By* ${}_\mu p_{x_{\nu-\mu}}^{(\nu)}$ *we denote the realized survival probability from time* $(\nu - \mu)$ *to time* $\nu$ *of* $x_{\nu-\mu}$ *year olds at time* $(\nu - \mu)$*.*

**Definition 2.2** *A* $(\tau, T, x_\tau)$*-Bond is a financial security paying* ${}_{T-\tau}p_{x_\tau}^{(T)}$ *at time* $T$*. We denote the time* $t$ *price of this bond by* $\Pi_t(\tau, T, x_\tau)$*,* $t \leq T$*.*

This immediately leads to the property

$$\Pi_T(\tau, T, x_\tau) \;=\; {}_{T-\tau}p_{x_\tau}^{(T)}$$

Furthermore, assuming a non-deflational environment, since ${}_{T-\tau}p_{x_\tau}^{(T)}$ are proportions, we obtain

$$\Pi_t(\tau, T, x_\tau) \leq 1 \text{ and } \Pi_t(\tau, T, x_\tau) < 1 \text{ if } T > \tau,\ t \leq T.$$

Besides, we have ${}_{T-\tau}p_{x_\tau}^{(T)} \to 1$ for $\tau \to T-$,[2] and therefore

$$\Pi_\tau(\tau, T, x_\tau) \to 1 \text{ for } \tau \to T- . \tag{2}$$

In order to define and analyze the market of longevity bonds, we assume the existence of a *financial* bond market:

**Assumption 2.1** *We assume the existence of an arbitrage-free, frictionless bond market as, e.g., introduced in Bjørk (1999), Chapter 20.*

Thus, the value of a payoff of 1 at time $T$ is worth $p(t, T)$ at time $t \leq T$, where $p(t, T)$ denotes the time $t$-price of a zero-coupon bond with maturity $T$. Thus, property (2) implies

$$\Pi_t(\tau, T, x_\tau) \to p(t, T) \text{ for } \tau \to T- .$$

Note that in comparison to the financial bond (FB in what follows), the longevity bonds (LB in what follows) are characterized by one major difference: Here the

[2] By $a \to b-$, we denote that $a$ is approaching $b$ from the left, i.e. $a \to b$ with $a \leq b$. $a \to b+$ is defined analogously.

payoffs are not deterministic but stochastic, since ${}_{T-\tau}p_{x_\tau}^{(T)}$ is not known prior to time T.[3] However, there are also some differences which motivate the definition of the mortality linked securities as bonds. We will come back to this issue later in this section.

The following lemma summarizes the results from above and brings together some more basic properties of longevity bonds.

**Lemma 2.1** *1.* $\Pi_T(\tau, T, x_\tau) = {}_{T-\tau}p_{x_\tau}^{(T)}$.

*2.* $\Pi_t(\tau, T, x_\tau) \leq 1$ *and* $\Pi_t(\tau, T, x_\tau) < 1$ *if* $T > \tau,\ t \leq T$.

*3.* $\Pi_\tau(\tau, T, x_\tau) \to 1$ *for* $\tau \to T-$.

*4.* $\Pi_t(\tau, T, x_\tau) \to p(t, T)$ *for* $\tau \to T-$.

*5.* $\Pi_t(\tau_1, T, x_{\tau_1}) \leq \Pi_t(\tau_2, T, x_{\tau_2}),\ \tau_1 \leq \tau_2 \leq T,\ t \leq T$.

*6.* $\Pi_t(\tau, T_1, x_\tau) \geq \Pi_t(\tau, T_2, x_\tau),\ \tau \leq T_1 \leq T_2,\ t \leq T_1$.

*proof:* 1. - 4. : see above.

5. : At time $T$, we have ${}_{T-\tau_1}p_{x_{\tau_1}}^{(T)} = {}_{\tau_2-\tau_1}p_{x_{\tau_1}}^{(\tau_2)}\ {}_{T-\tau_2}p_{x_{\tau_2}}^{(T)}$, where the first factor on the right is smaller or equal to one due to 2. Thus, the payoff of a $(\tau_2, T, x_{\tau_2})$-Bond exceeds the payoff of a $(\tau_1, T, x_{\tau_1})$-Bond in any case, implying that the value is smaller.

6. : At time $T_2$, we have ${}_{T_2-\tau}p_{x_\tau}^{(T_2)} = {}_{T_1-\tau}p_{x_\tau}^{(T_1)}\ {}_{T_2-T_1}p_{x_{T_1}}^{(T_2)}$ – with similar arguments as in 5., the claim follows.

□

Note that properties 5. and 6. implicitly depend on the fact that there is only one cohort, i.e. that (1) holds. Furthermore, our "proofs" partially rely on our interpretation rather than mathematical axioms. However, since distinguishing between a proof and a natural interpretation is a technicality rather, we will continue not distinguishing "mathematical" from "interpretation" proofs.

Our target is to define and asses the properties of an *arbitrage-free family of longevity bonds*. We will follow Bjørk (1999), Chapter 20, who in contrast to the discussion here considered financial bonds and not longevity bonds. Similar introductions can be found in many other textbooks, as for example Bingham and Kiesel (2003) or Musiela and Rutkowski (1997). In order to guarantee the existence of a sufficiently rich and regular bond market, we make the following assumption:

**Assumption 2.2** *1. There exists a frictionless market for $(\tau, T, x_\tau)$-Bonds for every $0 \leq \tau \leq T$.*[4]

*2. For each fixed t and $\tau$, the bond price $\Pi_t(\tau, T, x_\tau)$ is differentiable with respect to $T$.*

We will refer to $T$ as the maturity of a $(T, \tau, x_\tau)$-Bond and to $\tau$ as the inception of a $(T, \tau, x_\tau)$-Bond. The bond price is thus a stochastic object in three variables

[3] Actually, even the assumption that ${}_{T-\tau}p_{x_\tau}^{(T)}$ is known at time $T$ is quite questionable, since it has to be assessed statistically. However, this is a practical issue rather, and thus we will not address it here.

[4] Note that this assumption can be generalized toward many age groups, i.e. for every $x_\tau$. Since we only consider one cohort at this point, $x_\tau$ is determined by $\tau$, cf. (1).

$t, \tau, T$, and for each outcome in the underlying sample space[5], the dependence upon these variables is different.

We thus see the similarities of the LB-market to the FB-market: Here also, there exists an infinite number of assets (one bond for each inception $\tau$ and maturity $T$). Our goal is therefore to relate the LB-market to the FB-market, which offers means to investigate the relations between the bonds. Similarly to Bjørk (1999), we pose the following general problems to be studied:

- What is a reasonable model for the LB-market?
- Which relations must hold between the prices respectively price processes for bonds of different inceptions and maturities in order to guarantee an arbitrage free bond market?

Before we continue assessing these problems, we want to point out that for a given fixed time $t$ and $\tau = t$, $\frac{\Pi_t(t,T,x_t)}{p(t,T)}$ contains the "market expectation toward survivor rates", i.e. how the market expects ${}_{T-t}p_{x_t}^{(T)}$ at time $T$. Thus, the function $S(T) = \frac{\Pi_t(t,T,x_t)}{p(t,T)}$ is related to the so-called survivor function

$$P\left(\text{"time of death of } x_t \text{ year old at } t\text{"} \geq T \mid \text{"alive at time } t\text{"}\right).$$

We will further investigate this relationship in the next section.

# 3 Survival Probabilities and Interest Rates

In this section, we will restrict our considerations to only one inception date of longevity bonds, namely 0. Thus, similar to Cairns et al. (2005b), we will consider only $(0, T, x_0)$-Bonds for different maturities which Cairns et al. (2005b) simply refer to as $(T, x_0)$-Bonds.[6]

As noted earlier, we take the existence of an arbitrage-free (financial) bond market for granted. Thus, we use the standard notations as for example introduced in Bjørk (1999) without properly defining them.

In the following, we will define various survival probabilities. It is important to note that these may not equal the best-estimate or other survival probabilities used in actuarial practice, since they are defined via bond prices, i.e. prices of financial instruments rather than historic time series and therefore could contain adjustments or risk premiums. This is similar to credit risk modeling, where risk-neutral default probabilities are derived from market prices, which, in general, do not equal the "real" or "physical" default probabilities (For an introduction to credit risk modeling, see, e.g., Schoenbucher (2003)).

