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ABSTRACT. We construct interpolating divergence-free multiwavelets based on cubic Hermite splines. We give characterizations of the relevant function spaces and indicate their use for analyzing experimental data of incompressible flow fields. We also show that the standard interpolatory wavelets, based on the Deslauriers-Dubuc interpolatory scheme or on interpolatory splines, cannot be used to construct compactly supported divergence-free interpolatory wavelets.

1. INTRODUCTION

Wavelets are by now a widely accepted tool in signal and image processing as well as in numerical simulation. Divergence-free vector fields do have obvious possible applications for the analysis and numerical simulation of incompressible flows typically modeled by the Stokes system or by the incompressible Navier-Stokes equations. This has led to the construction of divergence-free wavelets. The first construction was introduced in [3] leading to an orthonormal wavelet basis for the space of divergence-free vector fields on \mathbb{R}^n . These functions were used in [19] to show an existence result for the instationary incompressible Navier-Stokes equations on \mathbb{R}^n . For numerical purposes, in particular for bounded domains, these wavelet bases are not appropriate since they are globally supported. In fact, it was shown in [30] that no orthonormal compactly supported divergence-free wavelet basis exists.

In [29, 38] compactly supported *biorthogonal* wavelets have been constructed and these functions have been used for the analysis of the Navier-Stokes equations [31], the numerical simulation of the Stokes equations on rectangular domains [37, 39, 41] and also for the analysis of experimental data of incompressible turbulent flows [1]. There are also generalizations to multiwavelets [27, 28].

However, the application of divergence-free wavelets in numerical simulations is so far restricted to rather academic simple domains like rectangles. This is mainly due to the necessity of a rather technically complicated construction of biorthogonal divergence-free wavelet bases on general domain geometries. This is one reason for considering collocation methods, see e.g. [24, 25, 26, 42]. In this case, *interpolatory* wavelets are required, which can easier be adapted to complicated domains. However, to our knowledge, there are no *divergence-free* interpolatory wavelets available, which is the reason why, e.g. in [25], an additional projection method has to be performed.

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Another motivation for our investigation comes from applications of wavelets in data analysis. Since wavelets are a widely accepted tool for signal and image processing, denoising and compression, it is a natural idea to use wavelet methods also for the analysis of incompressible flow data. Of course, for incompressible flows divergence-free wavelets would be appropriate. However, since typically the data in terms of point values is *approximated* by scaling function coefficients this does not only invent an approximation error, it also adds artificial compressibility (like in [1]). This could be avoided if point values and expansion coefficients coincide, which is the case for interpolatory functions.

The remainder of this paper is organized as follows. In Section 2 we collect some preliminaries, notation, and basics on interpolatory wavelets and we review the main construction idea for divergence-free wavelets. It is a natural idea to consider the maybe most frequently used interpolatory wavelet bases as a starting point for the construction of interpolatory divergence-free wavelets. We show in Section 3 that this approach does not work. Thus, we consider a multiwavelet construction based upon cubic Hermite splines in Section 4. We show in Section 5 how our construction can be used within the lifting scheme. The further generalization to arbitrary grids and domains as well as the numerical applications are devoted to a forthcoming paper.

2. PRELIMINARIES

In this section, we collect some preliminary facts that are needed in the remainder of the paper. Let us first set our notation. Let $\Omega \subseteq \mathbb{R}^n$ be a domain, $L_p(\Omega)$ the standard Lebesgue space, and

$$(2.1) \quad \mathbf{L}_p(\Omega) := L_p(\Omega)^n := \{\mathbf{v} = (v_1, \dots, v_n)^T : v_i \in L_p(\Omega), i = 1, \dots, n\}$$

the corresponding space of vector fields. We will always denote vector fields by boldface characters and real-valued functions in the standard font. For example, $\mathbf{H}^k(\Omega)$ is the space of n -dimensional vector fields whose components are functions in $H^k(\Omega)$.

