Stochastic representations of max-type functionals from random walk

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Formulation of the problem

Consider the random walk

$$S_k = \xi_1 + \ldots + \xi_k, \ \mathsf{P}(\xi_i = 1) = \mathsf{P}(\xi_i = -1) = 1/2$$

on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ with natural filtration $\mathcal{F}_n = \sigma(\xi_i, i \leqslant n)$, $\mathcal{F}_\infty = \sigma(\bigcup_{n \geqslant 0} \mathcal{F}_n)$.

For functionals $F = F(\omega)$ from random walk it required to make a

- **single representations**

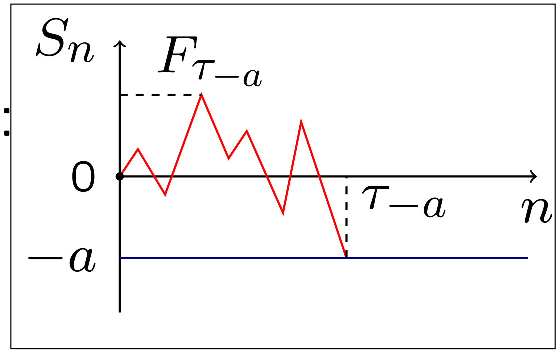
  $$\boxed{F = c_F + \sum_k u_{k-1} \Delta S_k,}$$

  where $\Delta S_k = S_k - S_{k-1}$, $u_{k-1} \sim \mathcal{F}_{k-1}$ – measurable process.

- **multiple representations**

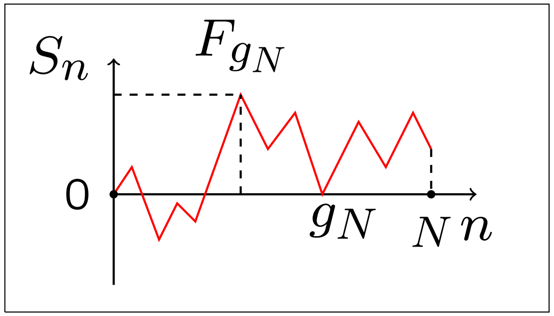
$$F = c_F + \sum_{k\geqslant 1} \sum_{t_1<\ldots<t_k} c_k(t_1,\ldots,t_k)\Delta S_{t_1}\ldots\Delta S_{t_k},$$

where $c_k(t_1,\ldots,t_k)$ is **deterministic** function.



There were considered the following functionals:

- $F_N = \max_{0\leqslant k\leqslant N} S_k$;
- $F_{\tau_{-a}} = \max_{0\leqslant k\leqslant \tau_{-a}} S_k$,
  where $\tau_{-a} = \inf\{k \geqslant 0 : S_k = -a\}$, $a \in \mathbb{N}$.
- $F_{g_N} = \max_{0\leqslant k\leqslant g_N} S_k$,
  where $g_N = \sup\{0 < k \leqslant N : S_k = 0\}$.



Single representations

1. Case $F_N = \max_{0 \leqslant k \leqslant N} S_k$

Theorem 1. *For* $F_N = \max_{0 \leqslant k \leqslant N} S_k$ *we have the following stochastic representation:*

$$\boxed{\max_{0 \leqslant n \leqslant N} S_n = \mathsf{E}(\max_{0 \leqslant n \leqslant N} S_n) + \sum_{k=1}^{N} G_{N-k+1}(F_{k-1} - S_{k-1})\Delta S_k,} \qquad (1)$$

where

$$G_{N-k+1}(F_{k-1}-S_{k-1}) = \sum_{r=F_{k-1}-S_{k-1}+1}^{N-k+1} \sum_{l=0}^{\left[\frac{N-k+1-r}{2}\right]} (-1)^{\left[\frac{N-k+1-r}{2}\right]-l} \times \binom{-1/2}{[(N-k+1-r)/2]-l} \varphi_{r+2l}(r),$$

$$\varphi_k(m) = \frac{m}{k2^k}\binom{k}{\frac{m+k}{2}}, \quad \mathsf{E}(\max_{0\leqslant n\leqslant N} S_n) \sim \sqrt{\frac{2N}{\pi}} \quad \text{by} \quad N\to\infty.$$

In continuous case for $F_t = \max_{0\leqslant s\leqslant t} B_s$, $t \leqslant T$

$$\max_{0\leqslant t\leqslant T} B_t = \mathsf{E}(\max_{0\leqslant t\leqslant T} B_t) + 2\int_0^T \left[1-\Phi\left(\frac{F_t-B_t}{\sqrt{T-t}}\right)\right] dB_t$$

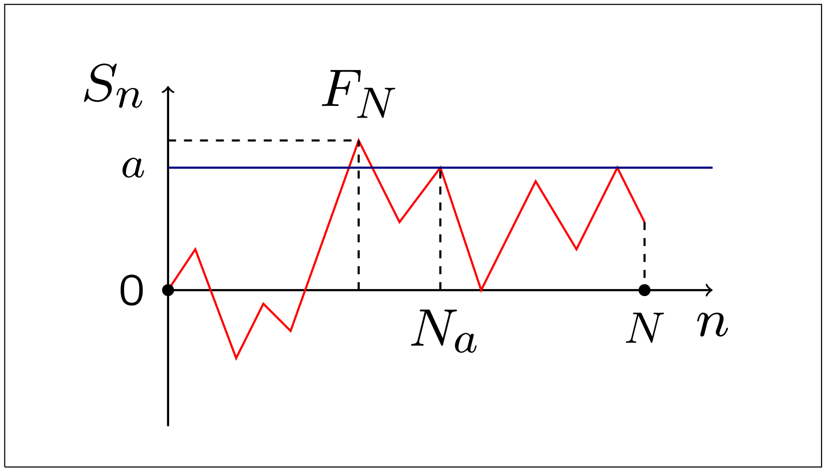
(**A. N. Shiryaev, M. Yor**)

Proof.

- $\xi \geqslant 0 \Rightarrow \xi = \int_0^{+\infty} I(\xi \geqslant a)\,da$. Hence

$$F_N = \int_0^{+\infty} I(\max_{0 \leqslant k \leqslant N} S_k \geqslant a)\,da = \int_0^{+\infty} I(N_a \leqslant N)\,da,$$

where $N_a = \inf\{k \geqslant 0 : S_k \geqslant a\}, \quad a \geqslant 0.$



Conclusion: sufficiently to receive the representation for $I(N_a \leqslant N)$.

• **Discrete stochastic exponent** for $\lambda \in (0,1)$:

$$\mathcal{E}_n(S) = \prod_{k=1}^{n} (1 + \lambda \Delta S_k) = (\sqrt{1-\lambda^2})^n \left(\sqrt{\frac{1+\lambda}{1-\lambda}}\right)^{S_n}, \ \mathcal{E}_0(S) := 1.$$

• $\Delta \mathcal{E}_n(S) = \lambda \mathcal{E}_{n-1}(S) \Delta S_n$ and $\mathcal{E}_n(S)$ – martingale with respect to $(\mathcal{F}_n)_{n \geqslant 0}$.

