On the Pricing of Longevity-Linked Securities☆

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Abstract

For annuity providers, longevity risk, i.e. the risk that future mortality trends differ from those anticipated, constitutes an important risk factor. In order to manage this risk, new financial products, so-called *longevity derivatives*, may be needed, even though a first attempt to issue a longevity bond in 2004 was not successful.

While different methods of how to price such securities have been proposed in recent literature, no consensus has been reached. This paper reviews, compares and comments on these different approaches. In particular, we use data from the United Kingdom to derive prices for the proposed first longevity bond and an alternative security design based on the different methods.

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1. Introduction

Longevity risk, i.e. the risk that the realized future mortality trend exceeds current assumptions, constitutes an important risk factor for annuity providers and pension funds. This risk is increased by the current problems of state-run pay-as-you-go pension schemes in many countries: The reduction of future benefits from public pension systems and tax incentives for annuitization of private wealth implemented by many governments may lead to an increasing demand for annuities (cf. Kling et al. (2008)). One prevalent way of managing this risk is securitization, i.e. isolating the cash flows that are linked to longevity risk and repackaging them into cash flows that

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are traded in capital markets (see Cowley and Cummins (2005) for an overview of securitization in life insurance).

In academic literature, several supporting instruments such as so-called longevity or survivor bonds (cf. Blake and Burrows (2001)) or survivor swaps (cf. Dowd et al. (2006)) have been proposed (see Blake et al. (2006a) for an overview). However, a first attempt to issue a longevity-linked security in 2004 failed (see Section 4 for more information). Nevertheless, the general consensus amongst practitioners appears to be that whilst a breakthrough is yet to come, "*betting on the time of death is set*",[1] and several investment banks such as *JPMorgan* or *Goldman Sachs* have installed trading desks for longevity risk.

Aside from the question of how to appropriately engineer such longevity-linked securities or *longevity derivatives*, there is an ongoing debate of how to price a given instrument. This debate is twofold: On one hand, there is a quest for actuarial or economic methods of how to derive prices "conceptually" and, on the other hand, there is the problem of how to derive prices given a certain methodology – i.e. what data to use for calibration purposes.

In this paper, we address all of these issues. First, we review and compare different pricing methods proposed in recent literature. Subsequently, we empirically compare different approaches based on the established methods using data from the United Kingdom (UK). Moreover, we discuss the financial engineering aspect of longevity securitization and, based on this discussion, introduce and analyze a longevity derivative with option-type payoff. Thus, in addition to a long-needed theoretical and empirical survey on proposed pricing approaches for longevity-linked securities, this paper provides several new ideas which could be helpful in building a market in longevity risk.

The pricing of longevity derivatives is of course closely related to modeling the stochastic evolution of mortality. So far, several stochastic mortality models have been proposed – for a detailed overview and a categorization see Cairns et al. (2006b). We rely on a continuous-time stochastic modeling framework for mortality. Milevsky and Promislow (2001) were among the first to propose a stochastic hazard rate or force of mortality, Dahl (2004) presents a general stochastic model for the mortality intensity, and in Biffis (2005), affine jump-diffusion processes are employed to model both, financial and demographic risk factors. However, while in these articles the spot force of mortality is modeled, we adopt the forward modeling approach for mortality (see Cairns et al. (2006b), Miltersen and Persson (2005), or Bauer et al. (2008)).

The remainder of the paper is organized as follows. In Section 2, we review pricing approaches presented in recent literature, and formalize as well as compare them in a common setting in Section 3. In particular, we point out technical problems with some of these approaches. Subsequently, we present an empirical comparison based on the first announced – but never issued – longevity bond, the so-called EIB/BNP-Bond, in Section 4. Aside from providing valuable insights on the adequacy of the different methods, our results reveal why different authors have come to different appraisals of "how good of a deal" the EIB/BNP-Bond was. However, beyond the pricing, the financial engineering of the Bond may also have been a reason for its failure. We discuss this matter and, in Section 5, introduce and analyze an alternative security design with an option-type payoff structure. Finally, Section 6 concludes.

[1] The Business, 08/15/2007, "Betting on the time of death is set", by P. Thornton.

2. Different Approaches for Pricing Longevity-Linked Securities

The price a party is willing to pay for some longevity derivative depends on both, the estimate of uncertain future mortality trends and the level of uncertainty.[2] This uncertainty is likely to induce a *mortality risk premium* that should be priced by the market (cf. Milevsky et al. (2005)). However, so far, if at all, there are no liquidly traded securities. Therefore, it is not possible to rely on market data for pricing purposes.[3] As a consequence, different methods for pricing mortality risk have been proposed.

Friedberg and Webb (2007) apply the *Capital Asset Pricing Model* (CAPM) and the *Consumption Capital Asset Pricing Model* (CCAPM) to quantify risk premiums for potential investors in longevity bonds, which turn out to be very low. However, the authors acknowledge that there is likely to exist a "mortality premium puzzle" similar to the well-known "equity premium puzzle" (cf. Mehra and Prescott (1985)) implying higher mortality risk premiums than these economic models would suggest. Thus, their quantitative results seem to be of limited applicability and we will not further discuss this approach.

Milevsky et al. (2005) and Bayraktar et al. (2008) develop a theory for pricing non-diversifiable mortality risk in an incomplete market: They postulate that an issuer of a life contingency requires compensation for this risk according to a pre-specified *instantaneous Sharpe ratio* (see also Bayraktar and Young (2008)). They show that their pricing approach has several appealing properties, which are particularly intuitive in a discrete-time setting (cf. Milevsky et al. (2006)). The basic idea is that an additional return in excess of the risk-free rate is paid on the insurer's asset portfolio, which is determined as a multiple λ – the instantaneous Sharpe ratio – of the remaining standard deviation after all diversifiable risk is hedged away. While for a finite number of insureds this approach leads to a non-linear partial differential equation for the price of an insurance contract, the pricing rule becomes linear as the portfolio size approaches infinity (see Milevsky et al. (2005)) – as it is clearly the case for an index-linked longevity derivative. Then, it can be shown that their approach coincides with a change of the probability measure from the physical measure P to a "risk-adjusted" measure Q_λ invoked by a constant "market price of risk" process (cf. Bauer (2008), Subs. 3.2.1).

Similarly, other structures of the market price of mortality risk could be considered, too; basically, we may choose any adapted, predictable process. However, as pointed out by Blake et al. (2006b), "*one can argue that more sophisticated assumptions about the dynamics of the market price of longevity risk are pointless given that, at the time, there was* [is] *just a single item of price data available for a single date (and even that is no longer valid)*." Even if a constant market price of risk is chosen, the question of how to calibrate it remains open, although Milevsky et al. (2006) state that the "*Sharpe ratio from stocks as an asset class*" is approximately equal to $\lambda^{S\&P} = 0.25$ (Loeys et al. (2007) also report a level of 25%).

In order to account for a risk premium, Lin and Cox (2005, 2008) apply the so-called 1-factor and 2-factor Wang transform, respectively, to distort the best estimate cumulative distribution func-

[2]In order to keep our presentation concise, we turn our attention to pricing approaches for index-linked longevity derivatives. In particular, we do not consider indemnity securitization transactions, where small sample risk may need to be taken into account.

[3]Cairns et al. (2006b) calibrate their 2-factor model to published pricing data of the EIB/BNP-Bond. However, we believe that relying on the published data is problematic since pricing may have been an issue for its failure.

tion of the future lifetime of an individual aged x. In order to find a suitable transform for pricing particular longevity securities, i.e. to calibrate the transform parameters, the authors rely on market prices of annuity contracts.

However, as pointed out by Pelsser (2008), the Wang transform does not provide "*a universal framework for pricing financial and insurance risks*" (cf. Wang (2002)), so the adequacy is questionable.[4] Moreover, it is not clear whether annuity prices offer an adequate starting point when pricing longevity derivatives, and it is unclear "*how different transforms for different cohorts and terms to maturity relate to one another and form a coherent whole*" (cf. Cairns et al. (2006b)).