The spaces of vector-fields induced by the divergence operator are denoted by

$$(2.2) \quad \mathbf{H}(\text{div}; \Omega) := \{\mathbf{v} \in \mathbf{L}_2(\Omega) : \text{div } \mathbf{v} \in L_2(\Omega)\},$$

$$(2.3) \quad \mathbf{V}(\text{div}; \Omega) := \{\mathbf{v} \in \mathbf{H}(\text{div}; \Omega) : \text{div } \mathbf{v} = 0\}.$$

Since we consider interpolatory functions, we will also need the spaces $\mathbf{C}^k(\Omega)$ and

$$(2.4) \quad \mathbf{C}(\text{div}; \Omega) := \{\mathbf{v} \in \mathbf{C}(\Omega) : \text{div } \mathbf{v} \in C(\Omega)\},$$

$$(2.5) \quad \mathbf{N}(\text{div}; \Omega) := \{\mathbf{v} \in \mathbf{C}(\text{div}; \Omega) : \text{div } \mathbf{v} = 0\}.$$

We will frequently use the notation $A \lesssim B$ to abbreviate that A is bounded by a constant multiple of B . $A \gtrsim B$ is defined as $B \lesssim A$ and $A \sim B$ means $A \lesssim B$ and $A \gtrsim B$.

2.1. How to Construct Divergence-free Wavelets. A standard tool for constructing wavelet bases for $L_2(\mathbb{R})$, i.e., on the whole real line, is a multiresolution analysis (MRA), which is a nested sequence of spaces $V_j \subset V_{j+1} \subset L_2(\mathbb{R})$, $j \in \mathbb{Z}$, whose union is dense in $L_2(\mathbb{R})$, while the intersection contains only the zero function. Moreover, V_0 is shift-invariant and V_j arises from V_0 by scaling in the sense

that $v \in V_0$ if and only if $v(2^j \cdot) \in V_j$. Finally, these spaces are generated by a *scaling function* ξ in the sense that

$$V_j = \text{clos}_{L_2} \text{span}\{\xi_{j,k} : k \in \mathbb{Z}\},$$

where $\xi_{j,k}(x) := 2^{j/2} \xi(2^j x - k)$, $x \in \mathbb{R}$, is a scaled and shifted version of ξ . From $V_0 \subset V_1$ it follows immediately that ξ is refinable, i.e.,

$$\xi(x) = \sum_{k \in \mathbb{Z}} a_k \xi(2x - k), \quad x \in \mathbb{R}.$$

The translates of ξ should be orthonormal or at least stable, in the sense that

$$\left\| \sum_{k \in \mathbb{Z}} c_k \xi(2x - k) \right\|_{L_2(\mathbb{R})} \sim \|\mathbf{c}\|_{\ell_2(\mathbb{Z})}, \quad \mathbf{c} = (c_k)_{k \in \mathbb{Z}} \in \ell_2(\mathbb{Z}).$$

A MRA can also be constructed on bounded domains $\Omega \subset \mathbb{R}^n$, involving more technicalities and renouncing shift-invariance, of course, see, e.g. [6, 7, 8, 11, 13, 14, 15].

Once a MRA (or two in the case of biorthogonal wavelets) is known, there are several general techniques to construct a corresponding wavelet basis.

Thus, it would be an immediate idea to construct a divergence-free wavelet basis from a MRA in $\mathbf{V}(\text{div}; \Omega)$. However, this is not appropriate since $\mathbf{V}(\text{div}; \Omega)$ is a closed subspace of $\mathbf{L}_2(\Omega)$, [29]. Instead, one uses an appropriate MRA $\{\mathbf{V}_j : j \in \mathbb{Z}\}$ in $\mathbf{H}(\text{div}; \Omega)$ and then splits the detail spaces

$$(2.6) \quad \mathbf{W}_j := \mathbf{V}_{j+1} \ominus \mathbf{V}_j = \mathbf{W}_j^\nabla \oplus \mathbf{W}_j^\Delta$$

into the divergence-free part $\mathbf{W}_j^\nabla = \mathbf{W}_j \cap \mathbf{V}(\text{div}; \Omega)$ and a stable complement \mathbf{W}_j^Δ . Then, one needs to construct a wavelet basis $\{\psi_{j,k}^\nabla : k\}$ for \mathbf{W}_j^∇ and the collection over all levels j gives the desired basis of divergence-free wavelets.