• From the **optional sampling theorem** (P-a. s.)

$$\mathsf{E}(\mathcal{E}_{N_a} | \mathcal{F}_n) = \mathcal{E}_{N_a \wedge n} = 1 + \lambda \sum_{k=1}^{N_a \wedge n} \mathcal{E}_{k-1} \Delta S_k, \quad (2)$$

$$\mathsf{E}\mathcal{E}_{N_a} = 1 \Rightarrow \mathsf{E}(\sqrt{1-\lambda^2})^{N_a} = \left(\sqrt{\frac{1+\lambda}{1-\lambda}}\right)^{-\lceil a \rceil}, \quad (3)$$

$$\left(\sqrt{\frac{1+\lambda}{1-\lambda}}\right)^{-\lceil a \rceil} = \sum_{l=0}^{+\infty} (\sqrt{1-\lambda^2})^{\lceil a \rceil + 2l} \varphi_{\lceil a \rceil + 2l}(\lceil a \rceil). \quad (4)$$

From (2), (3), (4) we get

$$\mathsf{E}((\sqrt{1-\lambda^2})^{N_a}|\mathcal{F}_n) = \mathsf{E}(\sqrt{1-\lambda^2})^{N_a} + $$
$$+\lambda \sum_{k=1}^{N_a \wedge n} \sum_{l=0}^{+\infty} (\sqrt{1-\lambda^2})^{\lceil a \rceil - S_{k-1} + 2l + k - 1} \varphi_{\lceil a \rceil - S_{k-1} + 2l}(\lceil a \rceil - S_{k-1}) \Delta S_k,$$

- Denote $s := \sqrt{1-\lambda^2}$. Then for all $s \in (0,1)$ we have

$$\mathsf{E}(s^{N_a}|\mathcal{F}_n) = \mathsf{E}s^{N_a} + \sum_{k=1}^{N_a \wedge n} \sum_{l,m=0}^{+\infty} (-1)^m C_{\frac{1}{2}}^m \varphi_{\lceil a \rceil - S_{k-1} + 2l}(\lceil a \rceil - S_{k-1})$$
$$\times s^{\lceil a \rceil - S_{k-1} + 2(m+l) + k - 1} \Delta S_k \quad (5)$$

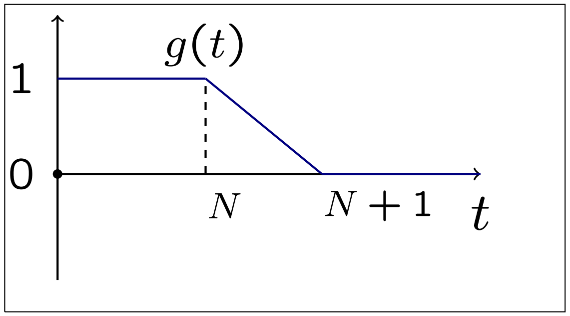
$$\left.\begin{array}{l} f_s(x)=s^x,\ x\geqslant 0, s\in(0,1) \\ s=e^{-n},\ \ n=1,2,..., \end{array}\right\} \Rightarrow \text{(5) is valid for } \mathcal{L}=\{e^{-nx}\}_{n\geqslant 0}.$$

$$C_0[0,+\infty) = \{f\in C[0,+\infty) : f(x)\to 0 \textbf{ by } x\to+\infty\}$$

$$\Downarrow_{(*)}$$

$$\mathcal{L}=\{e^{-nx}\}_{n\geqslant 0} \textbf{ is complete in } C_0[0,+\infty).$$

Consider the function



$$\left.\begin{array}{l} g(t)\in C_0[0,+\infty) \\ g(n) = I(n\leqslant N) \end{array}\right\} \Rightarrow \text{(5) is valid for } I(n\leqslant N).$$

(*) Sedletski A. M. *Classes of analityc Fourier transformations and their exponential approximations.* Moscow: Physmatlit, 2005, 504 p.

Lemma 1. *For $I(N_a \leqslant N)$ we have the representation* (P-*a.s.*)

$$I(N_a \leqslant N) = \mathsf{P}(N_a \leqslant N) + \sum_{k=1}^{N_a \wedge N} \sum_{l,m:\ l+m \leqslant \left[\frac{N-k+1}{2}\right]} (-1)^m C_{\frac{1}{2}}^m$$

$$\times \varphi_{\lceil a \rceil - S_{k-1} + 2l}(\lceil a \rceil - S_{k-1}) I(\lceil a \rceil - S_{k-1} \leqslant N - k + 1 - 2(l+m)) \Delta S_k.$$

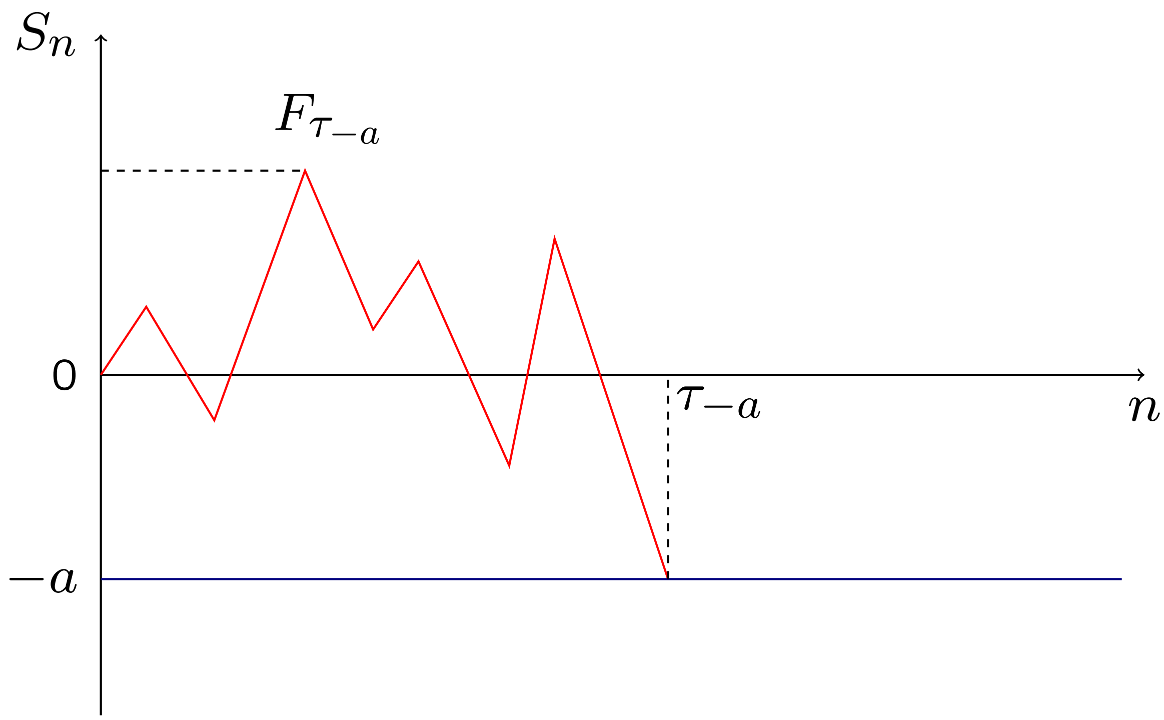
The decision formula is established by integrating on a.