**Definition 3.1** *We define*

$$ {}_{T-t}\tilde{p}_{x_t}^{(t):t\to T} := \frac{\Pi_t\left(0,T,x_0\right)}{\Pi_t\left(0,t,x_0\right)\, P(t,T)} = \frac{\Pi_t\left(0,T,x_0\right)}{{}_tp_{x_0}^{(t)}\, P(t,T)}$$

*as the spot survival probability at $t$ for $(T-t)$ years.*

---

[5] As usual, we fix a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\mathbf{F} = \mathcal{F}_{t\geq 0}$, where $\mathbf{F}$ satisfies the usual conditions (cf. Karatzas and Shreve (2000), Definition 1.1.2.25) for our considerations.

[6] Thus, we will come to similar results as Cairns et al. (2005b). However, in contrast to their study, we are not assuming independence of financial and biometric events.

We choose this rather complicated and unhandy notation, so that generalizations can be incorporated consistently. Thus, we have

$$\begin{aligned} {}_t p_{x_0}^{(t)} \, {}_{T-t}\tilde{p}_{x_t}^{(t):t\to T} \, P(t,T) &= \Pi_t\left(0,T,x_0\right) \\ \text{and consequently } P(0,T) \, {}_T\tilde{p}_{x_0}^{(0):0\to T} &= \Pi_0\left(0,T,x_0\right). \end{aligned}$$

The next step is to define forward survival probabilities.

**Definition 3.2** *For $T_2 \geq T_1 > t$, we define the forward survival probability for $T_1$ to $T_2$ at time $t$ as*

$$ {}_{T_2-T_1}\tilde{p}_{x_{T_1}}^{(t):T_1\to T_2} := \frac{\Pi_t\left(0,T_2,x_0\right) \, P(t,T_1)}{\Pi_t\left(0,T_1,x_0\right) \, P(t,T_2)}. $$

This definition yields

$$ {}_{T_2-T_1}\tilde{p}_{x_{t_1}}^{t:T_1\to T_2} \, \Pi_t\left(0,T_1,x_0\right) \exp\left\{-\int_{T_1}^{T_2} f(t,u)\,du\right\} = \Pi_t\left(0,T_2,x_0\right), $$

where $f(t,x)$ is the instantaneous forward rate with maturity $x$ contracted at $t$. Finally, we fix the case when $t > T_1$:

**Definition 3.3** *Let $T_2 \geq t \geq T_1$. Then we define the survival probability for $T_1$ to $T_2$ at time $t$ as*

$$ {}_{T_2-T_1}\tilde{p}_{x_{t_1}}^{t:T_1\to T_2} := \frac{\Pi_t\left(0,T_2,x_0\right)}{{}_{T_1}p_{x_0}^{(T_1)} \, P(t,T_2)}. $$

Note that in the above definitions, the denominator may well be zero, implying that the survival probabilities may not be well defined. However, due to Lemma 2.1, we have

$$ \Pi_t\left(0,T_1,x_0\right) \geq \Pi_t\left(0,T_2,x_0\right) \text{ if } T_1 \leq T_2, $$

implying that the numerator is zero if the denominator is. Thus, we define the probabilities to be zero whenever the denominator is, or, equivalently, whenever the numerator is.

Now, similarly to interest rate modeling, we can and will define certain rates which later will provide the starting point of our modeling considerations.

**Definition 3.4**
- *We define the forward force of mortality with maturity $T$ as from time $t$ ($t \leq T$) as*

$$ \tilde{\mu}_t\left(0,T,x_0\right) := \lim_{T^*\to T+} -\frac{\log\left\{{}_{T^*-T}\tilde{p}_{x_T}^{(t):T\to T^*}\right\}}{T^*-T}. $$

- *We define the spot force of mortality at time $t$ as*

$$ \tilde{\mu}_t^s\left(0,x_0\right) := \tilde{\mu}_t\left(0,t,x_0\right). $$

- *For convenience, we define the force of mortality for $T \leq t$ as*

$$ \tilde{\mu}_t\left(0,T,x_0\right) := \lim_{T^*\to T+} -\frac{\log\left\{{}_{T^*-T}p_{x_T}^{(T^*)}\right\}}{T^*-T}. $$

Again, we choose this rather cryptic notation to allow for generalizations. The following proposition shows the power of Definition 3.4.

**Proposition 3.1** *1. For $t \leq T$: $\tilde{\mu}_t\left(0,T,x_0\right) = -\frac{\partial}{\partial T}\log\left\{\frac{\Pi_t(0,T,x_0)}{P(t,T)}\right\}$.*

*2. For $t \leq T$:*

$$
\begin{aligned}
{}_{T-t}\tilde{p}_{x_t}^{(t):t\to T} = \frac{\Pi_t\left(0,T,x_0\right)}{P(t,T)\;{}_t p_{x_0}^{(t)}} &= \exp\left\{-\int_t^T \tilde{\mu}_t\left(0,u,x_0\right)\,du\right\} \\
\Leftrightarrow \Pi_t\left(0,T,x_0\right) &= {}_t p_{x_0}^{(t)}\;\exp\left\{-\int_t^T \tilde{\mu}_t\left(0,u,x_0\right)\,+\,f(t,u)\,du\right\}.
\end{aligned}
$$

*3. For $T_1 \leq T_2$,*

$$
{}_{T_2-T_1}\tilde{p}_{x_{t_1}}^{t:T_1\to T_2} = \frac{\Pi_t\left(0,T_2,x_0\right)\;P(t,T_1)}{\Pi_t\left(0,T_1,x_0\right)\;P(t,T_2)} = \exp\left\{-\int_{T_1}^{T_2} \tilde{\mu}_t\left(0,u,x_0\right)\,du\right\}.
$$

*proof:* 1. is a direct consequence from Definition 3.2 and Assumption 2.2, 2. is a special case of 3., and 3. follows when using 1. in the right hand side and some simple computations.

□

At this point, we want to note that mortality intensities in the stochastic sense can also be defined from an individual's perspective (an individual's death) rather than using longevity bonds (see Wang (2006) for a thorough treatment). This path was also taken by Miltersen and Persson (2005). Even though our way of defining the respective quantities might be a little synthetic, it enables us to derive results by arbitrage arguments.

If we want to specify a model for the LB-market, it could be done in various ways:[7]

- Specify the dynamics of the spot force (see, e.g., Dahl (2004)).
- Specify the dynamics of all possible longevity bonds.
- Specify the dynamics of all possible forward forces.

In order to assess the relationship between these approaches, in the following we will consider generic dynamics of the bond model. We consider dynamics of the following form:

- **Spot force dynamics**

$$
\begin{aligned}
d\tilde{\mu}_t^s\left(0,x_0\right) &= a^L(t)\,dt \;+\; b^L(t)\,dW(t) \\
dr_t &= a^F(t)\,dt \;+\; b^F(t)\,dW(t)
\end{aligned}
\tag{3}
$$

- **Bond price dynamics**

$$
\begin{aligned}
d\Pi_t\left(0,T,x_0\right) &= \Pi_t\left(0,T,x_0\right)\,\left(m^L(t,T)\,dt \;+\; v^L(t,T)\,dW(t)\right) \\
dp(t,T) &= p(t,T)\,\left(m^F(t,T)\,dt \;+\; v^F(t,T)\,dW(t)\right)
\end{aligned}
\tag{4}
$$

- **Forward force dynamics**

$$
\begin{aligned}
d\tilde{\mu}_t\left(0,T,x_0\right) &= \alpha^L(t,T)\,dt \;+\; \sigma^L(t,T)\,dW(t) \\
df(t,T) &= \alpha^F(t,T)\,dt \;+\; \sigma^F(t,T)\,dW(t)
\end{aligned}
\tag{5}
$$

[7] We assume that the FB-market is modeled/defined adequately.