Biorthogonal wavelets, differentiation, and integration. Next, we need to describe what an ‘appropriate’ MRA in $\mathbf{H}(\text{div}; \Omega)$ is. In order to avoid unnecessary technicalities we restrict ourselves to the case $\Omega = \mathbb{R}^n$ and refer to [37, 40, 41] for the construction on general bounded domains. Keeping in mind that any divergence-free wavelet $\psi_{j,k}^\nabla$ is a linear combination of the basis functions of \mathbf{V}_{j+1} , the spaces

$$V_j^{\text{div}} := \text{div}(\mathbf{V}_j) \subset L_2(\mathbb{R}^n)$$

play an important role in the construction. Recall that \mathbf{V}_j is a vector-valued multiresolution space, i.e., the spaces are of the form

$$\mathbf{V}_j = V_j^1 \times \cdots \times V_j^n,$$

where each V_j^i is a real-valued multiresolution space. Hence, a natural condition is to require that

$$(2.7) \quad \frac{\partial}{\partial x_i} V_j^i = V_j^{\text{div}}, \quad i = 1, \dots, n,$$

where V_j^{div} is a multiresolution space, too.

In turns, if we use a standard tensor product construction to obtain a MRA on \mathbb{R}^n from n MRA on \mathbb{R} , (2.7) reduces to a relation of the form

$$(2.8) \quad \frac{d}{dx} V_j^+ = V_j^-$$

for two multiresolution spaces V_j^+ , V_j^- of functions $f_j : \mathbb{R} \rightarrow \mathbb{R}$. Indeed, one can choose $V_j^{\text{div}} := V_j^- \otimes \dots \otimes V_j^-$ and $V_j^i := V_j^- \otimes \dots \otimes V_j^- \otimes V_j^+ \otimes V_j^- \otimes \dots \otimes V_j^-$ with V_j^+ as the i -th factor in the above tensor product.

For the corresponding compactly supported scaling functions this implies the relation

$$\frac{d}{dx} \xi^+(x) = \sum_{k=-d}^d \alpha_k \xi^-(x-k),$$

for some finite $d \in \mathbb{N}$ and coefficients $\alpha_k \in \mathbb{R}$. We will use ξ to indicate a univariate scaling function while φ will always denote a multivariate function.

It is proven in [29] that for a biorthogonal pair $\xi^+ \in H^1(\mathbb{R})$, $\tilde{\xi}^+ \in L_2(\mathbb{R})$ of compactly supported scaling functions there always exists another biorthogonal pair $\xi^- \in L_2(\mathbb{R})$, $\tilde{\xi}^- \in H^1(\mathbb{R})$ of compactly supported scaling functions such that

$$(2.9) \quad \frac{d}{dx} \xi^+(x) = \xi^-(x) - \xi^-(x-1), \quad \frac{d}{dx} \tilde{\xi}^-(x) = \tilde{\xi}^+(x+1) - \tilde{\xi}^+(x).$$

Moreover, the corresponding biorthogonal wavelets satisfy the relation

$$(2.10) \quad \frac{d}{dx} \eta^+(x) = 4\eta^-(x), \quad \frac{d}{dx} \tilde{\eta}^-(x) = -\frac{1}{4} \tilde{\eta}^+(x).$$

From the above remarks, it should be clear that the latter equations are the heart of the construction of divergence-free wavelets. We will come back to this point later in Section 3.

Now, we define for any subset of indices $I \subseteq \{1, \dots, n\}$

$$(2.11) \quad V_j^I := \bigotimes_{\nu=1}^n V_j^{(I, \nu)}, \quad \text{where } V_j^{(I, \nu)} := \begin{cases} V_j^+, & \text{if } \nu \in I, \\ V_j^-, & \text{if } \nu \notin I, \end{cases}$$

In particular, we set, as already mentioned above, $V_j^{\text{div}} := V_j^{\emptyset} = V_j^- \otimes \dots \otimes V_j^-$ and $V_j^i := V_j^{\{i\}}$. The corresponding scaling functions are

$$(2.12) \quad \varphi^I(x_1, \dots, x_n) := \prod_{\nu=1}^n \xi_\nu^I(x_\nu), \quad \text{where } \xi_\nu^I := \begin{cases} \xi^+, & \text{if } \nu \in I, \\ \xi^-, & \text{if } \nu \notin I. \end{cases}$$