$$F_N = \mathsf{E} F_N + \sum_{k=1}^{N} G_{N-k+1}(F_{k-1} - S_{k-1}) \Delta S_k.$$

$\square$

2. Case $F_{\tau_{-a}} = \max_{0 \leqslant k \leqslant \tau_{-a}} S_k$,

$$\tau_{-a} = \inf\{k \geqslant 0 : S_k = -a\}.$$



Theorem 2. *For $F_{\tau_{-a}}$ we have a stochastic representation*

$$\boxed{\max_{0\leqslant k\leqslant \tau_{-a}} S_k = -\sum_{k=2}^{\tau_{-a}} H(F_{k-1})\Delta S_k,}$$

$$\text{where } H(n) = \sum_{l=1}^{n} \frac{1}{a+l}, \quad F_k = \max_{0\leqslant l\leqslant k} S_l.$$

Proof.

$$I(F_{\tau_{-a}} \geqslant z) = I(N_z \leqslant \tau_{-a}),\ z > 0.$$

$$\Downarrow$$

$$\boxed{(a+\lceil z\rceil)I(F_{\tau_{-a}} \geqslant z) = a + S_{\tau_{-a}\wedge N_z}.}$$

Then

$$S_{\tau_{-a}\wedge N_z} + a = -\sum_{k=2}^{\tau_{-a}} I(F_{k-1} \geqslant z)\Delta S_k,$$

$$I(F_{\tau_{-a}} \geqslant z) = -\sum_{k=2}^{\tau_{-a}} \frac{I(F_{k-1} \geqslant z)}{a+\lceil z\rceil}\Delta S_k.$$

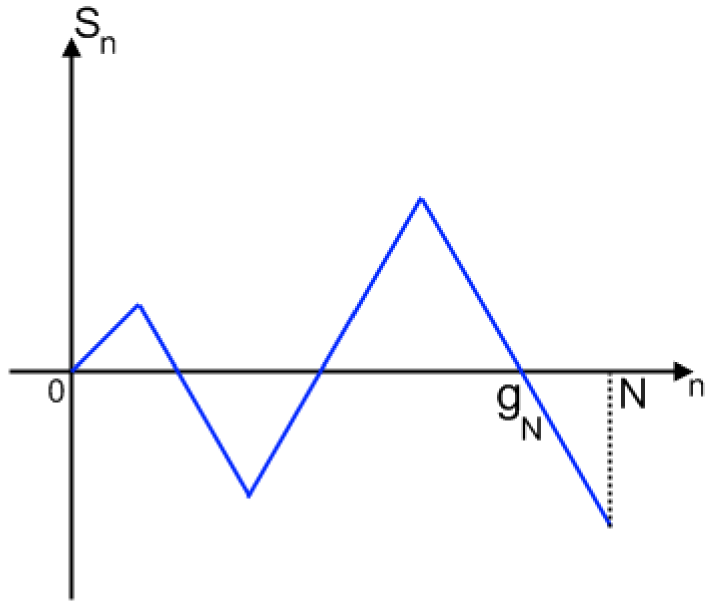
Integrating on z, we receive

$$F_{\tau_{-a}} = \int_0^{+\infty} I(F_{\tau_{-a}} \geqslant z)dz = -\sum_{k=2}^{\tau_{-a}} H(F_{k-1})\Delta S_k.$$

$\square$

3. Case $F_{g_N} = \max_{0 \leqslant k \leqslant g_N} S_k$,

$$g_N = \sup\{0 < k \leqslant N : S_k = 0\}.$$



Theorem 3. *For F_{g_N} we have the stochastic representation*

$$\max_{0 \leqslant k \leqslant g_N} S_k = \sum_{k=1}^{\left[\frac{N}{2}\right]} \mathsf{P}(F_N \geqslant 2k) + \sum_{k=1}^{N} L_{N-k+1}(F_{k-1}, S_{k-1}) \Delta S_k - \sum_{k=1}^{N} M_{N-k+1}(S_{k-1})(F_{k-1} - F_{g_{k-1}}) \Delta S_k,$$

where

$$L_{N-k+1}(F_{k-1}, S_{k-1}) = \sum_{n=F_{k-1}-S_{k-1}+1}^{N-k+1} \sum_{m=0}^{\left[\frac{N-k+1-n}{2}\right]} \sum_{l=0}^{N-k+1-2m} (-1)^m \binom{\frac{1}{2}}{m}$$

$$\times \varphi_l(2n + S_{k-1}),$$

$$M_{N-k+1}(S_{k-1}) = \sum_{l,m:\ l+m \leqslant \left[\frac{N-k+1}{2}\right]} (-1)^m \binom{\frac{1}{2}}{m} \varphi_{S_{k-1}+2l}(S_{k-1})$$

$$\times I(S_{k-1} \leqslant N - k + 1 - 2(l + m)).$$

Multiple representations

Denote

$$B^n = \{\bar{i} = (i_1 \dots i_n) |\ i_k \in \{-1, 1\},\ k = 1 \dots n\},$$

$$U_n^m = \{(t_1, \dots, t_m) : 1 \leqslant t_1 < \dots < t_m \leqslant n,\ t_i \in \mathbb{N}\}.$$

Definition 1. *Let* $h\colon \mathbb{R}^n \to \mathbb{R}$.
The difference quotient of order n

$$D^{(1)}h(x_1, \dots, x_n) = \frac{h(1, x_2, \dots, x_n) - h(-1, x_2, \dots, x_n)}{2},$$

$$D^{(k)}h(x_1, \dots, x_n) = \frac{1}{2} D^{(k-1)}(h(x_1, \dots, x_{k-1}, 1, x_{k+1}, \dots, x_n)$$

$$-h(x_1, \dots, x_{k-1}, -1, x_{k+1}, \dots, x_n)),\ k \leqslant n.$$

For instance, $D^{(2)}h(x_1, x_2) = 1/4(h(1,1) - h(-1,1) - h(1,-1) + h(-1,-1))$.

Proposition 1. *For any* $n \in \mathbb{N}$ *and for any function* $f: \mathbb{R}^n \to \mathbb{R}$ *we have*

$$\boxed{\frac{1}{2^n} \sum_{\bar{i} \in B^n} i_{t_1} \ldots i_{t_k} f(\bar{i}) = D^{(k)} \left(\mathsf{E} \left[f(\bar{\xi}) \,|_{\xi_{t_1}=x_1,\ldots,\ \xi_{t_k}=x_k} \right] \right),}$$

where $1 \leqslant t_1 < \ldots < t_k \leqslant n$, $\bar{\xi} = (\xi_1, \ldots, \xi_n)$.

Proof induction on k.

Lemma 2. *Let $f : \mathbb{R}^n \to \mathbb{R}$. Then we have*

$$f(\bar\xi) = \mathsf{E} f(\bar\xi) + \sum_{k=1}^{n} \sum_{\bar t \in U_n^k} D^{(k)} \left(\mathsf{E} \left[f(\bar\xi) \mid_{\xi_{t_1}=x_1,\ldots,\ \xi_{t_k}=x_k} \right] \right) \Delta S_{t_1} \ldots \Delta S_{t_k},$$

where $\bar\xi = (\xi_1, \ldots, \xi_n)$.