Here, $W(\cdot)$ is a finite dimensional Wiener process and, in particular, the same for financial and demographic processes. Thus, no independence is assumed! All involved processes are adapted and of adequate dimension. Note that for both, the bond price dynamics (4) and the forward force dynamics (5), we have an infinite dimensional system of SDEs rather than one SDE.

The following assumptions are similar to those in Bjørk (1999) and provide the technical necessities:

**Assumption 3.1**

- *For each fixed $\omega$ and $t$, all objects $m^F(t,T)$, $m^L(t,T)$, etc. are assumed to be differentiable in the $T$ variable.*
- *All processes are assumed to be regular enough to allow us to differentiate under the integral sign as well as to interchange the order of integration.*

The following proposition provides one of the main results. They hold regardless of the measure under consideration (in particular, no arbitrage-free market is assumed).

**Proposition 3.2**

1. *For the discounted price process of the longevity bond, we have*

$$d\left(\frac{\Pi_t(0,T,x_0)}{P(t,T)}\right) = \left(\frac{\Pi_t(0,T,x_0)}{P(t,T)}\right)\Big\{\Big(m^L(t,T) - m^F(t,T) + \big(v^F(t,T)\big)^2 - v^F(t,T)\,v^L(t,T)\Big)\,dt + \big(v^L(t,T) - v^F(t,T)\big)\,dW(t)\Big\}$$

2. *If $p(t,T)$ and $\Pi_t(0,T,x_0)$ satisfy (4), then for the forward rate dynamics we have (5) where*

$$\begin{aligned}
\alpha^F(t,T) &= v_T^F(t,T)\,v^F(t,T) \; - \; m_T^F(t,T),\\
\sigma^F(t,T) &= -v_T^F(t,T),\\
\alpha^L(t,T) &= m_T^F(t,T) \; - \; v_T^F(t,T)\,v^F(t,T) \; - \; m^L(t,T) \; + \; v_T^L(t,T)\,v^L(t,T),\\
\sigma^L(t,T) &= v_T^F(t,T) - v_T^L(t,T).
\end{aligned}$$

3. *If $f(t,T)$ and $\tilde{\mu}_t(0,T,x_0)$ satisfy (5), then for the spot force dynamics we have (3) where*

$$\begin{aligned}
a^F(t) &= f_T(t,t) \; + \; \alpha^F(t,t),\\
b^F(t) &= \sigma^F(t,t),\\
a^L(t) &= \frac{\partial}{\partial T}\tilde{\mu}_t(0,t,x_0) \; + \; \alpha^L(t,t),\\
b^L(t) &= \sigma^L(t,t).
\end{aligned}$$

4. *If $f(t,T)$ and $\tilde{\mu}_t(0,T,x_0)$ satisfy (5), then*

$$\begin{aligned}
dp(t,T) &= p(t,T)\left\{r_t \; + \; A^F(t,T) \; + \; \frac{1}{2}\left\|S^F(t,T)\right\|^2\right\}dt\\
&\quad + p(t,T)\,S^F(t,T)\,dW(t)\\
d\Pi_t(0,T,x_0) &= \Pi_t(0,T,x_0)\left\{r_t \; + \; A^L(t,T) \; + \; \frac{1}{2}\left\|S^L(t,T)\right\|^2\right\}dt\\
&\quad + \Pi_t(0,T,x_0)\;S^L(t,T)\,dW(t)
\end{aligned}$$

*where* $\|\cdot\|$ *denotes the Euclidean norm and*

$$\begin{aligned} A^F(t,T) &= -\int_t^T \alpha^F(t,s)\,ds, \\ S^F(t,T) &= -\int_t^T \sigma^F(t,s)\,ds, \\ A^L(t,T) &= -\int_t^T \alpha^F(t,s) \;+\; \alpha^L(t,s)\,ds, \\ S^L(t,T) &= -\int_t^T \sigma^F(t,s) \;+\; \sigma^L(t,s)\,ds. \end{aligned}$$

*proof:*

1. Let $f(x,y) = \frac{x}{y}$, $f_x = \frac{1}{y}$, $f_y = -\frac{x}{y^2}$, $f_{xx} = 0$, $f_{yy} = \frac{2x}{y^3}$, and $f_{xy} = -\frac{1}{y^2}$. With Itô's Lemma (see, e.g., Bingham and Kiesel (2003), Theorem 5.6.2), we have that

$$df(X_t, Y_t) = f_x\,dX_t \;+\; f_y\,dY_t \;+\; \frac{1}{2} f_{yy}\,d\langle Y, Y\rangle_t + f_{xy}\,d\langle X, Y\rangle_t.$$

   For $X_t = \Pi_t\,(0,T,x_0)$ and $Y_t = p(t,T)$, the claim follows.

2. The first two equations follow from Bjørk (1999), Proposition 20.5. For the remaining equations consider $f(x) = \log x$, with $f_x = \frac{1}{x}$ and $f_{xx} = -\frac{1}{x^2}$. Thus, again by Itô's lemma and using *1.*, we obtain

$$\begin{aligned} d\log\left\{\frac{\Pi_t\,(0,T,x_0)}{P(t,T)}\right\} &= \Bigg(m_L(t,T) \;-\; m^F(t,T) \;+\; \frac{1}{2}\left(v^F(t,T)\right)^2 \\ &\qquad -\; \frac{1}{2}\left(v^L(t,T)\right)^2\Bigg)\,dt \\ &\quad +\; \left(v^L(t,T) - v^F(t,T)\right)\,dW(t) \end{aligned}$$

   With Proposition 3.1, we obtain

$$\begin{aligned} \tilde{\mu}_t\,(0,T,x_0) &= -\frac{\partial}{\partial T}\log\left\{\frac{\Pi_t\,(0,T,x_0)}{P(t,T)}\right\} \\ &= \int_0^t m_T^F(s,T) \;-\; v_T^F(s,T)\,v^F(s,T) \;-\; m_T^L(s,T) \\ &\qquad +\, v_T^L(s,T)\,v^L(s,T)\,ds \\ &\quad + \int_0^t \left(v_T^F(s,T) - v_T^L(s,T)\right)\,dW(s), \end{aligned}$$

   and thus the claim follows.

3. Again, the first couple of equations follows from Bjørk (1999), Proposition 20.5. For the second couple consider

$$\begin{aligned} \mu_t^s(0,x_0) &= \tilde{\mu}_t\,(0,t,x_0) \\ &= \tilde{\mu}_0\,(0,t,x_0) \;+\; \int_0^t \alpha^L(u,t)\,du \;+\; \int_0^t \sigma^L(u,t)\,dW(u), \end{aligned}$$

and we can write

$$
\begin{aligned}
\alpha^L(u,t) &= \alpha^L(u,u) \;+\; \int_u^t \alpha_T^L(u,s)\,ds, \\
\sigma^L(u,t) &= \sigma^L(u,u) \;+\; \int_u^t \sigma_T^L(u,s)\,ds. \\
\Rightarrow \mu_t^s(0,x_0) &= \tilde{\mu}_0\left(0,t,x_0\right) \\
&\quad + \int_0^t \alpha^L(u,u)\,du \;+\; \int_0^t\int_u^t \alpha_T^L(u,s)\,ds\,du \\
&\quad + \int_0^t \sigma^L(u,u)\,dW(u) \;+\; \int_0^t\int_u^t \sigma_T^L(u,s)\,ds\,dW(u),
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^t\int_u^t \alpha_T^L(u,s)\,ds\,du \;+\; \int_0^t\int_u^t \sigma_T^L(u,s)\,ds\,dW(u) \\
&= \int_0^t \left(\int_0^s \alpha_T^L(u,s)\,du \;+\; \int_0^s \sigma_T^L(u,s)\,dW(u)\right)\,ds \\
&= \int_0^t \frac{\partial}{\partial T}\left(\int_0^s \alpha^L(u,s)\,du \;+\; \int_0^s \sigma^L(u,s)\,dW(u)\right)\,ds \\
&= \int_0^t \frac{\partial}{\partial T}\tilde{\mu}_s\left(0,s,x_0\right)\,ds.
\end{aligned}
$$