Obviously, the vector fields

$$(2.13) \quad \varphi_i := \varphi^{\{i\}} \delta_i,$$

are scaling functions in $\mathbf{H}(\text{div}; \mathbb{R}^n)$, where $\delta_i := (\delta_{1,i}, \dots, \delta_{n,i})^T$ denotes the i -th canonical vector in \mathbb{R}^n . In fact,

$$\mathbf{V}_j := \text{clos span} \{ \varphi_{i;j,k}, i = 1, \dots, n, k \in \mathbb{Z}^n \}$$

is the desired MRA, where

$$\varphi_{i;j,k}(x) := 2^{(nj)/2} \varphi_i(2^j x - k) \quad x \in \mathbb{R}^n, j \in \mathbb{Z}, k \in \mathbb{Z}^n.$$

Divergence-free wavelets. Now, we consider the corresponding wavelets. Let η^+ , η^- be (orthogonal or biorthogonal) wavelets according to ξ^+ and ξ^- , respectively. Then, the wavelets generated by φ^I are defined for

$$E_n := \{0, 1\}^n, \quad \text{and} \quad E_n^* := E_n \setminus \{0\}$$

as follows

$$(2.14) \quad \psi_e^I := \bigotimes_{\nu=1}^n \vartheta_{e_\nu, \nu}^I, \quad \text{where} \quad \vartheta_{\ell, \nu}^I := \begin{cases} \xi_\nu^I, & \text{if } \ell = 0, \\ \eta_\nu^I, & \text{if } \ell = 1, \end{cases}$$

with η_ν^I defined analogously to ξ_ν^I in (2.12). The vector-valued wavelets in $\mathbf{H}(\text{div}; \mathbb{R}^n)$ are scaled translates $\psi_{i,e;j,k}$ of the mother wavelets

$$\psi_{e,i} := \psi_e^{\{i\}} \delta_i, \quad e \in E_n^*.$$

Note that for any $e \in E_n^*$, there exists an index

$$i_e \in \{1, \dots, n\} \quad \text{such that} \quad e_{i_e} = 1,$$

i.e., at least one factor in the tensor product is a univariate wavelet. This index plays a crucial role in the definition of divergence-free wavelets. In fact, the additional constraint $\text{div } \mathbf{u} = 0$ fixes one degree of freedom. This means that we have $|E_n^*| (n-1) = (2^n - 1)(n-1)$ wavelets instead of $n |E_n^*|$ (which is the number of wavelets in $\mathbf{L}_2(\Omega)$). Then, we define divergence-free wavelets for $e \in E_n^*$ and $i \neq i_e$ component-wise by

$$(2.15) \quad \psi_{e,i;\nu}^\nabla := \begin{cases} \psi_e^{\{i\}}, & \text{if } \nu = i; \\ -\frac{1}{4} \frac{\partial}{\partial x_i} \psi_e^{\{i, i_e\}}, & \text{if } \nu = i_e; \\ 0 & \text{otherwise.} \end{cases}$$

We always denote by $\mathbf{f}_{:i}$ the i -th component of a vector field \mathbf{f} . Note that $\#\{(e, i) : e \in E^*, i \neq i_e\} = |E^*| (n-1) = (2^n - 1)(n-1)$, so that we have in fact the correct number of divergence-free wavelets. It is easily seen that these functions are divergence-free, i.e.,

$$\begin{aligned} \text{div } \psi_{e,i}^\nabla &= \frac{\partial}{\partial x_i} \psi_{e,i;i}^\nabla + \frac{\partial}{\partial x_{i_e}} \psi_{e,i;i_e}^\nabla \\ &= \frac{\partial}{\partial x_i} \psi_e^{\{i\}} - \frac{1}{4} \frac{\partial}{\partial x_{i_e}} \frac{\partial}{\partial x_i} \psi_e^{\{i, i_e\}} \\ &= \frac{1}{4} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_{i_e}} \psi_e^{\{i, i_e\}} - \frac{1}{4} \frac{\partial}{\partial x_{i_e}} \frac{\partial}{\partial x_i} \psi_e^{\{i, i_e\}} = 0. \end{aligned}$$

Note, that the index i_e is chosen so that we can apply equality (2.10). Moreover, it was shown in [29, 38] that the system

$$\Psi^\nabla := \{\psi_{e,i}^\nabla : e \in E^*, i \neq i_e\}$$

generates a Riesz basis for the space $\mathbf{V}(\text{div}; \mathbb{R}^n)$ of all divergence-free vector fields on \mathbb{R}^n . Generalizations to more general domains can be found in [37, 40, 41].