Corollary 1. ***For*** $F_N = \max_{0 \leqslant k \leqslant N} S_k$ ***we have the representation***

$$F_N = \mathsf{E} F_N + \sum_{k=1}^{N} \sum_{1 \leqslant t_1 < \ldots < t_k \leqslant N} D^{(k)} \left(\mathsf{E} \left[F_N \mid_{\xi_{t_1}=x_1,\ldots,\ \xi_{t_k}=x_k} \right] \right) \Delta S_{t_1} \ldots \Delta S_{t_k}.$$

1. Multiple representation of functional $F_N = \max_{0 \leqslant k \leqslant N} S_k$

Theorem 4. *For* $F_N = \max_{0 \leqslant k \leqslant N} S_k$ *we have the following multiple stochastic representation:*

$$F_N = \mathsf{E} F_N + \sum_{k=1}^{N} \mathsf{E}\left[G_{N-k+1}(F_{k-1} - S_{k-1})\right] \Delta S_k + \sum_{m=2}^{N} \sum_{1 \leqslant k_1 < \ldots < k_m \leqslant N} c(k_1, \ldots, k_m) \Delta S_{k_1} \ldots \Delta S_{k_m},$$

where

$$c(k_1, \ldots, k_m) = D^{(m-1)}\left(\mathsf{E}\left[G_{N-k_m+1}(F_{k_m-1} - S_{k_m-1})|_{\xi_{k_1}=x_1, \ldots,\ \xi_{k_{m-1}}=x_{m-1}}\right]\right).$$

Proof.

Using the theorem 1, we have

$$F_N = \mathsf{E}F_N + \sum_{k=1}^{N} G_{N-k+1}(F_{k-1} - S_{k-1})\Delta S_k. \qquad (6)$$

Apply lemma 3 for $G_{N-k+1}(F_{k-1} - S_{k-1})$:

$$G_{N-k+1}(F_{k-1} - S_{k-1}) = \mathsf{E}[G_{N-k+1}(F_{k-1} - S_{k-1})]$$

$$+ \sum_{l=2}^{k} \sum_{\bar{t} \in U_{k-1}^{l-1}} c(t_1, \ldots t_{l-1}, k)\Delta S_{t_1} \ldots \Delta S_{t_{l-1}}.$$

Substituting the computed representation in (6), we get the statement of the theorem.

$\square$

2. Multiple representation of functional $F_{\tau_{-a}} = \max\limits_{0 \leqslant k \leqslant \tau_{-a}} S_k$

Theorem 5. *For $F_{\tau_{-a}}$ we have the stochastic representation:*

$$F_{\tau_{-a}} = -\sum_{k=2}^{\tau_{-a}} \mathsf{E}[H(F_{k-1})]\Delta S_k + -\sum_{m=2}^{\tau_{-a}} \sum_{1 \leqslant k_1 < \ldots < k_m} h(k_1, \ldots, k_m) I(\tau_{-a} \geqslant k_m)\Delta S_{k_1} \ldots \Delta S_{k_m},$$

where

$$h(k_1, \ldots, k_m) = D^{(m-1)}\left(\mathsf{E}\left[H(F_{k_m-1})|_{\xi_{k_1}=x_1, \ldots,\ \xi_{k_{m-1}}=x_{m-1}}\right]\right).$$

Thank you for your attention

Appendix

Single representations

1. Case $F_N = \max_{0 \leqslant k \leqslant N} S_k$

Theorem 6. *For* $F_N = \max_{0 \leqslant k \leqslant N} S_k$ *we have the following stochastical representation:*

$$\boxed{\max_{0 \leqslant n \leqslant N} S_n = \mathsf{E}\left(\max_{0 \leqslant n \leqslant N} S_n\right) + \sum_{k=1}^{N} G_{N-k+1}(F_{k-1} - S_{k-1}) \Delta S_k,} \qquad (7)$$

where

$$G_{N-k+1}(F_{k-1}-S_{k-1}) = \sum_{r=F_{k-1}-S_{k-1}+1}^{N-k+1} \sum_{l=0}^{\left[\frac{N-k+1-r}{2}\right]} (-1)^{\left[\frac{N-k+1-r}{2}\right]-l} \times C_{-\frac{1}{2}}^{\left[\frac{N-k+1-r}{2}\right]-l} \varphi_{r+2l}(r),$$

$$\varphi_k(m) = \frac{m}{k2^k} C_k^{\frac{m+k}{2}}, \quad \mathsf{E}(\max_{0\leqslant n\leqslant N} S_n) \sim \sqrt{\frac{2N}{\pi}} \quad \text{by} \quad N \to \infty.$$

Доказательство. Let $\xi \geqslant 0$ is a random variable. Then $\xi = \int_0^{+\infty} I(\xi \geqslant a)da$. Hence

$$F_N = \int_0^{+\infty} I(\max_{0\leqslant k\leqslant N} S_k \geqslant a)da = \int_0^{+\infty} I(N_a \leqslant N)da,$$

where $N_a = \inf\{k \geqslant 0 : S_k \geqslant a\} = \inf\{k \geqslant 0 : S_k = \lceil a \rceil\} = N_{\lceil a \rceil}, \quad a \geqslant 0.$

Consider the **discrete stochastical exponent** defines for $\lambda \in (0, 1)$:

$$\mathcal{E}_n(S) = \prod_{k=1}^{n} (1 + \lambda \Delta S_k) = (\sqrt{1 - \lambda^2})^n \left(\sqrt{\frac{1+\lambda}{1-\lambda}} \right)^{S_n}, \ \mathcal{E}_0(S) := 1.$$

$\Delta \mathcal{E}_n(S) = \lambda \mathcal{E}_{n-1}(S) \Delta S_n$, and $\mathcal{E}_n(S)$ – martingale with respect to filtration $(\mathcal{F}_n)_{n \geqslant 0}$. From **the optional sampling theorem** (P-a. s.)

$$\mathsf{E}(\mathcal{E}_{N_a} | \mathcal{F}_n) = \mathcal{E}_{N_a \wedge n} = 1 + \sum_{k=1}^{N_a \wedge n} \Delta \mathcal{E}_k = 1 + \lambda \sum_{k=1}^{N_a \wedge n} \mathcal{E}_{k-1} \Delta S_k. \qquad (8)$$

Because of $S_{N_a} = \lceil a \rceil$, then

$$\mathsf{E}(\mathcal{E}_{N_a} | \mathcal{F}_n) = \left(\sqrt{\frac{1+\lambda}{1-\lambda}} \right)^{\lceil a \rceil} \mathsf{E}\left[(\sqrt{1-\lambda^2})^{N_a} | \mathcal{F}_n \right]. \qquad (9)$$

$\mathsf{E}\mathcal{E}_{N_a} = \mathsf{E}\mathcal{E}_n = 1$, therefore

$$\mathsf{E}(\sqrt{1-\lambda^2})^{N_a} = \left(\sqrt{\frac{1+\lambda}{1-\lambda}}\right)^{-\lceil a\rceil}. \qquad (10)$$