Here

$$
\frac{\partial}{\partial T}\tilde{\mu}_s\left(0,s,x_0\right) = \left.\frac{\partial}{\partial T}\tilde{\mu}_s\left(0,T,x_0\right)\right|_{T=s}.
$$

Thus, altogether we obtain

$$
\begin{aligned}
d\tilde{\mu}_t^s(0,x_0) &= \left(\alpha^L(t,t) \;+\; \frac{\partial}{\partial T}\tilde{\mu}_t\left(0,t,x_0\right)\right)\,dt \\
&\quad + \sigma^L(t,t)\,dW(t)
\end{aligned}
$$

4. Once again, the formulas for the FB are covered by Prop. 20.5 in Bjørk (1999). For the LB formulas, let

$$
Y(t,T) := -\int_t^T \tilde{\mu}_t\left(0,s,x_0\right) + f(t,s)\,ds,
$$

which leads to

$$
\Pi_t\left(0,T,x_0\right) =_t p_{x_0}^{(t)} \exp\left\{Y(t,T)\right\}.
$$

Therefore, by the multiplication rule and Itô's Lemma,

$$
\begin{aligned}
d\Pi_t\left(0,T,x_0\right) &= d\left({}_tp_{x_0}^{(t)}\exp\left\{Y(t,T)\right\}\right) \\
&= \exp\left\{Y(t,T)\right\}\, d_tp_{x_0}^{(t)} \;+\; {}_tp_{x_0}^{(t)}\, d\exp\left\{Y(t,T)\right\} \\
&= -\exp\left\{Y(t,T)\right\}\, {}_tp_{x_0}^{(}t)\,\tilde{\mu}_t\left(0,t,x_0\right)\,dt \\
&\quad + {}_tp_{x_0}^{(t)}\,\exp\left\{Y(t,T)\right\}\,dY(t,T) \\
&\quad + \frac{1}{2}\,{}_tp_{x_0}^{(t)}\,\exp\left\{Y(t,T)\right\} d\left\langle Y(\cdot,T),Y(\cdot,T)\right\rangle_t \\
&= \Pi_t\left(0,T,x_0\right)\left(dY(t,T) \;+\; \frac{1}{2}\,d\left\langle Y(\cdot,T),Y(\cdot,T)\right\rangle_t\right. \\
&\qquad\left. -\;\tilde{\mu}_t\left(0,t,x_0\right)\,dt\right), \qquad (6)
\end{aligned}
$$

since
$$\frac{\partial}{\partial t}\,{}_tp_{x_0}^{(t)} = \frac{\partial}{\partial t}\,e^{-\int_0^t \tilde{\mu}_s(0,s,x_0)ds} = -\tilde{\mu}_t\,(0,t,x_0)\;{}_tp_{x_0}^{(t)}.$$
There remains to compute
$$dY(t,T) = -d\int_t^T \tilde{\mu}_t\,(0,s,x_0) + f(t,s)\,ds,$$
We use a technique from Heath et al. (1992) – we start by writing the forward force dynamics in integrated form:
$$\begin{aligned}
f(t,s) \;+\; \tilde{\mu}_t\,(0,s,x_0) \;&=\; f(0,s) \;+\; \tilde{\mu}_0\,(0,s,x_0) \\
&\quad + \int_0^t \alpha^F(u,s) \;+\; \alpha^L(u,s)\,du \\
&\quad + \int_0^t \sigma^F(u,s) \;+\; \sigma^L(u,s)\,dW(u)
\end{aligned}$$
Thus, we have
$$\begin{aligned}
Y(t,T) \;&=\; -\int_t^T f(0,s) \;+\; \tilde{\mu}_0\,(0,s,x_0)\;ds \\
&\quad - \int_t^T \int_0^t \alpha^F(u,s) \;+\; \alpha^L(u,s)\,du\,ds \\
&\quad - \int_t^T \int_0^t \sigma^F(u,s) \;+\; \sigma^L(u,s)\,dW(u)\,ds \\
&=\; -\int_0^T f(0,s) \;+\; \tilde{\mu}_0\,(0,s,x_0)\;ds \\
&\quad - \int_0^t \int_u^T \alpha^F(u,s) \;+\; \alpha^L(u,s)\,ds\,du \\
&\quad - \int_0^t \int_u^T \sigma^F(u,s) \;+\; \sigma^L(u,s)\,ds\,dW(u) \\
&\quad + \int_0^t f(0,s) \;+\; \tilde{\mu}_0\,(0,s,x_0)\;ds \\
&\quad + \int_0^t \int_u^t \alpha^F(u,s) \;+\; \alpha^L(u,s)\,ds\,du \\
&\quad + \int_0^t \int_u^t \sigma^F(u,s) \;+\; \sigma^L(u,s)\,ds\,dW(u).
\end{aligned}$$
Changing the order of integration in last three summands leads to
$$\begin{aligned}
Y(t,T) \;&=\; Y(0,T) \;-\; \int_0^t \int_u^T \alpha^F(u,s) \;+\; \alpha^L(u,s)\,ds\,du \\
&\quad - \int_0^t \int_u^T \sigma^F(u,s) \;+\; \sigma^L(u,s)\,ds\,dW(u) \\
&\quad + \int_0^t f(0,s) \;+\; \tilde{\mu}_0\,(0,s,x_0)\;ds \\
&\quad + \int_0^t \int_0^s \alpha^F(u,s) \;+\; \alpha^L(u,s)\,du\,ds \\
&\quad + \int_0^t \int_0^s \sigma^F(u,s) \;+\; \sigma^L(u,s)\,dW(u)\,ds,
\end{aligned}$$

which, with (5) and $f(s,s) = r_s$ yields

$$\begin{aligned} Y(t,T) &= Y(0,T) + \int_0^t r_s \ + \ \tilde{\mu}_s\,(0,s,x_0)\ ds \\ &\quad - \int_0^t \int_u^T \alpha^F(u,s) \ + \ \alpha^L(u,s)\, ds\, du \\ &\quad - \int_0^t \int_u^T \sigma^F(u,s) \ + \ \sigma^L(u,s)\, ds\, dW(u) \\ &= Y(0,T) + \int_0^t r_s \ + \ \tilde{\mu}_s\,(0,s,x_0) \ + \ A^L(s,T)\, ds \\ &\quad + \int_0^t S^L(s,T)\, dW(s). \end{aligned}$$

Thus, together with (6), the claim follows.

□

As we pointed out earlier, if we want to specify a model of the LB-market, we can use a specification of any of the stochastic differential equations (3), (4), or (5). Then, Proposition 3.2 gives the other dynamics. In Section 5, we will take a closer look at the forward force modeling approach.

# 4 Generalization Toward Multiple Inception Dates

In Section 3, only longevity bonds with inception date 0 were considered. In a market where we additionally have $(\tau, T, x_\tau)$-Bonds for $0 < \tau < T$, some of the definitions in Section 3 could have been formulated differently. For example, we could the spot survival probability as

$$_{T-t}\tilde{p}_{x_t}^{(t):t\to T} = \frac{\Pi_t\,(t,T,x_t)}{P(t,T)},\ t \le T \tag{7}$$

or we could have defined the forward survival probability as

$$_{T_2-T_1}\tilde{p}_{x_{T_1}}^{(t):T_1\to T_2} = \frac{\Pi_t\,(T_1,T_2,x_{T_1})}{P(t,T_2)},\ t \le T_1 \le T_2. \tag{8}$$

However, these alternative definitions may add ambiguity – we will address this issue later in this section.