As already mentioned, the above system is a biorthogonal one. Let us denote by the accent ‘ \sim ’ the dual system corresponding to all functions involved in the construction. Then, the dual wavelets according to $\psi_{e,i}^\nabla$ are defined by

$$(2.16) \quad \tilde{\psi}_{e,i}^\nabla := \tilde{\psi}_e^{\{i\}} \delta_i, \quad e \in E^*, i \neq i_e.$$

Note that these functions are *not* divergence-free which is by no means a contradiction. It comes from the splitting of the complement spaces in (2.6).

2.2. Interpolatory Wavelets. As above, the starting point for the construction of interpolatory wavelets is a MRA generated by an interpolatory scaling function ξ , i.e., $\xi(k) = \delta_{0,k}$ for all $k \in \mathbb{Z}$. Thus, any function $f_j \in V_j$ has an expansion of the form

$$(2.17) \quad f_j(x) = \sum_{k \in \mathbb{Z}} c_{j,k} \xi(2^j x - k), \quad c_{j,k} = f(2^{-j}k),$$

i.e., the expansion coefficients are given in terms of *point values (samples)* of f rather than by a scalar product $(f, \xi_{j,k})_{0,\Omega}$.

It is readily seen that

$$(2.18) \quad \eta(x) := 2\xi(2x - 1)$$

is a possible choice for such a wavelet with interpolatory properties, [18]. This is also the reason, why in several papers the term ‘interpolatory wavelet’ is used for the scaling function ξ . From (2.18) it is obvious, that interpolatory wavelets do not have vanishing moments. On the other hand, one still has fast algorithms for decomposition and reconstruction of a signal (Fast Wavelet Transform). Finally, one cannot hope to obtain a Riesz basis for L_2 . However, in [18], norm equivalences and characterizations for Besov spaces embedded in $C(\mathbb{R})$ are proven (the range of Besov spaces depends of course on the regularity and polynomial exactness of ξ), see also Theorem 2.4 below.

2.2.1. Subdivision Schemes and Interpolatory Wavelets. One way, to construct interpolatory scaling functions is to consider iterative interpolation processes. In this section, we review some known facts on subdivision schemes and interpolatory wavelets. A subdivision scheme based on a *mask* $\{a_k : k \in \mathbb{Z}\}$ starts (for planar curves) with a sequence of *control points* $\boldsymbol{\lambda} := \{\lambda_k : k \in \mathbb{Z}\}$. These control points are interpreted as values of a signal on the integer lattice \mathbb{Z} and one step of the subdivision scheme refines these data to the finer grid $\mathbb{Z}/2 = \{k/2 : k \in \mathbb{Z}\}$ using the following definition

$$(2.19) \quad \lambda_k^1 := (S\boldsymbol{\lambda})_k := \sum_{m \in \mathbb{Z}} a_{k-2m} \lambda_m, \quad k \in \mathbb{Z},$$

and recursively $\boldsymbol{\lambda}^i := S^i \boldsymbol{\lambda} = S(S^{i-1} \boldsymbol{\lambda})$. In practice, one usually assumes that the mask is *compactly supported*, i.e., there are only finitely many non-zero coefficients a_k so that the sum in (2.19) is finite.

A subdivision operator $S : \ell_\infty(\mathbb{Z}) \rightarrow \ell_\infty(\mathbb{Z})$ is called *convergent* if for every sequence $\boldsymbol{\lambda}$ there exists a limit function $f = f_\boldsymbol{\lambda} \in C(\mathbb{R})$ such that

$$\lim_{i \rightarrow \infty} \|(S^i \boldsymbol{\lambda} - f(k/2^i))_{i \in \mathbb{Z}}\|_{\ell_\infty(\mathbb{Z})} = 0,$$

and $f \not\equiv 0$ for at least one $\boldsymbol{\lambda}$.

We are in particular interested in a special subclass of subdivision schemes, namely *interpolatory* ones, where the limit function interpolates the original control points, i.e. $f(k) = \lambda_k$, $k \in \mathbb{Z}$, (see, e.g. [32]).

Remark 2.1. One particular example are subdivision schemes that leave the previous control points unchanged while adding new intermediate control points. This is the case if

$$(2.20) \quad a_{2k} = \delta_k, \quad k \in \mathbb{Z}.$$

Note that condition (2.20) is sufficient but *not* necessary.