From (8), (9), (10) we get

$$\mathsf{E}((\sqrt{1-\lambda^2})^{N_a}|\mathcal{F}_n) = \mathsf{E}(\sqrt{1-\lambda^2})^{N_a} +$$
$$+\lambda \sum_{k=1}^{N_a\wedge n} \left(\sqrt{\frac{1+\lambda}{1-\lambda}}\right)^{-(\lceil a\rceil - S_{k-1})} (\sqrt{1-\lambda^2})^{k-1}\Delta S_k.$$

$\mathsf{P}(N_a = k) = \varphi_k(\lceil a\rceil)$, so for all $a \geqslant 0$

$$\sum_{l=0}^{+\infty} (\sqrt{1-\lambda^2})^{\lceil a\rceil+2l}\varphi_{\lceil a\rceil+2l}(\lceil a\rceil) = \left(\sqrt{\frac{1+\lambda}{1-\lambda}}\right)^{-\lceil a\rceil}. \qquad (11)$$

Denote $\sqrt{1-\lambda^2} =: s$

$$\Rightarrow \quad \lambda = \sqrt{1-s^2} = \sum_{m=0}^{+\infty} (-1)^m C_{\frac{1}{2}}^m s^{2m}. \qquad (12)$$

From (11), (12) for all $s \in (0,1)$ we have

$$\mathsf{E}(s^{N_a}|\mathcal{F}_n) = \mathsf{E}s^{N_a} + \sum_{k=1}^{N_a \wedge n} \sum_{l,m=0}^{+\infty} (-1)^m C_{\frac{1}{2}}^m \varphi_{\lceil a \rceil - S_{k-1} + 2l}(\lceil a \rceil - S_{k-1})$$

$$\times\, s^{\lceil a \rceil - S_{k-1} + 2(m+l) + k - 1} \Delta S_k \quad (13)$$

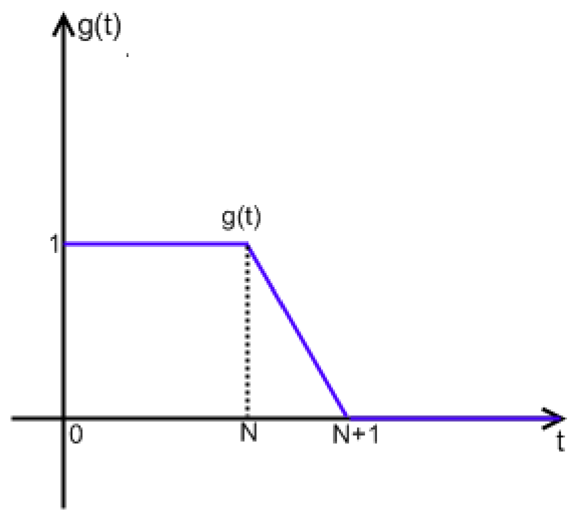
$$\left.\begin{array}{l} f_s(x) = s^x,\ x \geqslant 0,\ s \in (0,1) \\ s = e^{-n},\ \ n = 1, 2, \ldots, \end{array}\right\} \Rightarrow \text{(13) is valid for } \mathcal{L} = \{e^{-nx}\}_{n \geqslant 0}.$$

$$C_0[0,+\infty] = \{f \in C[0,+\infty] : f(x) \to 0 \text{ при } x \to +\infty\}$$

$$\Downarrow_{(*)}$$

$$\mathcal{L}=\{e^{-nx}\}_{n\geqslant 0} \text{ is complete in } C_0[0,+\infty].$$

Consider the function



(*) Sedletski A. M. *Classes of analityc Fourier transformations and exponential approximations*. Moscow: Physmatlit, 2005, 504 p.

$$\left.\begin{array}{l} g(t) \in C_0[0,+\infty) \\ g(n) = I(n \leqslant N) \end{array}\right\} \Rightarrow (13) \text{ is valid for } I(n \leqslant N).$$

Lemma 3. *For $I(N_a \leqslant N)$ we have the representation (*P*-a.s.)*

$$I(N_a \leqslant N) = \mathsf{P}(N_a \leqslant N) + \sum_{k=1}^{N_a \wedge N} \sum_{l,m:\ l+m \leqslant \left[\frac{N-k+1}{2}\right]} (-1)^m C_{\frac{1}{2}}^m$$

$$\times \varphi_{\lceil a \rceil - S_{k-1} + 2l}(\lceil a \rceil - S_{k-1}) I(\lceil a \rceil - S_{k-1} \leqslant N - k + 1 - 2(l+m)) \Delta S_k.$$

$$F_N = \int\limits_0^{+\infty} I(N_a \leqslant N)da = \int\limits_0^{+\infty} \mathsf{P}(F_N \geqslant a)da +$$

$$+ \sum_{k=1}^{N} \sum_{l+m \leqslant \left[\frac{N-k+1}{2}\right]} \int\limits_0^{S_{k-1}+N-k+1-2(l+m)} (-1)^m C_{\frac{1}{2}}^m \varphi_{\lceil a \rceil - S_{k-1}+2l}(\lceil a \rceil - S_{k-1})$$

$$\times I(k \leqslant N_a)da \Delta S_k =$$

$$\mathsf{E}F_N + \sum_{k=1}^{N} \sum_{l+m \leqslant \left[\frac{N-k+1}{2}\right]} (-1)^m C_{\frac{1}{2}}^m \left(\int\limits_{F_{k-1}-S_{k-1}}^{N-k+1-2(l+m)} \varphi_{\lceil a \rceil+2l}(\lceil a \rceil)da \right) \Delta S_k.$$

$$F_N = \mathsf{E}F_N + \sum_{k=1}^{N} \left(\sum_{l+m \leqslant \left[\frac{N-k+1}{2}\right]} \sum_{r=F_{k-1}-S_{k-1}+1}^{N-k+1-2(l+m)} (-1)^m C_{\frac{1}{2}}^m \varphi_{r+2l}(r) \right) \Delta S_k$$

$$= \mathsf{E}F_N + \sum_{k=1}^{N} \sum_{r=F_{k-1}-S_{k-1}+1}^{N-k+1} \left(\sum_{l+m \leqslant \left[\frac{N-k+1-r}{2}\right]} (-1)^m C_{\frac{1}{2}}^m \varphi_{r+2l}(r) \right) \Delta S_k.$$

For all $x \in \mathbb{R}$

$$(-1)^n C_{x-1}^n = \sum_{m=0}^{n} (-1)^m C_x^m$$

$\Rightarrow$ theorem 6 is proved.