With these alternative formulations in mind, we may generalize our definition of the force of mortality (Definition 3.4):

**Definition 4.1** • *We define the forward force of mortality with maturity $T$ as from time $t \le T$ using inception date $\tau \le T$ as*

$$\tilde{\mu}_t\,(\tau,T,x_\tau) := \lim_{T^*\to T+} \frac{-\log\left\{\frac{\Pi_t(\tau,T^*,x_\tau)\,P(t,T)}{\Pi_t(\tau,T,x_\tau)\,P(t,T^*)}\right\}}{T^*-T}.$$

- *We define the spot force of mortality at time $t$ using inception date $\tau \le t$ as*

$$\tilde{\mu}_t^s(\tau,x_\tau) := \tilde{\mu}_t\,(\tau,t,x_\tau) = \lim_{T\to t} \frac{-\log\left\{\frac{\Pi_t(\tau,T,x_\tau)}{P(t,T)\ {}_{t-\tau}p_{x_\tau}^{(t)}}\right\}}{T-t}.$$

- *For $\tau \le T \le t$, we let $\tilde{\mu}_t(\tau, T, x_\tau) := \tilde{\mu}_t(0, T, x_0)$.*

This definition shows, why the rather cryptic notation was chosen in Definition 3.4: It allows us to consistently define forces of mortality for different inception dates. Similar to Proposition 3.1, we can express the longevity bond prices in terms of forward rates for all inception dates.

The question whether a redefinition of the survival probabilities as in (7) and (8) adds ambiguity is equivalent to the question whether the forces of mortality depend on the inception dates. The following proposition answers this question for the situation when $\tau \le t$ in an arbitrage-free market:

**Proposition 4.1** *Under the assumption that there are no arbitrage possibilities in the LB-market, we have*

1. *for $T_2 \ge t \ge T_1$, $\tau \le t$*

$$
\begin{aligned}
{}_{T_2-T_1}\tilde{p}^{(t):T_1\to T_2}_{x_{T_1}} &= \frac{\Pi_t(0, T_2, x_0)}{{}_{T_1}p^{T_1}_{x_0}\, P(t, T_2)} \\
&= \begin{cases} \dfrac{\Pi_t(\tau, T_2, x_\tau)}{{}_{T_1-\tau}p^{(T_1)}_{x_\tau}\, P(t,T_2)} & , T_1 \ge \tau \\ \dfrac{\Pi_t(\tau, T_2, x_\tau)\, {}_{\tau-T_1}p^{(\tau)}_{x_{T_1}}}{P(t,T_2)} & , T_1 < \tau \end{cases} \\
&= \frac{\Pi(T_1, T_2, x_{T_1})}{P(t, T_2)};
\end{aligned}
$$

2. *for $T_2 \ge T_1 \ge t$, $\tau \le t$*

$$
\begin{aligned}
{}_{T_2-T_1}\tilde{p}^{(t):T_1\to T_2}_{x_{T_1}} &= \frac{\Pi_t(0, T_2, x_0)\; P(t, T_1)}{\Pi_t(0, T_1, x_0)\; P(t, T_2)} \\
&= \frac{\Pi_t(\tau, T_2, x_\tau)\; P(t, T_1)}{\Pi_t(\tau, T_1, x_\tau)\; P(t, T_2)};
\end{aligned}
$$

3. *for $T \ge t \ge \tau$*

$$
{}_{T-t}\tilde{p}^{(t):t\to T}_{x_t} = \frac{\Pi_t(t, T, x_t)}{P(t, T)} = \frac{\Pi_t(\tau, T, x_\tau)}{P(t, T)\; {}_{t-\tau}p^{(t)}_{x_\tau}};
$$

4. $\forall\, \tau \le t :\ \tilde{\mu}^s_t(0, x_0) = \tilde{\mu}^s_t(\tau, x_\tau)$.

*proof:* The first equalities from 1. and 2. are just the definitions, the last equality of 1. as well as 3. are special cases of the second equality of 1., and 4. is a direct consequence of 3. and Definition 4.1, so the only things left to prove are the second equalities of 2. and 1., respectively. The latter is equivalent to

$$
\begin{aligned}
\frac{\Pi_t(\tau, T_2, x_\tau)}{P(t, T_2)} &= \frac{\Pi_t(0, T_2, x_0)}{P(t, T_2)\; {}_{\tau}p^{(\tau)}_{x_0}},\ T \ge t \ge \tau \\
\Leftrightarrow \Pi_t(\tau, T_2, x_\tau) &= \frac{\Pi_t(0, T_2, x_0)}{{}_{\tau}p^{(\tau)}_{x_0}},\ T \ge t \ge \tau
\end{aligned}
$$

Consider the following arbitrage table:

| **time** | $t$ | $T_2$ |
|---|---|---|
| | Buy $\frac{1}{{}_{\tau}p^{(\tau)}_{x_0}}$ $(0, T_2, x_0)$ −Bonds<br>Sell a $(\tau, T_2, x_\tau)$-Bond | Receive $\frac{{}_{T_2}p^{(T_2)}_{x_0}}{{}_{\tau}p^{(\tau)}_{x_0}} = {}_{T_2-\tau}p^{(T_2)}_{x_\tau}$<br>Pay out ${}_{T_2-\tau}p^{(T_2)}_{x_\tau}$ |
| Net | $\Pi_t(\tau, T_2, x_\tau) - \frac{\Pi_t(0,T_2,x_0)}{{}_{\tau}p^{(\tau)}_{x_0}}$, $T \ge t \ge \tau$ | 0 |

To exclude arbitrage opportunities, the price of the $(\tau, T_2, x_\tau)$-Bond is thus given by

$$\Pi_t(\tau, T_2, x_\tau) = \frac{\Pi_t(0, T_2, x_0)}{{}_\tau p_{x_0}^{(\tau)}}.$$

In order to prove 2., for $i \in \{1, 2\}$ use the relationship

$$\Pi_t(\tau, T_i, x_\tau) = \frac{\Pi_t(0, T_i, x_0)}{{}_\tau p_{x_0}^{(\tau)}}$$

which follows from 1. in the left hand side.

□

Thus, for $\tau \leq t$, the redefinition does not add ambiguity. However, for $\tau > t$, the situation is more complicated. The following lemma provides a condition which is both, necessary and sufficient:

**Lemma 4.1**

$$\Pi_t(T_1, T_2, x_{T_1}) = \frac{\Pi_t(0, T_2, x_0)\ P(t, T_1)}{\Pi_t(0, T_1, x_0)}\ \forall t \leq T_1 \leq T_2, \tag{9}$$

*if and only if*

$$\frac{\Pi_t(0, T_2, x_0)\ P(t, T_1)}{\Pi_t(0, T_1, x_0)} = \frac{\Pi_t(\tau, T_2, x_\tau)\ P(t, T_1)}{\Pi_t(\tau, T_1, x_\tau)}\ \forall t \leq T_1 \leq T_2,\ \tau \leq T_1.$$

*proof:* For $t \leq \tau$ plug condition (9) into the right-hand side of the latter equation, for $t > \tau$ use Proposition 4.1 in the right-hand side of the latter equation, and evaluate. This proves the sufficiency. Necessity is trivial since it is a special case.

□

Thus, (9) provides a condition for (8) or, equivalently, for the forces of mortality as introduced in Definition 4.1 to be independent of the inception date. Unfortunately, this condition does not hold automatically in an arbitrage free market, i.e. it cannot be proved using arbitrage arguments. Hence, it is of interest under which conditions (9) is fulfilled.

Let us consider $\{\Pi_t(T_1, T_2, x_{T_1}) \mid t \leq T_1\}$ for arbitrary $0 \leq T_1 \leq T_2$. Then, at time $T_1$ we have with Proposition 4.1

$$\begin{aligned}\Pi_{T_1}(T_1, T_2, x_{T_1}) &= \frac{\Pi_{T_1}(0, T_2, x_0)}{{}_{T_1} p_{x_0}^{(T_1)}} \\ &= \frac{\Pi_{T_1}(0, T_2, x_0)\ P(T_1, T_1)}{\Pi_{T_1}(0, T_1, x_0)}\end{aligned}$$

Assuming that there exists a equivalent martingale measure $Q$, condition (9) is therefore equivalent to

$$E_Q\left[e^{-\int_t^{T_1} r_s\, ds} \frac{\Pi_{T_1}(0, T_2, x_0)\ P(T_1, T_1)}{\Pi_{T_1}(0, T_1, x_0)} \middle| \mathcal{F}_t\right] = \frac{\Pi_t(0, T_2, x_0)\ P(t, T_1)}{\Pi_t(0, T_1, x_0)} \tag{10}$$

or, in other words, equivalent to the discounted price process of

$$\frac{\Pi_t(0, T_2, x_0)\ P(t, T_1)}{\Pi_t(0, T_1, x_0)}$$

to be a $Q$-martingale.