3. Methodological Comparison of the Pricing Approaches

In the previous section, we introduced pricing approaches for longevity-linked securities. In the current section, we establish them in a common framework: We adopt the setup from Bauer et al. (2008), which is summarized in the first subsection. The second subsection then presents pricing formulae for simple longevity bonds under the different approaches. Finally, the last subsection points out potential shortcomings of the transform-based approach from Lin and Cox (2005, 2008).

3.1. The Forward Mortality Framework

For the remainder of this paper, we fix a time horizon T^* and a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, P)$, where $\mathbf{F} = (\mathcal{F}_t)_{0 \leq t \leq T^*}$ is assumed to satisfy the usual conditions. Furthermore, we fix a (large) underlying population of individuals, where each age cohort is denoted by the age x_0 at time zero. We assume that the *best estimate forward force* of mortality with maturity T as from time t,

$$\hat{\mu}_t(T, x_0) := -\frac{\partial}{\partial T} \log \left\{ E_P \left[{}_T p_{x_0}^{(T)} \middle| \mathcal{F}_t \right] \right\} \stackrel{T \geq t}{=} -\frac{\partial}{\partial T} \log \left\{ E_P \left[{}_{T-t} p_{x_0+t}^{(T)} \middle| \mathcal{F}_t \right] \right\}, \tag{1}$$

is well defined, where ${}_{T-t} p_{x_0+t}^{(T)}$ denotes the proportion of $(x_0 + t)$-year olds at time $t \leq T$ who are still alive at time T, i.e. the survival rate or the "realized survival probability". Moreover, we assume that $(\hat{\mu}_t(T, x_0))_{0 \leq t \leq T}$ satisfy the system of stochastic differential equations

$$d\hat{\mu}_t(T, x_0) = \hat{\alpha}(t, T, x_0)\, dt + \hat{\sigma}(t, T, x_0)\, dW_t,\ \hat{\mu}_0(T, x_0) > 0,\ x_0, T \geq 0, \tag{2}$$

where $W = (W_t)_{t \geq 0}$ is a d-dimensional standard Brownian motion independent of the financial market, and $t \mapsto \hat{\alpha}(t, T, x_0)$ as well as $t \mapsto \hat{\sigma}(t, T, x_0)$ are continuous functions. Hence, we are in the "Gaussian case", i.e. the forward force of mortality could become negative for extreme scenarios, but we regard this as an acceptable shortcoming for practical considerations. The drift condition (cf. Bauer et al. (2008)) yields

$$\hat{\alpha}(t, T, x_0) = \hat{\sigma}(t, T, x_0) \times \int_t^T \hat{\sigma}(t, s, x_0)'\, ds,$$

[4] Several other authors also relied on the Wang transform for pricing longevity derivatives. See e.g. Denuit et al. (2007) or Loisel and Serant (2007).

and, by the definition (1), for the $(T-t)$-year best estimate survival probability for an (x_0+t)-year old at time t, we have

$$E_P\left[\left._{T-t}p_{x_0+t}^{(T)}\right|\mathcal{F}_t\right] = E_P\left[\left.e^{-\int_t^T \hat{\mu}_s(s,x_0)\,ds}\right|\mathcal{F}_t\right] = e^{-\int_t^T \hat{\mu}_t(s,x_0)\,ds}.$$

The price of any security is now given as the expected value of its payoff under some equivalent martingale measure (see Harrison and Kreps (1979) or Duffie and Skiadas (1994)), which in turn is defined by its Radon-Nikodym density, say

$$\left.\frac{\partial Q}{\partial P}\right|_{\mathcal{F}_t} = \exp\left\{\int_0^t \lambda(s)'\,dW_s - \frac{1}{2}\int_0^t \|\lambda(s)\|^2\,ds\right\},$$

where – for simplicity – we restrict ourselves to deterministic choices of the "market price of risk" process $(\lambda(t))_{t\geq 0}$. Hence, we have

$$\begin{aligned}
& E_Q\left[\left._{T-t}p_{x_0+t}^{(T)}\right|\mathcal{F}_t\right] \\
= \; & E_Q\left[\left.e^{-\int_t^T \hat{\mu}_s(s,x_0)\,ds}\right|\mathcal{F}_t\right] = E_Q\left[\left.e^{-\int_t^T \hat{\mu}_t(s,x_0)+\int_t^s \hat{\alpha}(u,s,x_0)\,du+\int_t^s \hat{\sigma}(u,s,x_0)\,dW_u\,ds}\right|\mathcal{F}_t\right] \\
= \; & e^{-\int_t^T\int_t^s \hat{\sigma}(u,s,x_0)\,\lambda(u)\,du\,ds} \times E_Q\left[\left.e^{-\int_t^T \hat{\mu}_t(s,x_0)+\int_t^s \hat{\alpha}(u,s,x_0)\,du+\int_t^s \hat{\sigma}(u,s,x_0)\,(dW_u-\lambda(u)\,du)\,ds}\right|\mathcal{F}_t\right] \\
= \; & e^{-\int_t^T\int_t^s \hat{\sigma}(u,s,x_0)\,\lambda(u)\,du\,ds} \times E_P\left[\left._{T-t}p_{x_0+t}^{(T)}\right|\mathcal{F}_t\right],
\end{aligned} \tag{3}$$

since $\tilde{W} = \left(\tilde{W}_t\right)_{t\geq 0}$ with $\tilde{W}_t = W_t - \int_0^t \lambda(u)\,du$ is a Q-Brownian motion by Girsanov's Theorem (see e.g. Theorem 3.5.1 in Karatzas and Shreve (1991)). Therefore, for the "risk-neutral" forward force of mortality with maturity T as from time t,

$$\tilde{\mu}_t(T,x_0) := -\frac{\partial}{\partial T}\log\left\{E_Q\left[\left._{T-t}p_{x_0+t}^{(T)}\right|\mathcal{F}_t\right]\right\} = \hat{\mu}_t(T,x_0) + \int_t^T \hat{\sigma}(s,T,x_0)\,\lambda(s)\,ds, \tag{4}$$

we also have

$$d\tilde{\mu}_t(T,x_0) = \hat{\alpha}(t,T,x_0)\,dt + \hat{\sigma}(t,T,x_0)\,d\tilde{W}_t \tag{5}$$

by Itô's Formula. In particular, for the time zero "price" of a basic (T,x_0)-Longevity Bond, which pays ${}_Tp_{x_0}^{(T)}$ at time T (cf. Cairns et al. (2006b)), we have

$$\Pi_0(T,x_0) := p(0,T)\,E_Q\left[{}_Tp_{x_0}^{(T)}\right] = e^{-\int_0^T \tilde{\mu}_0(s,x_0)+f(0,s)\,ds},$$

where $p(0,T)$ is the time zero price of a zero-coupon bond with maturity T and $f(0,s)$ is the forward force of interest (see e.g. Björk (1999)).

From a practical point of view, this definition could be problematic: In reality, at time T, ${}_Tp_{x_0}^{(T)}$ can only be approximated from a finite amount of data and this approximation may not be available until months or even years after time T (cf. Cairns et al. (2006b)). However, similarly to the EIB/BNP-Bond, payments could be postponed by a deferral period. Moreover, several investment banks are working on establishing real-time longevity indices such as Goldman Sachs' *QxX index* (see http://www.qxx-index.com/) or JPMorgan's *LifeMetrics Index* (see http://www.lifemetrics.com).