Lemma 4. *For* $G_{N-k+1}(F_{k-1}-S_{k-1})$ *we have the representation (*P*-a.s.)*

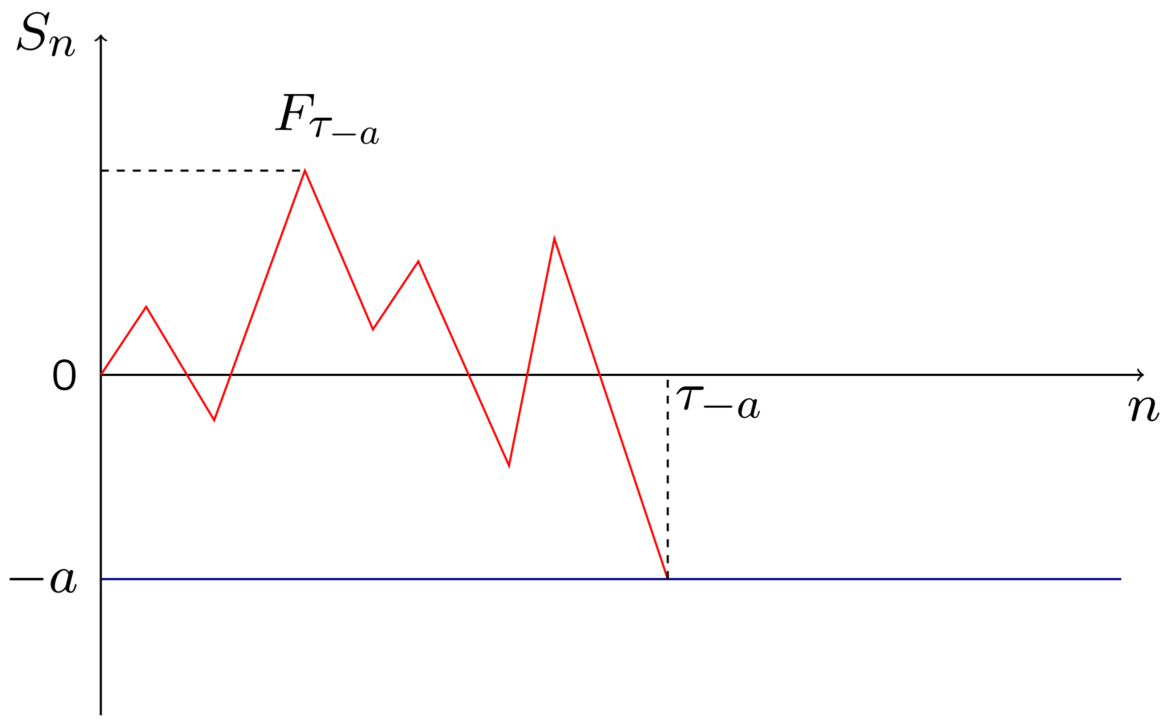
$$G_{N-k+1}(F_{k-1}-S_{k-1}) = \int_{F_{k-1}-S_{k-1}}^{N-k+1} (\mathsf{P}(F_{N-k+1} \geqslant a) - \mathsf{P}(F_{N-k+1-2\eta} \geqslant a))da,$$

where η *is a random variable, independent from* $\mathcal{F}_N$*, with distribution*

$$\mathsf{P}(\eta{=}m) = (-1)^{m-1} C_{\frac{1}{2}}^{m},\ m = 1,2\dots$$

2. Case $F_{\tau_{-a}} = \max_{0 \leqslant k \leqslant \tau_{-a}} S_k$,

$$\tau_{-a} = \inf\{k \geqslant 0 : S_k = -a\}.$$



Theorem 7. *For $F_{\tau_{-a}}$ we have the stochastical representation*

$$\boxed{\max_{0 \leqslant k \leqslant \tau_{-a}} S_k = -\sum_{k=2}^{\tau_{-a}} H(F_{k-1}) \Delta S_k,}$$

$$\textit{where } H(n) = \sum_{l=1}^{n} \frac{1}{a+l}, \qquad F_k = \max_{0 \leqslant l \leqslant k} S_l.$$

Proof.

$$I(F_{\tau_{-a}} \geqslant z) = I(N_z \leqslant \tau_{-a}), \; z > 0.$$

$$\Downarrow$$

$$\boxed{(a+\lceil z\rceil)I(F_{\tau_{-a}} \geqslant z) = a + S_{\tau_{-a}\wedge N_z},} \tag{14}$$

$$\Downarrow$$

$$S_{\tau_{-a}\wedge N_z} + a = \sum_{k=1}^{\tau_{-a}\wedge N_z} \Delta S_k + a = \sum_{k=1}^{\tau_{-a}} I(k \leqslant N_z)\Delta S_k + a$$

$$= \sum_{k=1}^{\tau_{-a}} I(F_{k-1} < z)\Delta S_k + a = -\sum_{k=2}^{\tau_{-a}} I(F_{k-1} \geqslant z)\Delta S_k.$$

Therefore, from (14) we have

$$I(F_{\tau_{-a}} \geqslant z) = -\sum_{k=2}^{\tau_{-a}} \frac{I(F_{k-1} \geqslant z)}{a + \lceil z \rceil} \Delta S_k.$$

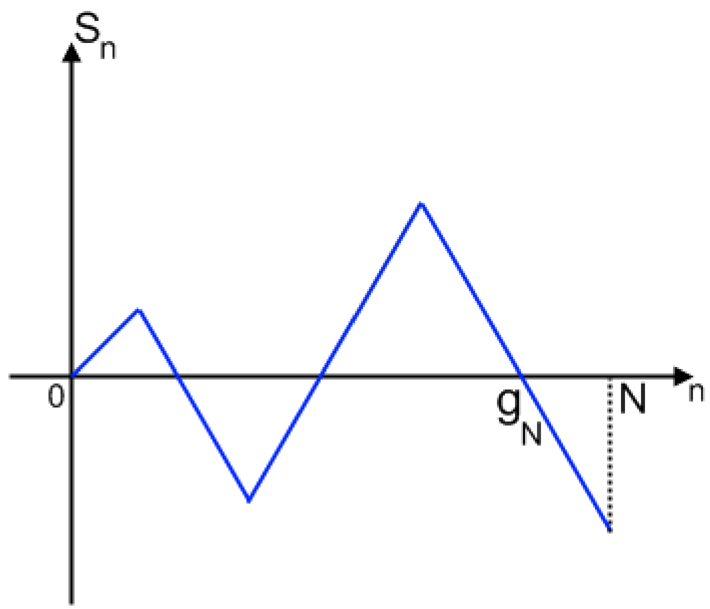
$$F_{\tau_{-a}} = \int_0^{+\infty} I(F_{\tau_{-a}} \geqslant z) dz = -\int_0^{+\infty} \sum_{k=2}^{\tau_{-a}} \frac{I(F_{k-1} \geqslant z)}{a + \lceil z \rceil} \Delta S_k dz$$

$$= -\sum_{k=2}^{\tau_{-a}} \left(\int_0^{F_{k-1}} \frac{dz}{a + \lceil z \rceil} \right) \Delta S_k = -\sum_{k=2}^{\tau_{-a}} H(F_{k-1}) \Delta S_k.$$