Using the bond price dynamics and applying Itô's formula to the function $f(x,y,z) = \frac{xy}{z}$, we obtain for $t \leq T_1$

$$\begin{aligned} d\left(\frac{\Pi_t\left(0,T_2,x_0\right)\ P(t,T_1)}{\Pi_t\left(0,T_1,x_0\right)}\right) &= \frac{\Pi_t\left(0,T_2,x_0\right)\ P(t,T_1)}{\Pi_t\left(0,T_1,x_0\right)} \\ &\quad \{\big(m^F(t,T_1)\ +\ m^L(t,T_1)\ -\ m^L(t,T_1) \\ &\quad + v^L(t,T_2)\,v^F(t,T_1)'\ -\ v^L(t,T_1)\,v^F(t,T_1)' \\ &\quad - v^L(t,T_2)\,v^L(t,T_1)'\ +\ v^L(t,T_1)\,v^L(t,T_1)'\big)\ dt \\ &\quad + \big(v^F(t,T_1)\ +\ v^L(t,T_1)\ +\ v^L(t,T_1)\big)\ dW(t)\ \} \end{aligned}$$

Since we know that discounted asset prices are local martingales under $Q$, we have (when modeling under $Q$)

$$\begin{aligned} m^F(t,T_1) = m^L(t,T_2) = m^L(t,T_1) &= r_t \\ \Rightarrow m^F(t,T_1)\ +\ m^L(t,T_2)\ -\ m^L(t,T_1) &= r_t \end{aligned}$$

Thus, in order for (10) to hold, we need

$$\begin{aligned} 0 &= v^L(t,T_2)\,v^F(t,T_1)'\ -\ v^L(t,T_1)\,v^F(t,T_1)' \\ &\quad - v^L(t,T_2)\,v^L(t,T_1)'\ +\ v^L(t,T_1)\,v^L(t,T_1)' \\ \Leftrightarrow 0 &= \big(v^L(t,T_1)\ -\ v^L(t,T_2)\big)\,\big(v^L(t,T_1)\ -\ v^F(t,T_1)\big)' \end{aligned} \tag{11}$$

Using Proposition 3.2, this condition can be alternatively formulated in terms of the forward force dynamics. Thus, assume that the forward force dynamics satisfy (5), then (11) is equivalent to

$$\begin{aligned} \big(S^L(t,T_1)\ -\ S^L(t,T_2)\big)\,\big(S^L(t,T_1)\ -\ S^F(t,T_1)\big)' &= 0\ \forall t \leq T_1 \leq T_2 \\ \Leftrightarrow \left(\int_{T_1}^{T_2} \sigma^F(t,s)\ +\ \sigma^L(t,s)\,ds\right)\left(-\int_t^{T_1} \sigma^L(t,s)'\,ds\right) &= 0\ \forall t \leq T_1 \leq T_2 \end{aligned}$$

Note that in case of independence of financial and biometric events, this condition simplifies to

$$\int_{T_1}^{T_2} \sigma^L(t,s)\,ds \int_t^{T_1} \sigma^L(t,s)'\,ds = 0\ \forall t \leq T_1 \leq T_2.$$

The following proposition summarizes the foregoing findings.

**Proposition 4.2** *In an arbitrage-free market with a risk-neutral measure $Q$, in order for the force of mortality as defined in Definition 4.1 to be independent of the inception date, the following conditions are necessary and sufficient:*

1. *If we assume that (4) holds for the bond price dynamics,*

$$\big(v^L(t,T_1)\ -\ v^L(t,T_2)\big)\,\big(v^L(t,T_1)\ -\ v^F(t,T_1)\big)' = 0\ \forall\, t \leq T_1 \leq T_2.$$

2. *If we assume that (5) holds for the forward force dynamics,*

$$\left(\int_{T_1}^{T_2} \sigma^F(t,s)\ +\ \sigma^L(t,s)\,ds\right)\left(-\int_t^{T_1} \sigma^L(t,s)'\,ds\right) = 0\ \forall\, t \leq T_1 \leq T_2$$

*If these restrictions are satisfied, the forward force can be denoted by*

$$\tilde{\mu}_t(T,x_T) := \tilde{\mu}_t\left(0,T,x_0\right) = \tilde{\mu}_t\left(\tau,T,x_\tau\right),\ \tau \leq t. \tag{12}$$

In the following, we will adopt this notation; however, we will not generally assume that the conditions of Proposition 4.2 are satisfied. We will rather impose that one of the following three alternatives hold:

1. We assume that there are only $(0, T, x_0)$-Bonds in the market and we define $\tilde{\mu}_t(T, x_T) := \tilde{\mu}\,(0, T, x_0)$.

2. We assume that at time $t$, there are only $(\tau, T, x_\tau)$-Bonds with inception date $\tau \leq t$, i.e. there are no bonds with inception dates in the future, and we set $\tilde{\mu}_t(T, x_T) := \tilde{\mu}\,(t, T, x_t) = \tilde{\mu}\,(\tau, T, x_\tau)\,,\ \tau \leq t$. Proposition 4.1 provides that there is no ambiguity.

3. We assume that the volatility restriction from Proposition 4.2 is satisfied and $\mu_t(T, x_T)$ is defined as in (12).

Thus, if not stated otherwise, we will either assume 1. or 2. or 3. to hold, but accept this ambiguity in order to deal with all cases at the same time.

# 5 Forward Force Modeling – The Heath-Jarrow-Morton Framework

As pointed out in Section 3, we could have also considered spot force or bond price dynamics. However, we choose the forward modeling approach due to its generality: the Heath-Jarrow-Morton (HJM) framework (see Heath et al. (1992)) basically unifies all continuous interest rate models and, as such, all possible movements of the terms structure of interest rates (cf. Filipovic (2001)). Analogously, the framework also unifies all continuous force of mortality models, in particular all models of the spot force of mortality. Therefore, the question whether the framework is adequate is obsolete. Also, we want to point out that the idea of applying the HJM framework to mortality modeling is not new, but was considered by authors in less general or different settings (see Cairns et al. (2005b) or Miltersen and Persson (2005)). Again, we will follow the outline of Bjørk (1999) for interest rate modeling.

**Assumption 5.1** *We assume that for every fixed $T$, the forward forces $f(\cdot, T)$ and $\tilde{\mu}_\cdot(T, x_T)$ have stochastic differentials which, under the objective measure $P$ are given by*

$$\begin{aligned} df(t,T) &= \alpha^F(t,T)\,dt \ + \ \sigma^F(t,T)\,d\bar{W}(t), \\ f(0,T) &= f^*(0,T), \\ d\tilde{\mu}_t(T,x_T) &= \alpha^L(t,T)\,dt \ + \ \sigma^L(t,T)\,d\bar{W}(t), \\ \tilde{\mu}_0(T,x_T) &= \tilde{\mu}_0^*(T,x_T), \end{aligned} \tag{13}$$

*where $\bar{W}$ is a finite dimensional $P$-Wiener process and $\alpha^F(\cdot, T)$, $\alpha^L(\cdot, T)$, $\sigma^F(\cdot, T)$, and $\sigma^L(\cdot, T)$ are adapted processes.*

Note that the observed forward rate curve $\{f^*(0,T);\ T \geq 0\}$ as well as the observed mortality term structure $\{\tilde{\mu}_0^*(T, x_T)\}$ are used as initial conditions. For the FB market, this setup gives us a perfect fit between model and market prices. Theoretically, the same should hold for the LB market. However, there are almost no observable prices, since there is no liquid LB market yet. Thus, while the forward rate curve is well-known, it is not obvious how the prevailing mortality term structure could be observed or derived – a possible starting point are *implied survival probabilities* derived from annuity or endowment prices. See Bauer and Russ (2006) for a further discussion of this issue.