It is important to note that we do not assume that (T, x_0)-Longevity Bonds are liquidly available in the market. In fact, when assuming that a sufficient number of (T, x_0)-Longevity Bonds were traded, by the same argumentation as in Heath et al. (1992) for financial bonds, under weak conditions on the volatility structure, the market would be complete. In particular, the measure Q would be uniquely determined, such that basically all longevity derivatives would have a well-determined price – and that is clearly unrealistic. Hence, the quantities $\Pi_0\left(T, x_0\right)$ should be interpreted as technical auxiliary definitions, which rather simplify the presentation than denote prices of securities that are actually traded; this is similar to interest rate modeling, where zero-coupon bonds are employed as the basic building blocks.

Their particular usefulness under our specification becomes evident when regarding Equation (3): When given best estimate dynamics (2), which can be inferred from historic data, it is sufficient to know the "market price of risk" process (function) $\left(\lambda(t)\right)_{t\geq 0}$ in order to specify the "risk-neutral" (pricing) model (cf. Eq. (4) and (5)). By Equation (3), we immediately get

$$\Pi_0\left(T, x_0\right) = p(0,T)\, {}_T\tilde{p}_{x_0} = e^{-\int_0^T \int_0^s \hat{\sigma}(u,s,x_0)\, \lambda(u)\, du\, ds}\, p(0,T)\, {}_T\hat{p}_{x_0}, \tag{6}$$

where ${}_T\hat{p}_{x_0} = E_P\left[{}_Tp_{x_0}^{(T)}\right]$ and ${}_T\tilde{p}_{x_0} = E_Q\left[{}_Tp_{x_0}^{(T)}\right]$. Thus, specifying the "market price of risk" is equivalent to specifying $\Pi_0\left(T, x_0\right)$, i.e. pricing (T, x_0)-Longevity Bonds.

3.2. *Pricing Simple Longevity Bonds*

For pricing a (T, x_0)-Longevity Bond, the idea of Lin and Cox (2005) corresponds to applying the 1-factor Wang transform to the best estimate mortality probability, i.e.

$$1 - {}_T\tilde{p}_{x_0} = \Phi\left(\Phi^{-1}\left(1 - {}_T\hat{p}_{x_0}\right) - \theta\right),$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard Normal distribution and θ is the parameter of the Wang transform. Hence, under this approach we have (cf. Eq. (6))

$$\Pi_0\left(T, x_0\right) = p(0,T)\, \left(1 - \Phi\left(\Phi^{-1}\left(1 - {}_T\hat{p}_{x_0}\right) - \theta\right)\right). \tag{7}$$

A similar equation can be found for the 2-factor Wang transform applied in Lin and Cox (2008).

As pointed out in Section 2, pricing (T, x_0)-Longevity Bonds using a "pre-specified instantaneous Sharpe ratio" $\bar{\lambda}$ in the 1-factor model from Milevsky et al. (2005) corresponds to a constant market price of risk. However, in our model setup, we allow for a multi-dimensional Brownian motion – and thus a multi-dimensional market price of risk process. By an application of Itô's product formula, we obtain

$$\lim_{h\to 0} \sqrt{\frac{1}{h}\mathrm{Var}_P\left[\hat{\mu}_{t+h}(T, x_0) \middle| \mathcal{F}_t\right]} = \sqrt{\sum_{i=1}^{d} \left(\hat{\sigma}_i(t, T, x_0)\right)^2},$$

where clearly $\hat{\sigma}_i$ is the i^{th} component of the volatility function $\hat{\sigma}$. Thus, in the multi-dimensional setting, this approach yields[5]

$$\sum_{i=1}^{d} \hat{\sigma}_i(t,T,x_0)\,\lambda_i(t) = -\bar{\lambda}\sqrt{\sum_{i=1}^{d}\left(\hat{\sigma}_i(t,T,x_0)\right)^2}, \tag{8}$$

which in turn yields (cf. Eq. (6))

$$\Pi_0\,(T,x_0) = p(0,T)\,e^{\bar{\lambda}\int_0^T\int_0^s \|\hat{\sigma}(u,s,x_0)\|\,du\,ds}\;{}_T\hat{p}_{x_0}. \tag{9}$$

Similarly, we could assume $\lambda_i(t)$ to be constant for each $1 \leq i \leq d$, i.e. allow for a different loading for each factor. However, this would yield prices with d rather than one free parameters, such that a comparison of the different models based on the same data would lead to ambiguities. Therefore, in order to be able to compare the approaches, but also to illustrate the basic effects, we limit ourselves to one free parameter and leave the analysis of variations of the loadings across ages and durations for future research.

In order to determine $\bar{\lambda}$, one could use the Sharpe ratio from equity markets. As aforementioned, Milevsky et al. (2006) and Loeys et al. (2007) identify a level of $\bar{\lambda} = 25\%$. However, it is questionable whether relying on equity market data yields a suitable choice. Empirical studies show that the risk premium for stocks is considerably higher than for other securities. Several authors have studied this *equity premium puzzle* (cf. Mehra and Prescott (1985)); for example, relatively recently, Barro (2006) and Weitzman (2007) – using different arguments – have proposed that a heavy-tailed distribution of (log) stock returns yields a coherent explanation. It is rather arguable that the distribution of future mortality is heavy-tailed on the left side of mortality improvements, and hence employing the equity market Sharpe ratio does not seem appropriate. Moreover, traditional financial economic theory attributes (equity) risk premiums to correlation with consumption and, again, it is doubtable that longevity shows similar patterns as stock indices. Hence, we do not share the opinion of, e.g., Loeys et al. (2007) that this approach yields a "*fair compensation for risk*".

Alternatively, we can calibrate the Sharpe ratio $\bar{\lambda}$ to a suitable annuity quote, which would be in the spirit of Lin and Cox (2005). However, again the question arises of whether annuity quotes offer an adequate starting point when pricing longevity bonds: In addition to systemic longevity risk, annuity providers are subject to non-systematic types of mortality risk arising from their finite portfolios of insureds as well as to other sources of risk. Therefore, the overall "per policy" risk of an annuity provider surmounts the per policy pure longevity risk, so that the risk premium within an annuity policy – if it exists – should be greater than or equal to the risk premium for a longevity derivative.

There is strong empirical evidence that market prices of annuities exceed their *actuarially fair price*, i.e. the (best estimate) expected present value. For example, Mitchell et al. (1999) report that in the mid 1990s in the United States (US), the average annuity policy delivers pay-outs of less than 91 cents per unit of annuity premium (Finkelstein and Poterba (2002) make similar ob-

[5]Note that the cumulative market price of risk should be negative since we assume that investors require compensation for taking longevity risk, i.e. the "risk-adjusted" forward force of mortality should be smaller than the "best estimate" counterpart.

servations for the UK). From the policyholder's point of view, the price of the annuity contract may be interpreted as the fair price plus some individual utility premium. From the insurer's perspective, on the other hand, the amount exceeding the fair price may consist of several components. While according to Mitchell et al. (1999) this "*transaction cost*" is primarily due to expenses, profit margins, and contingency funds, Milevsky and Young (2007) do not believe that these fees can be classified "*under the umbrella of transaction costs*". They rather think the fees are "*inseparably linked to aggregate mortality risk*". Weale and Van de Ven (2006) come to a similar conclusion: Annuitization costs due to aggregate longevity uncertainty arise endogenously in their two period overlapping generation general equilibrium model.

Therefore, we conclude that it is very plausible for life annuities to include a risk premium for longevity risk, and there is evidence that this risk premium accounts for a significant part of the amount exceeding the actuarially fair price. In particular, estimates of the longevity risk premium based on annuity prices will at least provide an upper bound for the risk premium to be included in longevity derivative pricing. The question of how close this bound will be depends on the relative influence of other factors on annuity prices, and particularly on the question of whether or not annuities include risk premiums for other sources of risk such as non-systematic mortality risk.