$\square$

3. Case $F_{g_N} = \max_{0 \leqslant k \leqslant g_N} S_k$,

$$g_N = \sup\{0 < k \leqslant N : S_k = 0\}.$$



Theorem 8. *For F_{g_N} we have the stochastical representation*

$$\max_{0\leqslant k\leqslant g_N} S_k = \sum_{k=1}^{\left[\frac{N}{2}\right]} \mathsf{P}(F_N \geqslant 2k) + \sum_{k=1}^{N} L_{N-k+1}(F_{k-1}, S_{k-1})\Delta S_k - \sum_{k=1}^{N} M_{N-k+1}(S_{k-1})(F_{k-1} - F_{g_{k-1}})\Delta S_k,$$

where

$$L_{N-k+1}(F_{k-1}, S_{k-1}) = \sum_{n=F_{k-1}-S_{k-1}+1}^{N-k+1} \sum_{m=0}^{\left[\frac{N-k+1-n}{2}\right]} \sum_{l=0}^{N-k+1-2m} (-1)^m C_{1/2}^m$$

$$\times \varphi_l(2n + S_{k-1}),$$

$$M_{N-k+1}(S_{k-1}) = \sum_{l,m:\ l+m \leqslant \left[\frac{N-k+1}{2}\right]} (-1)^m C_{\frac{1}{2}}^m \varphi_{S_{k-1}+2l}(S_{k-1})$$

$$\times I(S_{k-1} \leqslant N - k + 1 - 2(l + m)).$$

Proof.

$$\max_{0\leqslant n\leqslant g_N} S_n = \int_0^{+\infty} I(\max_{0\leqslant n\leqslant g_N} S_n\geqslant a)da$$

$$= \int_0^{+\infty} I(N_a \leqslant g_N)da = \int_0^{+\infty} I(d_{N_a}\leqslant N)da,$$

where

$$d_{N_a}=\inf\{k\geqslant N_a : S_k=0\}.$$

$$d_{N_a}=N_a+\inf\{k \geqslant 0 : S_{k+N_a}=0\} = N_a + \widehat{N}_{-a},$$

where

$$\widehat{N}_{-a} = \inf\{k \geqslant 0 : \widehat{S}_k = -\lceil a \rceil\},$$

$(\widehat{S}_k)_{k \geqslant 0}$ с $\widehat{S}_k = S_{N_a+k} - \lceil a \rceil$ – random walk that indpendent from $\mathcal{F}_{N_a}$.

$$d_{N_a} = N_a + \widehat{N}_{-a}, \qquad I(d_{N_a} \leqslant N) = I(\widehat{N}_{-a} \leqslant N - N_a).$$

$$\widehat{N}_{-a} \text{ и } N_a \text{ н. о. р.} \overset{\text{lemma } 4}{\Rightarrow} I(\widehat{N}_a(-\widehat{S}) \leqslant b) = I(\widehat{N}_{-a} \leqslant b);$$

Denote $b = N - N_a$.

$$I(d_{N_a}\leqslant N) = \sum_{l=0}^{+\infty} I(l \leqslant N - N_a)\varphi_l(\lceil a \rceil) - \sum_{k=(N_a+1)}^{d_{N_a}\wedge N} \sum_{l+m\leqslant\left[\frac{N-k+1}{2}\right]} (-1)^m$$

$$\times\, C_{\frac{1}{2}}^{m} \varphi_{S_{k-1}+2l}(S_{k-1}) I(S_{k-1} \leqslant N - k + 1 - 2(l+m)) \Delta S_k.$$

Using the relations below

$$\sum_{l=0}^{N} \mathsf{P}(N_a \leqslant N-l)\varphi_l(\lceil a \rceil) = \mathsf{P}(N_{2\lceil a \rceil} \leqslant N),$$

$$\sum_{l=0}^{\left[\frac{N-k+1-n}{2}\right]-m} \varphi_{n+2l}(n)\mathsf{P}(N_{n+x} \leqslant N-k+1-2m-n-2l)|_{x=S_{k-1}} =$$

$$= \sum_{l=0}^{N-k+1-2m} \varphi_l(2n+S_{k-1}),$$

we receive the statement of the theorem.

Multiple representations

Denote

$$B^n=\{\bar{i} = (i_1 \dots i_n)|\ i_k \in \{-1, 1\},\ k = 1 \dots n\},$$

$$U_n^m = \{(t_1, \dots, t_m) : 1 \leqslant t_1 < \dots < t_m \leqslant n,\ t_i \in \mathbb{N}\}.$$

Definition 2. *Let* $h\colon \mathbb{R}^n \to \mathbb{R}$.
Difference quotient of order n

$$D^{(1)}h(x_1, \dots, x_n) = \frac{h(1, x_2, \dots, x_n) - h(-1, x_2, \dots, x_n)}{2},$$

$$D^{(k)}h(x_1, \dots, x_n) = \frac{1}{2}D^{(k-1)}(h(x_1, \dots, x_{k-1}, 1, x_{k+1}, \dots, x_n)$$

$$-h(x_1, \dots, x_{k-1}, -1, x_{k+1}, \dots, x_n)),\ k \leqslant n.$$

Proposition 2. *For all* $n \in \mathbb{N}$ *and for any function* $f\colon \mathbb{R}^n \to \mathbb{R}$ *we have*

$$\boxed{\frac{1}{2^n} \sum_{\bar{i} \in B^n} i_{t_1} \ldots i_{t_k} f(\bar{i}) = D^{(k)} \left(\mathsf{E} \left[f(\bar{\xi}) \mid_{\xi_{t_1}=x_1,\ldots,\ \xi_{t_k}=x_k} \right] \right),}$$

where $1 \leqslant t_1 < \ldots < t_k \leqslant n, \ \ \bar{\xi} = (\xi_1, \ldots, \xi_n)$.

Proof. Induction on k.

When **k=1** we have

$$\frac{1}{2^n}\sum_{\bar{i}\in B^n} i_{t_1} f(\bar{i}) = \frac{1}{2^n}\sum_{\bar{i}\in B^n:\ i_{t_1}=1} f(\bar{i}) - \frac{1}{2^n}\sum_{\bar{i}\in B^n:\ i_{t_1}=-1} f(\bar{i}) =$$

$$= \frac{1}{2}\sum_{\bar{i}\in B^n:\ i_{t_1}=1} f(\bar{i})\mathsf{P}(\xi_1 = i_1, \ldots \widehat{\xi_{t_1}} \ldots, \xi_n = i_n) - \frac{1}{2}\sum_{\bar{i}\in B^n:\ i_{t_1}=-1} f(\bar{i})$$

$$\times\, \mathsf{P}(\xi_1 = i_1, \ldots \widehat{\xi_{t_1}} \ldots, \xi_n = i_n) = \frac{1}{2}\left(\mathsf{E}\left[f(\bar{\xi})|_{\xi_{t_1}=1}\right] - \mathsf{E}\left[f(\bar{\xi})|_{\xi_{t_1}=-1}\right]\right)$$

$$= D^{(1)}\left(\mathsf{E}\left[f(\bar{\xi})\ |_{\xi_{t_1}=x}\right]\right).$$

So,

$$\frac{1}{2^n}\sum_{\bar{i}\in B^n} i_{t_1}\dots i_{t_{k+1}} f(\bar{i}) = \frac{1}{2}\left(D^{(k)}\left(\mathsf{E}\left[f(\bar{\xi})\ |_{\xi_{t_1}=x_1,\dots,\xi_{t_k}=x_k,\ \xi_{t_{k+1}}=1}\right]\right)\right.$$

$$\left. - D^{(k)}\left(\mathsf{E}\left[f(\bar{\xi})\ |_{\xi_{t_1}=x_1,\dots,\xi_{t_k}=x_k,\ \xi_{t_{k+1}}=-1}\right]\right)\right) =$$

$$= D^{(k+1)}\left(\mathsf{E}\left[f(\bar{\xi})\ |_{\xi_{t_1}=x_1,\dots,\xi_{t_{k+1}}=x_{k+1}}\right]\right).$$