Suppose for now that we have specified the processes as well as the initial conditions from Assumption 5.1. Then we have specified the entire forward rate structure and thus

$$\begin{aligned} p(t,T) &= \exp\left\{ -\int_t^T f(t,s)\,ds \right\}, \\ \Pi_t\,(0,T,x_0) &= {}_t p_{x_0}^{(t)} \exp\left\{ -\int_t^T f(t,s) + \tilde{\mu}(s,x_s)\,ds \right\}. \end{aligned} \tag{14}$$

Since we have a finite number of random sources ($\dim(W)$) and an infinite number of traded assets, we clearly run the risk of having introduced arbitrage into the market. The following theorem answers the question how the processes must be related in order to not have arbitrage possibilities.

**Theorem 5.1** – *HJM drift condition*

*Assume the forward forces are given by (13) and there exists a risk-neutral martingale measure $Q$. Then there exists a $(\dim(W))$-dimensional column vector process $\lambda(t)$ with the property that for all $T \geq 0$ and for all $t \leq T$, we have*

$$\begin{aligned} \alpha^F(t,T) &= \sigma^F(t,T) \int_t^T \sigma^F(t,s)'\,ds \;+\; \sigma^F(t,T)\,\lambda(t), \\ \alpha^L(t,T) &= \sigma^F(t,T) \int_t^T \sigma^L(t,s)'\,ds \;+\; \sigma^L(t,T) \int_t^T \sigma^F(t,s)'\,ds \\ &\quad + \sigma^L(t,T) \int_t^T \sigma^L(t,s)'\,ds \;+\; \sigma^L(t,T)\,\lambda(t). \end{aligned}$$

*proof:* An equivalent measure is given via a Girsanov density, say

$$L(t) = \exp\left\{ -\int_0^t \lambda(s)'\,dW(s) \;-\; \frac{1}{2}\int_0^t \|\lambda(s)\|^2\,ds \right\}.$$

By applying Girsanov's Theorem (cf. Bingham and Kiesel (2003), Theorem 5.7.1) together with to Proposition 3.2, we have for the $Q$-dynamics of $p(t,T)$ and $\Pi_t(0,T,x_0)$

$$\begin{aligned} dp(t,T) &= p(t,T)\left\{ r_t \;+\; A^F(t,T) \;+\; \frac{1}{2}\left\|S^F(t,T)\right\|^2 \;-\; S^F(t,T)\,\lambda(t) \right\} dt \\ &\quad + p(t,T)\,S^F(t,T)\,d\tilde{W}(t), \\ d\Pi_t\,(0,T,x_0) &= \Pi_t\,(0,T,x_0)\,\Big\{ r_t \;+\; A^L(t,T) \;+\; \frac{1}{2}\left\|S^L(t,T)\right\|^2 \\ &\quad - S^L(t,T)\,\lambda(t) \Big\}\; dt \;+\; \Pi_t\,(0,T,x_0)\; S^L(t,T)\,d\tilde{W}(t), \end{aligned}$$

respectively, where $\tilde{W}(\cdot)$ is a $Q$-Brownian motion. Since for a risk-neutral martingale measure, the discounted price processes have to be martingales, we obtain:

$$\begin{aligned} 0 &= A^F(t,T) \;+\; \frac{1}{2}\left\|S^F(t,T)\right\|^2 \;-\; S^F(t,T)\,\lambda(t) \\ 0 &= A^L(t,T) \;+\; \frac{1}{2}\left\|S^L(t,T)\right\|^2 \;-\; S^L(t,T)\,\lambda(t), \end{aligned}$$

which, taking the derivative with respect to $T$ yields

$$-\sigma^F(t,T)\,\lambda(t) = -\alpha^F(t,T) - \sigma^F(t,T)\,S^F(t,T)'$$

and

$$\begin{aligned} -\left(\sigma^F(t,T) + \sigma^L(t,T)\right)\,\lambda(t) &= -\alpha^F(t,T) - \alpha^L(t,T) \\ &\quad - \left(\sigma^F(t,T) \;+\; \sigma^L(t,T)\right)\, S^L(t,T)', \end{aligned}$$

which, after rearranging, using the first equation in the second equation, and some computations, equals the claim.

□

We will now turn to martingale modeling, i.e. we specify the forward rates directly under the martingale measure $Q$ as

$$\begin{aligned} df(t,T) &= \alpha^F(t,T)\,dt \;+\; \sigma^F(t,T)\,W(t), \\ f(0,T) &= f^*(0,T), \\ d\tilde{\mu}_t(T,x_T) &= \alpha^L(t,T)\,dt \;+\; \sigma^L(t,T)\,dW(t), \\ \tilde{\mu}_0(T,x_T) &= \tilde{\mu}_0^*(T,x_T), \end{aligned}$$

where $W(\cdot)$ is a $Q$-Brownian motion. Since we are working under a martingale measure now, we no longer have the problem of arbitrage free prices. However, we have the following different formulas for bond prices:

$$\begin{aligned} p(0,T) &= \exp\left\{-\int_0^T f(0,s)\,ds\right\} \stackrel{!}{=} E_Q\left[e^{-\int_0^T r_s\,ds}\right] \\ \Pi_t\left(0,T,x_0\right) &= \exp\left\{-\int_0^T f(0,s)+\tilde{\mu}_0(s,x_s)\,ds\right\} \stackrel{!}{=} E_Q\left[e^{-\int_0^T r_s\,ds}\,{}_Tp_{x_0}^{(T)}\right]. \end{aligned}$$

In order for these formulas to hold simultaneously, we have to impose some consistency relation:

**Corollary 5.1** *Under the martingale measure $Q$, the processes $\alpha^F$, $\alpha^L$, $\sigma^F$, and $\sigma^L$ must satisfy the following relation for every $T \geq 0$ and $t \leq T$:*

$$\begin{aligned} \alpha^F(t,T) &= \sigma^F(t,T)\int_t^T \sigma^F(t,s)'\,ds \\ \alpha^L(t,T) &= \sigma^F(t,T)\int_t^T \sigma^L(t,s)'\,ds \;+\; \sigma^L(t,T)\int_t^T \sigma^F(t,s)'\,ds \\ &\quad + \sigma^L(t,T)\int_t^T \sigma^L(t,s)'\,ds. \end{aligned}$$

*proof:* Theorem 5.1 with $\lambda(t) \equiv 0$.

□

Here, we recovered a result from Miltersen and Persson (2005). According to Bjørk (1999), the *moral* of Corollary 5.1 is that when specifying the forward rate dynamics under $Q$, we may freely specify the volatility structure, and then apply the HJM-drift condition to obtain the drift.

When we assume that the volatility restriction from Section 4 is satisfied, we can even obtain the following result:

**Corollary 5.2** *Under the assumption Proposition that the volatility restriction from Proposition 4.2 is satisfied, under the martingale measure $Q$ the processes $\alpha^F$, $\alpha^L$, $\sigma^F$, and $\sigma^L$ must satisfy the following relation for every $T \geq 0$ and $t \leq T$:*

$$\begin{aligned} \alpha^F(t,T) &= \sigma^F(t,T)\int_t^T \sigma^F(t,s)'\,ds \\ \alpha^L(t,T) &= \sigma^L(t,T)\int_t^T \sigma^F(t,s)'\,ds \end{aligned}$$

*If we additionally assume independence of financial and biometric events, we even obtain*

$$\alpha^L(t,T) = 0.$$

*proof:* From Prop. 4.2 we know that

$$f_{(t,T)}(\tau) := \left(\int_T^\tau \sigma^F(t,s) + \sigma^L(t,s)\,ds\right)\left(\int_t^T \sigma^L(t,s)'\,ds\right) = 0\;\forall\,\tau \geq T,\; t < T$$

$$\Rightarrow \frac{\partial}{\partial\tau} f_{(t,T)}(\tau) = \left(\sigma^F(t,\tau) + \sigma^L(t,\tau)\right)\left(\int_t^T \sigma^L(t,s)'\,ds\right) = 0\;\forall\,\tau \geq T,\; t < T.$$

Using this for $\tau = T$ in the $\alpha^L(t,T)$ of Corollary 5.1 yields the claim.