While there does not seem to be a distinct answer, at least for non-systematic mortality risk, we see evidence that this is not the case: On one hand, Brown and Orszag (2006) note that "*insurers are extremely adept at using the LLN* (Law of Large Numbers) *to essentially eliminate the relevance of idiosyncratic risk facing any one individual that they insure*." On the other hand, if there were a charge for non-systematic mortality risk, this charge would clearly depend on the number of insureds within an insurer's portfolio, so that large insurers would have a significant advantage. This, in turn, would eventually lead to a market with only few, large insurers, which clearly is not the case in many developed markets.

Hence, although the validity of the idea by Lin and Cox (2005) to rely on annuity market data for deriving longevity derivative prices cannot be conclusively confirmed, we believe that this approach offers valuable insights at the current stage of the market with little or no longevity derivatives traded and considering the proposed alternatives such as relying on premiums for utterly different types of risks. In particular, the results will at least provide upper bounds for the prices of simple longevity bonds, so together with the actuarially fair price we obtain a price interval for any longevity derivative. However, we still need to address the question of whether their application of the Wang transform is adequate and leads to "*coherent*" prices (cf. Sec. 2).

3.3. Shortcomings of the Approach by Lin and Cox (2005, 2008)

As explained above, the pricing approaches of Lin and Cox (2005, 2008) can be interpreted as transforms of best estimate survival probabilities ${}_T\hat{p}_{x_0}$, $T, x_0 \geq 0$ (or equivalently, the forward force of mortality surface at time zero $(\hat{\mu}_0(T, x_0))_{T,x_0 \geq 0}$), to match observed insurance prices. However, it is not clear if the application of an arbitrary transform is suitable; in particular, the suitability of the Wang transform is of interest.

In order to keep this discussion simple, let us adapt the framework from Subsection 3.1 with the additional assumption that $d = 1$, i.e. that mortality is driven by a one-dimensional Brownian motion. Then, Equation (4) yields

$$\hat{\mu}_0(T, x_0) - \tilde{\mu}_0(T, x_0) = -\int_0^T \sigma(s, T, x_0)\, \lambda(s)\, ds. \tag{10}$$

On the other hand, if "risk-neutral" forward forces are derived from best estimate forward forces via some transform, say $O(\cdot)$, we have

$$O\left(\hat{\mu}_0(\cdot,\cdot)\right)(T,x_0) = \tilde{\mu}_0(T,x_0),$$

i.e. we are given the left-hand side of (10), and in order to derive the implied market price of risk we may solve for $(\lambda(t))_{t\geq 0}$.

For each fixed x_0, (10) is a *Volterra integral equation of the first kind*, and under weak regularity conditions, there exists a unique solution, say $(\lambda_{x_0}(t))_{t\geq 0}$ (see e.g. Subs. 7.2.1.2 in Polyanin and Manzhirov (1999)). Now consider another age group (cohort), say x_1. If there exists some T such that

$$O\left(\hat{\mu}_0(\cdot,\cdot)\right)(T,x_1) \neq \hat{\mu}_0(T,x_1) + \int_0^T \sigma(s,T,x_1)\,\lambda_{x_0}(s)\,ds,$$

then (10) does not have a solution, i.e. the market price of risk process for different ages – but for the same *source of risk* – will be different, which clearly leads to arbitrage opportunities as soon as longevity derivatives on both cohorts are traded. Hence, under the current assumptions, the only *suitable* transforms are of the form

$$O_{\lambda(\cdot)}\left(\hat{\mu}_0(\cdot,\cdot)\right)(T,x_0) := \hat{\mu}_0(T,x_0) + \int_0^T \sigma(s,T,x_0)\,\lambda(s)\,ds$$

for some given $\lambda(\cdot)$ or, equivalently, of the form (3) for the best estimate survival survival probabilities. In particular, the Wang transform does not present a suitable choice in this framework as soon as there are at least two different longevity derivatives traded based on at least two cohorts even when allowing for different transform parameters for different cohorts.

While this inconsistency is only shown under the current assumptions, our results indicate drawbacks associated with the methodology from Lin and Cox (2005, 2008). Nevertheless, for practical applications and when only pricing a longevity derivative based on a single cohort, the Wang transform still presents a theoretically valid approach – the question of whether it is adequate, however, prevails.

4. Empirical Comparison Based on the EIB/BNP Longevity Bond

In this section, we compare the pricing approaches introduced in Section 2 from an empirical perspective adopting the Gaussian forward mortality model proposed by Bauer et al. (2008). Alongside the Wang transform parameters derived by Lin and Cox (2005, 2008) and the instantaneous Sharpe ratio from equity markets (cf. Subs. 3.2), we use a time series of UK annuity market quotes to derive parameter estimates for different points in time.

4.1. "The Volatility of Mortality"

Bauer et al. (2008) propose a volatility specification and, hence, a model for the best estimate forward force of mortality $\hat{\mu}_t(T,x_0)$ in the framework introduced in Subsection 3.1. The model equations are provided in Appendix A; for details on the volatility structure, resulting characteristics of the model, and the calibration algorithm, we refer to their paper. However, since their estimates are based on US pensioner mortality data, for our objective it is necessary to recalibrate the model to UK data. We rely on UK life tables and projections for pension annuities as published on the website of the *Institute of Actuaries and the Faculty of Actuaries*.[6] Details on the employed tables as well as the resulting parameter estimates are also presented in Appendix A.

As explained in Section 3, only a market price of longevity risk $(\lambda(t))_{t\geq 0}$ is still needed to have a fully specified pricing model at hand. This gap will be filled in the following subsection.

4.2. Pricing Methods

In order to compare the pricing approaches by Lin and Cox (2005, 2008) and Milevsky et al. (2005) based on the model introduced in the previous subsection, we derive Sharpe ratios and Wang transform parameters from a monthly time series of UK pension annuity quotes from January 2000 until December 2006.[7] We consider quotes for single premium compulsory pension annuities payable monthly in advance on 65-year old male lives without guarantee period. For each date, we take the average of all available quotes.[8] In order to analyze the risk premium, administrative charges need to be eliminated. We assume up-front charges α between 1.5% and 2% and an investment fee of 7 basis points of the technical reserve p.a. (see also Bauer and Weber (2008)). Moreover, we rely on Woolhouse's summation formula (cf. Bowers et al. (1997)) to derive the corresponding quotes R for annually paying annuities. The resulting present values are then equated with their theoretical counterparts given a certain pricing approach. For example, under the assumption of a constant instantaneous Sharpe ratio, we obtain

$$\begin{aligned}
\pounds 10,000\,(1-\alpha) &= R\sum_{k=0}^{\infty}\frac{1}{(1+i_k-0.0007)^k}\,{}_T\tilde{p}_{x_0} \\
&= R\sum_{k=0}^{\infty}\frac{1}{(1+i_k-0.0007)^k}\,e^{\bar{\lambda}\int_0^k\int_0^s\|\hat{\sigma}(u,s,x_0)\|\,du\,ds}\,{}_T\hat{p}_{x_0}, \qquad (11)
\end{aligned}$$

where i_k is the annualized interest rate for k years, and the second equation follows from Equations (3) and (8).

For each annuity quote, the interest rates are taken from the UK government bonds yield curve for the first working day of the corresponding month as published by the *Bank of England*.[9] Since the yield curve is only given for maturities up to 25 years, we assume a flat yield curve thereafter.

The best estimate survival probabilities ${}_T\hat{p}_{x_0}$ for 2000 to 2006 are obtained from the following mortality tables: For the first set of annuity quotes (01/2000 - 11/2002), we use the 92 Life Office Pensioners' mortality table published by the *Institute of Actuaries and the Faculty of Actuaries* in 1999; in December 2002, adjusted mortality projections were published for this mortality table and, hence, from 12/2002 until 11/2005, we use the newly constructed medium cohort projection which implies stronger mortality improvements than the original projection; for the remaining quotes (12/2005 - 12/2006), we rely on an even stronger projection based on the average of five

[6] http://www.actuaries.org.uk.