□

Lemma 5. *Let* $f : \mathbb{R}^n \to \mathbb{R}$*. Then we have the stochastical representation*

$$f(\bar{\xi}) = \mathsf{E}f(\bar{\xi}) + \sum_{k=1}^{n} \sum_{\bar{t} \in U_n^k} D^{(k)} \left(\mathsf{E}\left[f(\bar{\xi}) \,|_{\xi_{t_1}=x_1,\dots,\ \xi_{t_k}=x_k} \right] \right) \Delta S_{t_1} \dots \Delta S_{t_k},$$

where $\bar{\xi} = (\xi_1, \dots, \xi_n)$.

Corollary 2. ***For*** $F_N = \max_{0 \leqslant k \leqslant N} S_k$ ***we have the stochastical representation***

$$F_N = \mathsf{E}F_N + \sum_{k=1}^{N} \sum_{1 \leqslant t_1 < \dots < t_k \leqslant N} D^{(k)} \left(\mathsf{E}\left[F_N \,|_{\xi_{t_1}=x_1,\dots,\ \xi_{t_k}=x_k} \right] \right) \Delta S_{t_1} \dots \Delta S_{t_k}.$$

1. Multiple representation of functional $F_N = \max_{0 \leqslant k \leqslant N} S_k$

Theorem 9. *For* $F_N = \max_{0 \leqslant k \leqslant N} S_k$ *we have the following multiple stochastic representation:*

$$F_N = \mathsf{E} F_N + \sum_{k=1}^{N} \mathsf{E}\left[G_{N-k+1}(F_{k-1} - S_{k-1})\right] \Delta S_k$$

$$+ \sum_{m=2}^{N} \sum_{1 \leqslant k_1 < \ldots < k_m \leqslant N} c(k_1, \ldots, k_m) \Delta S_{k_1} \ldots \Delta S_{k_m},$$

where

$$c(k_1, \ldots, k_m) = D^{(m-1)}(\mathsf{E}\left[G_{N-k_m+1}(F_{k_m-1} - S_{k_m-1})|_{\xi_{k_1}=x_1, \ldots, \ \xi_{k_{m-1}}=x_{m-1}}\right]).$$

Proof.

Apply lemma 6 for $G_{N-k+1}(F_{k-1} - S_{k-1})$.

$$G_{N-k+1}(F_{k-1} - S_{k-1}) = \mathsf{E}[G_{N-k+1}(F_{k-1} - S_{k-1})]$$

$$+\sum_{m=1}^{k-1} \sum_{\bar{t} \in U_{k-1}^{m}} D^{(m)} \mathsf{E}[G_{N-k+1}(F_{k-1} - S_{k-1}) \mid_{\xi_{t_1}=x_1,\ldots,\xi_{t_m}=x_m}] \Delta S_{t_1} \ldots \Delta S_{t_m}.$$

$$m = l - 1$$

$$G_{N-k+1}(F_{k-1} - S_{k-1}) = \mathsf{E}[G_{N-k+1}(F_{k-1} - S_{k-1})] \tag{15}$$

$$+\sum_{l=2}^{k} \sum_{\bar{t} \in U_{k-1}^{l-1}} c(t_1, \ldots t_{l-1}, k) \Delta S_{t_1} \ldots \Delta S_{t_{l-1}}.$$

Substituting (15) in formula (7), we have

$$F_N = \mathsf{E}F_N + \sum_{k=1}^{N} G_{N-k+1}(F_{k-1} - S_{k-1})\Delta S_k = \mathsf{E}F_N + \sum_{k=1}^{N} \mathsf{E}[G_{N-k+1}(F_{k-1}$$

$$-S_{k-1})]\Delta S_k + \sum_{k=1}^{N} \sum_{l=2}^{k} \sum_{\bar{t} \in U_{k-1}^{l-1}} c(t_1, \ldots t_{l-1}, k)\Delta S_{t_1} \ldots \Delta S_{t_{l-1}} \Delta S_k =$$

$$= \mathsf{E}F_N + \sum_{k=1}^{N} \mathsf{E}[G_{N-k+1}(F_{k-1} - S_{k-1})]\Delta S_k + \sum_{l=2}^{N}\sum_{k=l}^{N}\sum_{\bar{t}\in U_{k-1}^{l-1}} c(t_1,$$

$$\ldots, t_{l-1}, k)\Delta S_{t_1}\ldots\Delta S_{t_{l-1}}\Delta S_k = \mathsf{E}F_N + \sum_{k=1}^{N} \mathsf{E}[G_{N-k+1}(F_{k-1} - S_{k-1})]\Delta S_k$$

$$+ \sum_{l=2}^{N}\sum_{\bar{t}\in U_N^l} c(t_1, \ldots, t_l)\Delta S_{t_1}\ldots\Delta S_{t_l}.$$

$\square$

2. Multiple representation of functional $F_{\tau_{-a}} = \max_{0 \leqslant k \leqslant \tau_{-a}} S_k$

Theorem 10. *For* $F_{\tau_{-a}}$ *we have the multiple stochastic representation*

$$F_{\tau_{-a}} = -\sum_{k=2}^{\tau_{-a}} \mathsf{E}[H(F_{k-1})]\Delta S_k$$

$$-\sum_{m=2}^{\tau_{-a}} \sum_{1 \leqslant k_1 < \ldots < k_m} h(k_1, \ldots, k_m) I(\tau_{-a} \geqslant k_m) \Delta S_{k_1} \ldots \Delta S_{k_m},$$

where

$$h(k_1, \ldots, k_m) = D^{(m-1)} \left(\mathsf{E} \left[H(F_{k_m - 1})|_{\xi_{k_1} = x_1, \ldots,\ \xi_{k_{m-1}} = x_{m-1}} \right] \right).$$

Proof.

From lemma 6 we have

$$H(F_{k-1}) = \mathsf{E}[H(F_{k-1})] +$$

$$+ \sum_{m=1}^{k-1} \sum_{\bar{t} \in U_{k-1}^m} D^{(m)} \left(\mathsf{E} \left[H(F_{k-1})|_{\xi_{t_1}=x_1,\ldots,\ \xi_{t_m}=x_m} \right] \right) \Delta S_{t_1} \ldots \Delta S_{t_m} =$$

$$= \mathsf{E}[H(F_{k-1})] + \sum_{l=2}^{k} \sum_{\bar{t} \in U_{k-1}^{l-1}} h(t_1, \ldots, t_{l-1}, k) \Delta S_{t_1} \ldots \Delta S_{t_{l-1}}.$$

Using theorem 7, we get the statement of the initial theorem.