□

# 6 Considerations and Applications

As mentioned in the introduction, a possible reason why the BNP/EIB longevity bond was not successful could be the payoff structure: on the one hand, the coupon payments stop after a fixed maturity date, i.e. the bond does not enable its holder to secure the full tail risk, that is the risk that many people live very long, and, on the other hand, each coupon pays out the full ${}_kp_{x_0}^{(k)}$, whereas the insurers probably are more interested in securing "bad scenarios" and therefore may be more interested in a payoff of the form $\max\left\{{}_kp_{x_0}^{(k)} - K, 0\right\}$, where $K$ is some fixed "strike" price, which, e.g., could be some expected value. The latter argument also questions the design of the $(\tau, T, x_\tau)$-Bond as introduced here. However, in the fixed income market the most traded instruments are coupon bearing bonds, swaps, and swaptions, whereas most modeling frameworks take zero-coupon bonds as the starting point. Therefore, we believe taking the $(\tau, T, x_\tau)$-Bond as the basic instrument within our modeling framework is appropriate, even if we do not believe that these specific securities will ever be heavily traded, since they allow it for more complex payoff structures to be valuated consistently.

As explained above, the potential holders of longevity bonds, for example annuity providers or large corporations with a lot of pension obligations, may be interested in payoffs of an option type, for example of the form

$$\max\left\{{}_kp_{x_0}^{(k)} - K, 0\right\} = \left({}_kp_{x_0}^{(k)} - K\right)^+, \tag{15}$$

where K is some threshold, for example $K = (1+\epsilon)\,E_P\left[{}_kp_{x_0}^{(k)}\right]$ for some $\epsilon > 0$. This would enable them to still keep most of their funds and some of the risk in their own books, while they would be hedged against extreme scenarios.

Similarly to the valuation of bond options in the fixed income market (compare Bjørk (1999), Section 24.5), for the valuation of payoffs like the one in (15), the above model can be employed; by the risk-neutral pricing formula (cf. Bingham and Kiesel (2003), Theorem 4.4.2), for the arbitrage price we have

$$\begin{aligned}
V_0 &= E_Q\left[e^{-\int_0^T r_s\,ds}\left({}_kp_{x_0}^{(k)} - K\right)^+\right] \\
&= E_Q\left[e^{-\int_0^T r_s\,ds}\left(\Pi_T(0,T,x_0) - K\right)\,1_{\Pi_T(0,T,x_0)\geq K}\right] \\
&= E_Q\left[e^{-\int_0^T r_s\,ds}\Pi_T(0,T,x_0)\,1_{\Pi_T(0,T,x_0)\geq K}\right] \\
&\quad -K\,E_Q\left[e^{-\int_0^T r_s\,ds}\,1_{\Pi_T(0,T,x_0)\geq K}\right] \\
&= \Pi_0(0,T,x_0)\;E_{Q_\Pi}\left[1_{\Pi_T(0,T,x_0)\geq K}\right] - K\,p(0,T)\,E_{Q_T}\left[\,1_{\Pi_T(0,T,x_0)\geq K}\right],
\end{aligned}$$

where $Q_\Pi$ denotes the martingale measure with respect to the numéraire $\Pi_t\,(0,T,x_0)$ and $Q_T$ denotes the $T$-forward measure (cf. Bjørk (1999), Section 24.5).

Let now $v(t) := v^L(t,T) - v^F(t,T)$ be deterministic, where $v^F$ and $v^L$ are the volatility processes from (4). Since $\frac{\Pi_t(0,T,x_0)}{p(t,T)}$ is a martingale under $Q_T$, together with Proposition 3.2, 1., we have

$$E_{Q_T}\left[\, 1_{\Pi_T(0,T,x_0)\geq K}\right] = Q_T\left(\frac{\Pi_T\,(0,T,x_0)}{p(T,T)} \geq K\right) = \Phi(d_2),$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function, and

$$\begin{aligned} d_2 &= \frac{\log\left\{\frac{\Pi_0(0,T,x_0)}{K\,p(0,T)}\right\} - \frac{1}{2}\Sigma^2(T)}{\Sigma(T)}, \\ \Sigma^2(T) &= \int_0^T \|v(t)\|^2\, dt = \int_0^T \left\|\int_t^T \sigma^L(t,s)\,ds\right\|^2 dt. \end{aligned}$$

Similarly $\frac{p(t,T)}{\Pi_t(0,T,x_0)}$ is a martingale under $Q_\Pi$ and, by an application of Itô's Lemma, we find that its volatility process is

$$\frac{p(t,T)}{\Pi_t\,(0,T,x_0)}\,\left(v^F(t,T) - v^L(t,T)\right) = -\,\frac{p(t,T)}{\Pi_t\,(0,T,x_0)}\,v(t).$$

Thus, we have

$$E_{Q_\Pi}\left[1_{\Pi_T(0,T,x_0)\geq K}\right] = Q_\Pi\left(\frac{p(t,T)}{\Pi_T\,(0,T,x_0)} \leq \frac{1}{K}\right) = \Phi(d_1)$$

with

$$d_1 = d_2 + \Sigma(T).$$

This, altogether, leads to

$$V_0 = \Pi_0\,(0,T,x_0)\;\Phi(d_1)\;-\;K\,p(0,T)\,\Phi(d_2).$$

Analogously, other derivative instruments can be priced.

# 7 Conclusions

This paper introduces a family of simple longevity bonds which can be used to price and model various mortality derivatives. Similarly to fixed income markets modeling, we assume the existence of an exogenously given longevity bond market, i.e. we assume that longevity bonds of various maturities and various inception dates exist. Given these longevity bonds, we define survival probabilities and forces of mortality using their price processes.

Initially we restrict our considerations to a single inception date, namely 0. We explain how models of longevity bonds can be built specifying either spot force dynamics, bond price dynamics, or forward force dynamics and formulate the relationship between these approaches.

When considering a market of longevity bond with multiple inception dates, different survival rates and forces of mortality can be defined for virtually the same quantities. In order for these definitions to be independent of the inception dates, restrictions on the volatility structure of the mortality bonds are necessary.

Adopting these restrictions or, alternatively, restricting the amount of inception dates to either only longevity bonds starting at zero or longevity bonds with inception dates in the past, we discuss the forward force modeling approach in more detail. In particular, we find a no-arbitrage condition on the drift term which resembles the famous HJM-drift condition from fixed income modeling.

Finally, we briefly discuss appropriate payoff structures for longevity bonds and price an appropriate payoff assuming a Gaussian framework.

In comparison to earlier work on this subject, our approach is more general in various ways: We allow for an interdependence between financial and biometric events, we allow for various inception dates, and we do not focus on a single modeling approach, but rather present a framework for future work. However, there as still several open issues which need to be addressed. Therefore, there are multiple possible paths for future research:

- So far, we only consider a single cohort which the longevity bonds are based on. Therefore, we need to generalize our approach for simultaneously modeling the dynamics for various cohorts. Since an environmental change at time $t$ modeled by a shock of the underlying Brownian motion will not only affect bonds of various maturities but also of all underlying cohorts, we believe that we can use the dynamics as described here and allowing for age-dependent volatilities similar to Schrager (2006) together with high or even infinite dimensional Brownian motions could offer adequate models, however this has to be scrutinized in more detail.

- As pointed out in Section 5, it is not clear how to define and calibrate the initial mortality term structure. Bauer and Russ (2006) propose so-called implied survival probabilities and propose a parametric representation; however, it is not clear whether this representation is consistent in the sense of Filipovic (2001) or if there even are adequate consistent parameterizations.

- In Section 4, we found a volatility restriction in order to allow for all inception dates without adding arbitrage possibilities. However, it is not clear if volatility structures satisfying these conditions are adequate for modeling movements of the mortality term structure. Therefore, the *volatility of mortality* has to be assessed and adequate volatility structures need to be derived.

- Besides adequate volatility structures in the Brownian framework, it is also unclear if models governed by Brownian motion offer a reasonable choice, or if more general driving processes such as for example Lévy processes should be considered.[8]

In spite of these open issues, we believe that the proposed framework is general enough to serve as a solid base for our and other future considerations and provides insights on how continuous stochastic mortality models can be built.

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[8]For HJM models using Lévy drivers see for example Schoutens (2003).

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