[7] Data provided by *Moneyfacts.co.uk Ltd.*

[8] The number of available quotes falls between 9 and 14 for each month.

[9] Downloaded from http://www.bankofengland.co.uk/statistics/yieldcurve/index.htm on 01/13/2008. The interest rates are to reflect the default risk faced by a policyholder, i.e. the risk that he does not receive his annuity payments at some point. We use the UK government bonds yield curve here since such a scenario seems extremely unlikely as the *Financial Services Compensation Scheme* will step in if an insurer is not able to meet its obligations.

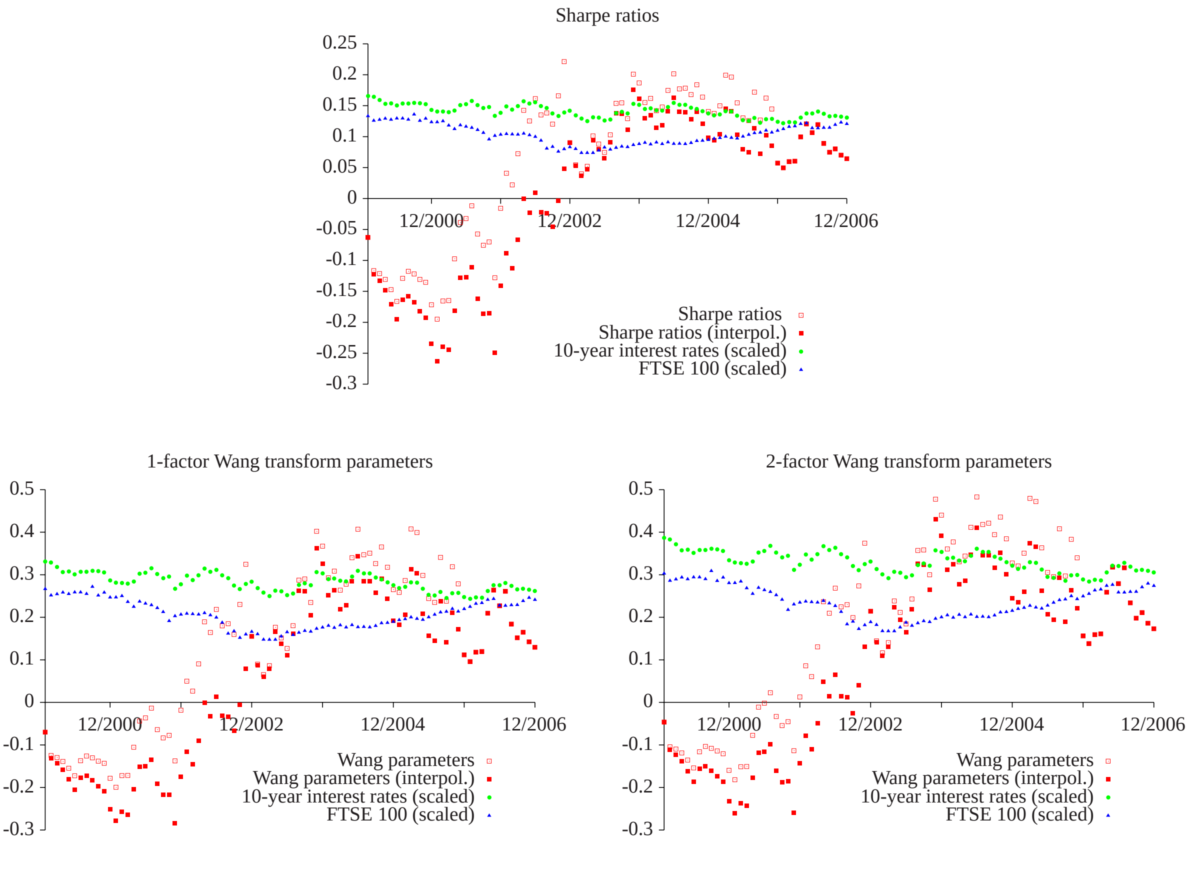

Figure 1: Evolution of the market price of longevity risk

projections used by different large insurance companies at that time as presented by Grimshaw (2007).

However, the underlying assumption of sudden changes in the best estimate survival probabilities when switching from one projection to the next is questionable in our setup: The derivation of a new mortality projection clearly indicates that the existing projection does not yield best estimate probabilities anymore. Therefore, in addition to the assumption of piecewise constant mortality projections, we also derive Sharpe ratios and Wang transform parameters under the assumption of gradual changes in expected mortality, i.e. we interpolate linearly between the original 92 Life Office Pensioners' mortality table for January 2000, the mortality table resulting from the medium cohort projection in December 2002, and the mortality table based on the average projection of large insurance companies in December 2005. In both cases, solving Equation (11) for $\bar{\lambda}$ for each annuity quote gives a time series of Sharpe ratios. For up-front charges of $\alpha = 1.5\%$, the time series are displayed in the top panel of Figure 1, together with the corresponding time series of scaled 10-year interest rates and scaled *FTSE 100* closing prices.[10]

Obviously, the approach of interpolated mortality tables leads to smaller Sharpe ratios due to larger best estimate probabilities and does not exhibit jumps at the regime breaks. Moreover, we observe that, in both cases, the Sharpe ratios implied by the first annuity quotes are negative, which

[10]FTSE 100 data downloaded from http://www.livecharts.co.uk/historicaldata.php on 03/03/2008.

	α	Sharpe ratios	1-factor Wang parameters	2-factor Wang parameters
11/2002	1.5%	0.0483	0.0787	0.1311
	1.75%	0.0427	0.0691	0.1210
	2.0%	0.0371	0.0597	0.1110
11/2004	1.5%	0.1209	0.2438	0.3008
	1.75%	0.1166	0.2333	0.2898
	2.0%	0.1122	0.2228	0.2788
11/2006	1.5%	0.0701	0.1424	0.1860
	1.75%	0.0660	0.1331	0.1764
	2.0%	0.0618	0.1239	0.1667

Table 1: Market prices of longevity risk implied by UK pension annuities

means that, at that time, insurers assumed lower survival probabilities than listed in the respective mortality table and/or higher expected investment returns than the risk-free interest rate: Until 2000, insurers made large profits from equity investments which may have led them to promise rather high yields to policyholders. However, from September 2000 to March 2003, the FTSE 100 crashed by approximately 46%, which then may have forced insurers to lower the offered yields resulting in an increase of the Sharpe ratios. The correlation between the Sharpe ratio and the FTSE 100 during this period is -0.800 in contrast to a correlation between Sharpe ratios and 10-year interest rates of only -0.182. In the subsequent time period until 2006, on the other hand, the correlation between Sharpe ratios and FTSE 100 drops to -0.496 and, hence, insurers seem to have significantly reduced – but not rendered – their exposure to equity returns.

If insurers never changed their annuity quotes, the correlation between the Sharpe ratio and the 10-year interest rate would of course be very high. If insurers used pricing rates that move parallel with interest rate shifts, on the other hand, this correlation would be very close to 0. The observed correlation for the time period from March 2003 to December 2006 is 0.907. To some extent, this might be simply due to a delayed reaction by insurers to changing interest rates as annuity rates usually are not adjusted on a daily basis. Another explanation might be drawn from the competition in the annuity market: In a scenario of falling interest rates, no insurer may want to be the first to reduce the annuity rate as the company would loose out on new business. On the other hand, if interest rates increase, the insurer might want to keep a larger profit margin by not immediately increasing its annuity rate. All these effects would result in annuity quotes somewhat smoothing out interest rate movements.

As mentioned above, the Sharpe ratios displayed in Figure 1 have been derived under the assumption of an up-front charge parameter $\alpha = 1.5\%$. Assuming larger up-front charges obviously results in smaller Sharpe ratios. However, the overall influence of the parameter α is rather small. In Table 1, the Sharpe ratios for November 2002, November 2004 and November 2006 are displayed for different choices of α. We find that that increasing α from 1.5% to 2.0% reduces the Sharpe ratio by only about 0.01, i.e. the influence of the up-front charge is not very pronounced. Moreover, we observe that, for all three dates, the Sharpe ratios implied by UK pension annuities are significantly smaller than the one of equity markets.

In an analogous fashion to the Sharpe ratio approach, values for the 1-factor and 2-factor Wang

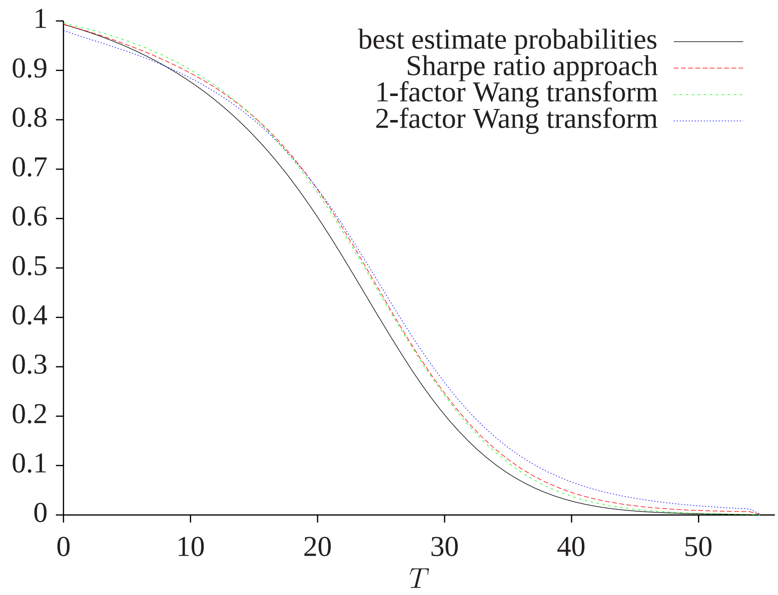

Figure 2: Comparison of risk-adjusted survival probabilities for a 65-year old

transform parameters, i.e. θ_1 and θ_2, can be derived from the UK pension annuity quotes. The respective values are also displayed in Figure 1 and Table 1. The observations regarding the correlation with the financial market as well as the influence of up-front charges are very similar to those based on Sharpe ratios.

In Section 3, we raised the question of whether or not the Wang transform is adequate for pricing longevity-linked securities. A first indication that the Wang transform may not be adequate is given in Figure 2, where T-year risk-adjusted survival probabilities for a 65-year old in 2006 are displayed.[11] We observe that the 1-factor Wang transform allocates much of the risk premium to shorter maturities and less to longer maturities, whereas for the 2-factor Wang transform, this relation is inversed as much of the risk premium is allocated to longer maturities. In turn, for the first years, the risk-adjusted survival probabilities are even smaller than the best estimates. For the Sharpe ratio, on the other hand, the allocated risk premium is proportional to the aggregate risk.

In what follows, we compare the different pricing approaches based on the announced – but never issued – EIB/BNP-Bond. By considering both pricing approaches with different parameters and also taking into account the best estimate value of the bond to obtain intervals for the bond's fair price (cf. Subs. 3.2), we apply the following seven pricing alternatives:

- **BE**: Best estimate valuation;
- **SRUK**: Sharpe Ratio approach / from UK annuity quotes;
- **SRLOE**: Sharpe Ratio approach / from equity markets (cf. Cairns et al. (2005) and Loeys et al. (2007));

[11]The graphs are based on the Sharpe ratio and the Wang transform parameters as implied by the last annuity quote with $\alpha = 1.5\%$, i.e. $\bar{\lambda} = 0.0644$, $\theta_1 = 0.1296$ and $\theta_2 = 0.1731$, and the best estimate survival probabilities for December 2006 as listed in the mortality table with the average projection of large insurance companies.

- **1WTUK**: 1-Factor Wang transform / from UK annuity quotes;
- **1WTLC**: 1-Factor Wang transform / from Lin and Cox (2005);
- **2WTUK**: 2-Factor Wang transform / from UK annuity quotes;
- **2WTLC**: 2-Factor Wang transform / from Lin and Cox (2008).

4.3. Comparison of Different Assessments of the EIB/BNP-Bond

In November 2004, *BNP* announced the issuance of the first publicly traded longevity bond, the so-called EIB/BNP-Bond. While it was withdrawn for redesign in 2005, it still has attracted considerable attention in academia as well as among practitioners.[12]

The notes were to be issued by the *European Investment Bank* (EIB), and the longevity risk was to be taken by the Bahama-based reinsurer *Partner Re*. BNP was the originator and structurer of the deal. The basic design is quite simple: Investors in the bond were entitled to receive variable coupon payments contingent on the mortality experience of English and Welsh males aged 65 in 2003. The coupons $C(t)$, $t = 1, 2, ..., 25$, were set to

$$C(t) = S(t)\ \pounds 50\text{mn, where } S(t) = S(t-1)\,(1 - m(64+t, 2002+t)) \text{ and } S(0) = 1.$$

Here, $m(x, z)$ denotes the central death rate for an x-year old in year z as published by the *Office for National Statistics*. Note that there is a time lag of two years between the end of the reference period and the payment date since the mortality experience has to be assessed statistically.

Thus, holding such a security is similar to holding a portfolio of $(k, 65)$-longevity bonds as introduced in Section 3 for $k = 1, ..., 25$ in 2003, but the payments are delayed by two years and the definition here is based on central death rates rather than mortality probabilities. Now, if more individuals survive than anticipated, coupon payments will be higher; thus, for the next 25 years, the notes serve as an almost perfect hedge for an annuity provider whose portfolio of insureds coincides with the reference population.

The total "value" of the issue was £540mn, and the offer price was determined by taking survival rates as projected by the *UK Government Actuary's Department* and discounting the projected coupon payments at LIBOR minus 35bps. Since, due to its S&P AAA rating, the EIB's yield curve averaged at approximately LIBOR-15, the remaining spread of about 20bps can be interpreted as the fee investors had to pay for the hedge. Lin and Cox (2008) believe that the charged risk premium is very high making the bond unattractive for potential investors. Contrarily, Cairns et al. (2005) figure that this price seems reasonable, even though "*it is difficult to judge precisely how good a deal the pension funds are* [were] *being offered*". It is worth noting that the authors base their conclusions on similar pricing methods to those introduced in Section 2: While Lin and Cox (2008) rely on comparisons of "risk premiums" implied by annuity prices and the EIB/BNP-Bond as parameters in corresponding Wang transforms, Cairns et al. (2005) take the EIB/BNP risk-premium relative to its "volatility" – i.e. the Sharpe ratio – and compare it to equity risk premiums. Thus, their method is in line with the ideas of Milevsky et al. (2005).

[12] The following short description of the security is based on Azzopardi (2005), Blake et al. (2006a), and Cairns et al. (2005). See their presentations for more details.

	11/2002	11/2004	11/2006
Actual	*na*	540	*na*
BE	512.80	528.85	548.15
SRUK	520.25	550.33	561.68
SRLOE	555.10	576.16	600.94
1WTUK	527.16	569.67	572.84
1WTLC	544.75	559.42	578.89
2WTUK	526.83	566.71	568.49
2WTLC	530.36	544.36	563.23

Table 2: Prices of EIB-type bonds using different pricing methods

In order to explain these different assessments of the EIB/BNP-Bond, we apply the seven pricing methods defined in the previous subsection. In addition, we price two "artificial" bonds of the same structure but assuming that they would have been offered two years earlier and two years later, i.e. in November 2002 and November 2006, respectively, in order to analyze the influence of different Sharpe ratios and Wang transform parameters on the bond price. The exact issue dates are assumed to be the 18th of the respective months and the corresponding Sharpe ratios and Wang transform parameters are shown in Table 1. The best estimate survival rates are taken from the mortality projections of the *UK Government Actuary's Department* from 2001, 2003 and 2004, i.e. the most recent mortality projection in each case, and the reference cohort is aged 65 in 2001, 2003 and 2005, respectively. The interest rates are taken from the EIB's yield curve at the respective dates.[13] The resulting bond prices are shown in Table 2.

We observe that the bond prices for the same pricing method differ considerably between the issue dates. This is due to different interest rates and changes in the mortality projections as well as to changes in the Sharpe ratio and Wang transform parameters in case these have been derived from the UK annuity quotes. There are also significant differences between the pricing methods for the same issue date resulting from the varying risk premium allocations and/or market prices of risk. However, the most striking observation is that, for all six "risk-adjusting" pricing methods, the price of the EIB/BNP-Bond exceeds the £540mn requested by BNP. Hence, our analyses indicate that the bond was indeed a rather "good deal". Even when assuming that only a part of the mark-up within annuity quotes can be attributed to aggregate longevity risk, the fact that the suggested price lies approximately in the middle of best estimate and "risk-adjusted" values based on annuity prices suggests that the pricing was at least reasonably fair. From this observation, two questions arise immediately:

- Why do Lin and Cox (2008) regard the bond as too expensive?
- And why was the bond not successfully placed?

The first question is quite simple to answer as Lin and Cox (2008) use different interest rates (gilt STRIPS) and, in particular, different best estimate survival rates, namely rates based on "*realized mortality rates of English and Welsh males aged 65 and over in 2003*". Especially in the long

[13]Data downloaded from Bloomberg on 10/24/2008.

term, these survival rates are significantly smaller than those projected by the UK Government Actuary's Department resulting in a lower "fair" price of the bond.

The second question is more difficult to answer. One reason might be the fact that the bond was based on population mortality experience of a specific cohort rather than annuitants' mortality experience implying basis risk (cf. Lin and Cox (2008)). However, as exhibited by Cairns et al. (2005), differences in mortality improvements between the general and the assured population are not very pronounced, even though this observation may differ for a particular insurer's portfolio. Other potential reasons have been pointed out, e.g. that the Bahama-based reinsurer Partner Re was not perceived to be a natural holder of UK longevity risk. However, probably the most striking explanations lie in the fixed maturity and the high upfront capital expense (see Cairns et al. (2005)): Due to the fixed maturity of 25 years, insurers and pension funds purchasing the bond would still be stuck with the tail risk, i.e. the longevity risk for high ages in the far future; moreover, the securitization of the "complete" survivor index takes capital away which may be used to hedge other sources of risk or to speculate in capital markets. After all, insurers are financial service providers meaning that investing in capital markets can be regarded as one of their core competences.

To sum up, aside from basis risk, the financial engineering of the EIB/BNP-Bond may present an important reason for its failure. Thus, in the next section, we propose a differently designed longevity derivative, which overcomes some of the stylized deficiencies.

5. An Option-Type Longevity Derivative

As explained in the previous section, the EIB/BNP-Bond may not seem very attractive to insurers. Therefore, we propose a differently designed derivative here: a call option-type longevity derivative with a payoff of the form[14]

$$C_T = \left({}_T p_{x_0}^{(T)} - K(T) \right)^+$$

at time T, where $K(T)$, $0 \leq K(T) \leq 1$, is some threshold or *strike*, for example

$$K(T) = (1+a)\, E_P \left[{}_T p_{x_0}^{(T)} \right],\ a > 0.$$

This security or a combination of securities of this type overcome several deficiencies of the EIB/BNP-Bond and could thus be more appealing to insurers. For example, the insurer keeps the "equity tranche" of the longevity risk exposure in its own books and only passes over the risk of extreme longevity. In particular, this will significantly decrease the committed capital. We refer to Bauer (2008), Section 5.1, for a more detailed discussion.

Within the setup introduced in Section 3, such a derivative can be priced by a Black-type formula (see Bauer (2008)). One only needs to provide the volatility of mortality, the best estimate and the risk-adjusted T-year survival probability for an x_0-year old today, and the price of a zero-coupon bond with maturity T. In Table 3, prices of this option-type longevity derivative are presented for a 65-year old in December 2006, the seven pricing methods defined in Subsection 4.1, and various combinations of maturity T and strike parameter a. The best estimate survival

[14]This derivative has already been introduced in Bauer (2008) and similar payoff structures can also be found in, e.g., Blake et al. (2006a) or Lin and Cox (2005).

a		$T=5$	$T=10$	$T=15$	$T=20$	$T=25$	$T=30$	Portfolio
2%	BE	0.00197	0.01513	0.02911	0.03424	0.02759	0.01555	0.67562
	SRUK	0.00247	0.01943	0.03912	0.04840	0.04122	0.02501	0.99593
	SRLOE	0.00450	0.03605	0.07968	0.11015	0.10587	0.07515	3.84770
	1WTUK	0.00422	0.02204	0.03936	0.04666	0.03978	0.02415	0.97330
	1WTLC	0.00524	0.02473	0.04336	0.05166	0.04495	0.02800	1.09830
	2WTUK	0.00073	0.01615	0.03679	0.04826	0.04425	0.02982	1.06686
	2WTLC	0.00063	0.01532	0.03537	0.04633	0.04217	0.02825	1.01921
5%	BE	0.00024	0.00934	0.02409	0.03082	0.02567	0.01473	0.57976
	SRUK	0.00033	0.01245	0.03299	0.04413	0.03875	0.02390	0.87722
	SRLOE	0.00078	0.02547	0.07048	0.10357	0.10194	0.07332	3.66316
	1WTUK	0.00071	0.01440	0.03321	0.04248	0.03737	0.02306	0.84660
	1WTLC	0.00097	0.01645	0.03681	0.04721	0.04235	0.02682	0.96031
	2WTUK	0.00006	0.01006	0.03091	0.04299	0.04167	0.02860	0.95853
	2WTLC	0.00005	0.00947	0.02964	0.04217	0.03966	0.02706	0.91453
10%	BE	0.00024	0.00368	0.01726	0.02576	0.02275	0.01345	0.46925
	SRUK	0.00033	0.00525	0.02442	0.03768	0.03492	0.02216	0.73538
	SRLOE	0.00078	0.01283	0.05663	0.09316	0.09559	0.07033	3.42618
	1WTUK	0.00071	0.00628	0.02460	0.03619	0.03362	0.02136	0.70126
	1WTLC	0.00097	0.00742	0.02757	0.04048	0.03831	0.02495	0.80209
	2WTUK	0.00006	0.00404	0.02272	0.03756	0.03767	0.02666	0.82222
	2WTLC	0.00005	0.00375	0.02170	0.03592	0.03577	0.02519	0.78280

Table 3: Prices of call-option-type payoff using different pricing approaches

probabilities and the interest rates are the same as in the analysis of the last annuity quote (from December 2006). "Portfolio" refers to a combination of longevity bonds for each maturity T, $1 \leq T \leq 55$, i.e. a portfolio which completely hedges a provider of a life-long annuity for a 65-year old with annual payments against extreme longevity risk.[15]

Obviously, the prices decrease with increasing strike parameter a. The equality of prices for $T = 5$ and $a = 5\%$ and 10% is due to the strike being capped at 1. As expected, the bond prices initially increase with maturity as the realized survival probability is more likely to significantly exceed today's expectation in the farther future. However, around $T = 20$ this trend is reversed since the absolute value of the survival probability becomes very small thereafter. We also observe that, compared to the Sharpe ratio approach, the 2-factor Wang transform leads to significantly lower prices for short maturities and higher prices for longer maturities as a result of the disproportional risk premium allocation (cf. Subs. 4.2). For instance, for $T = 5$, the prices obtained by the Sharpe ratio approach are almost five times as high as the prices obtained using the 2-factor Wang transform. Moreover, the 2-factor Wang transform prices are considerably smaller than the best estimate value of such a derivative which again questions the adequacy of the Wang transform for pricing longevity-linked securities. The differences between the 1-factor Wang transform and

[15]Note that we assume a limiting age of 120 as implied by the underlying mortality table.

the Sharpe ratio approach are not quite as strong but still significant.

In order to assess the question of "how good of a deal" this derivative is from the insurer's perspective, we compare the price of a portfolio of our option-type longevity derivatives with the price of a life-long immediate annuity. The average price of the annuity paying £1 annually in advance in December 2006 was £14.92. Hence, for a rather low strike parameter ($a = 2\%$) and a Sharpe ratio implied by UK pension annuities, an insurer would have to pay approximately 6.7% of the single premium for a full coverage against extreme longevity improvements. Since the company would hardly be exposed to longevity risk anymore while, at the same time, profiting from realized longevity improvements below today's expectation, the price appears to be very reasonable, particularly if the value is interpreted as the upper bound. In case the company is willing to accept a larger portion of the risk, e.g. for $a = 10\%$, the (maximal) cost for the hedge can be reduced to approximately 4.9% of the single premium.

Moreover, an insurer could reduce the cost of longevity coverage by incorporating the company's mortality exposure from the sale of life insurance policies. This idea of compensating longevity risk by mortality risk is often referred to as *natural hedging* (see, e.g., Bayraktar and Young (2007), Cox and Lin (2007), or Wetzel and Zwiesler (2008)). Although such a compensation is only partial since the age cohorts exposed to these risks are typically different and the maturities of annuities generally exceed those of life insurance policies, the number of longevity-linked securities necessary to reduce the longevity risk to a desired level can certainly be reduced.

6. Conclusion

The current paper analyzes and compares different approaches for pricing longevity-linked securities. Milevsky et al. (2006) postulate that an issuer of such a security should be compensated for taking longevity risk according to a pre-specified instantaneous Sharpe ratio. Lin and Cox (2005, 2008), on the other hand, apply the Wang transform to best estimate death probabilities in order to account for a risk premium.

However, as explained in detail in Section 3, the risk premium implied by the Wang transform is not consistent – arbitrage opportunities may arise as soon as multiple securities based on different cohorts are traded. Even if only one security is considered, the disproportional risk premium allocation contests the adequacy of an application of the Wang transform for pricing longevity derivatives (cf. Subs. 4.2 and Sec. 5).

Moreover, we discuss the derivation of an adequate market price of longevity risk, i.e. a Sharpe ratio or Wang transform parameters in our case. Milevsky et al. (2005) do not address this question but Loeys et al. (2007) propose to adopt a Sharpe ratio from equity markets. We believe that this approach is not appropriate since peculiarities of equity markets may not be present in the longevity market. In contrast, Lin and Cox (2005, 2008) obtain their Wang transform parameters from US annuity quotes which is more sound from our point of view.

We compare the different pricing approaches based on UK data. In particular, we derive a time series for the market price of risk within market annuity quotes and analyze the relationship to interest rates and the stock market. We find considerable correlations indicating that the independence assumption of the risk-adjusted mortality evolution and the development of the financial market may not be adequate.

We then apply the different approaches to assess the first announced – but never issued – longevity bond, the so-called EIB/BNP-Bond. For each of the considered "risk-adjusting" methods, the price exceeds the £540mn quoted by BNP, i.e. the Bond appears to have been a rather "good deal". Even if the "risk-adjusted" prices are interpreted as upper bounds for the EIB/BNP-Bond's fair value with the best estimate value as the lower bound, the price quoted by BNP appears at least reasonably fair. Consequently, there must have been other reasons for its failure, for example the financial engineering.

We propose a differently designed and potentially more suitable security, namely an option-type longevity derivative. Such a security allows an insurer to keep the "equity tranche" of the longevity risk in the company's own books. We derive prices for derivatives of this kind based on the different methods and compare the results. All in all, we believe that such option-type longevity derivatives could become successful tools for securitizing longevity risk.

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Appendix

A. The Gaussian Mortality Model from Bauer et al. (2008)

Model equations

As noted in Subsection 4.1, the forward model is fully specified by the corresponding volatility structure $(\sigma(t,T,x_0))$. The Gaussian model presented in Bauer et al. (2008) is a six factor model, i.e. the volatility structure is of the form $\sigma(t,T,x_0) = (\sigma_1(t,T,x_0),\ldots,\sigma_6(t,T,x_0))$, where the single components attributed to certain effects are of the following form:

$$\begin{array}{ll}
\text{general:} & \sigma_1(t,T,x_0) = c_1 \times \exp\left(a(x_0+T)+b\right); \\
\text{short-term:} & \sigma_2(t,T,x_0) = c_2 \times \exp\left(a(x_0+T)+b\right) \times \exp\left(\log(0.1)(T-t)\right); \\
\text{young age:} & \sigma_3(t,T,x_0) = c_3 \times \exp\left(a(x_0+T)+b\right) \\
 & \quad \times \exp\left(\frac{\log(0.5)}{20^2}(T-t-20)^2 + \frac{\log(0.5)}{17.5^2}(x_0+T-37.5)^2\right); \\
\text{middle age:} & \sigma_4(t,T,x_0) = c_4 \times \exp\left(a(x_0+T)+b\right) \\
 & \quad \times \exp\left(\frac{\log(0.5)}{20^2}(T-t-20)^2 + \frac{\log(0.5)}{12.5^2}(x_0+T-67.5)^2\right); \\
\text{old age:} & \sigma_5(t,T,x_0) = c_5 \times \exp\left(a(x_0+T)+b\right) \\
 & \quad \times \exp\left(\frac{\log(0.5)}{20^2}(T-t-20)^2 + \frac{\log(0.5)}{30^2}(x_0+T-110)^2\right); \\
\text{long-term:} & \sigma_6(t,T,x_0) = c_6 \times \exp\left(a(x_0+T)+b\right) \\
 & \quad \times \exp\left(\frac{\log(0.5)}{80^2}(T-t-120)^2\right).
\end{array}$$

Underlying Tables for the Calibration of the Mortality Model

- Period table PA(90)m projected backward (until 1968) and forward using the projection applied in constructing this table;[16]
- Basic table PMA80 and projection as published in the "80" Series of mortality tables;
- Basic table PMA92 and projection as published in the "92" Series of mortality tables;
- Basic table PMA92 and the medium cohort projection published in 2002;
- Basic table PNMA00 from the "00" Series of mortality tables and the Lee-Carter projection based on data up to 2003 (part of the *Library of mortality projections*);

[16] The PA(90)m table is a projected version of a period table from 1968. Hence, the best estimate mortality forecast in 1968 in form of a generation table are based on the period table for 1968 and the aforecited mortality projection.

- Basic table PNMA00 from the “00” Series of mortality tables and the Lee-Carter projection based on data up to 2004 (part of the *Library of mortality projections*);
- Basic table PNMA00 from the “00” Series of mortality tables and the Lee-Carter projection based on data up to 2005 (part of the *Library of mortality projections*).

Calibration results

According to the notation in Bauer et al. (2008), we deployed the data points

$$(T-t, x_0+T) \in \{(0,30),\ (0,70),\ (0,110),\ (30,70),\ (30,110),\ (90,110)\}\,.$$

Moreover, we chose the slope parameter of the correction term to be $a = 0.9$ since this is the average value over all tables of the slope parameter in Gompertz forms fitted to the cohort mortality of a 20-year old in the respective basis years. The parameter b is chosen as -10. The resulting values for the volatility parameters are:

Parameters	c_1	c_2	c_3	c_4	c_5	c_6
	0.007	0.287	0.144	0.284	0.021	0.058