We now quote two results from [32], which we will use later. For this purpose, we introduce the *symbol* $\mathbf{a}(z)$ of a mask \mathbf{a} defined by

$$(2.21) \quad \mathbf{a}(z) := \sum_{k \in \mathbb{Z}} a_k z^k, \quad z \in \mathbb{C}.$$

Theorem 2.2. [32, Thm. 2.1] *Suppose that the subdivision scheme S satisfies (2.20) and converges. Then*

$$(2.22) \quad \mathbf{a}(-1) = 0, \quad \mathbf{a}(1) = 2$$

and the limit f is given by

$$(2.23) \quad f(x) = \sum_{k \in \mathbb{Z}} \lambda_k \xi(x - k), \quad x \in \mathbb{R},$$

where ξ is a continuous function of compact support which satisfies the refinement equation

$$(2.24) \quad \xi(x) = \sum_{k \in \mathbb{Z}} a_k \xi(2x - k), \quad x \in \mathbb{R},$$

as well as $\sum_{k \in \mathbb{Z}} \xi(x - k) = 1$, $x \in \mathbb{R}$, and

$$(2.25) \quad \xi(k) = \delta_k, \quad k \in \mathbb{Z}.$$

Conversely, if $\xi \in C(\mathbb{R})$ is a solution of the refinement equation (2.24) satisfying (2.25), then (2.22) holds and the subdivision scheme converges to f in (2.23).

The following result gives some information on the regularity of ξ which is an important information for the construction of divergence-free functions.

Theorem 2.3. [32, Thm. 2.2] *Suppose \mathbf{a} is a finite mask satisfying (2.20) and the corresponding subdivision scheme converges. Then $\xi \in C^r(\mathbb{R})$, $r \geq 1$, if and only if*

$$\mathbf{a}(z) = \frac{(1+z)^{r+1}}{2^r} \mathbf{b}(z)$$

for some $\mathbf{b}(z)$ such that $\mathbf{b}(1) = 1$ and the subdivision scheme corresponding to $\mathbf{c}(z) := (1+z)\mathbf{b}(z)$ converges.

2.2.2. Characterization of Function Spaces. The following general characterization result is well-known. We will derive similar results for interpolatory divergence-free wavelets. Here $B_q^s(L_p(\mathbb{R}))$ denotes the standard Besov space.

Theorem 2.4. [18, Thm. 2.7] *Let $\xi \in C^r(\mathbb{R})$ be a compactly supported, refinable, interpolatory scaling function which is exact of order d , i.e.,*

$$x^m = \sum_{k \in \mathbb{Z}} \alpha_{mk} \xi(x - k), \quad 0 \leq m \leq d, \quad x \in \mathbb{R},$$

with appropriate coefficients $\alpha_{mk} \in \mathbb{R}$, where the convergence of the sum is to be understood locally. Then we have the following characterization for the interpolatory wavelets $\eta = \xi(2 \cdot -1)$ for $\frac{1}{p} < s < \min\{r, d\}$, $0 < p, q \leq \infty$

$$(2.26) \quad \left\| \sum_{j,k} d_{j,k} \eta_{j,k} \right\|_{B_q^s(L_p(\mathbb{R}))} \sim \left(\sum_j \left(2^{j(s+\frac{1}{2}-\frac{1}{p})} \sum_k |d_{j,k}|^p \right)^q \right)^{1/q}.$$

Note that the above result also holds for interpolatory spline functions (which are not compactly supported, but piecewise polynomials). In this case, one obviously has $r = d - 1$.

3. A DEAD END

The maybe most natural approach is to use the above mentioned interpolatory scaling functions for a construction of divergence-free interpolatory wavelets. Unfortunately, this is not possible, as we will show in this section.

At a first glance, there seems to be some hope from the following result.

Proposition 3.1. *Let $\xi^+ \in C^r(\mathbb{R})$ be a compactly supported, interpolatory scaling function such that (2.20) holds. Then, there exists a compactly supported scaling function $\xi^- \in C^{r-1}(\mathbb{R})$ defined by the mask*

$$(3.1) \quad \mathbf{a}^-(z) := \frac{2}{1+z} \mathbf{a}^+(z)$$

satisfying

$$(3.2) \quad \frac{d}{dx} \xi^+(x) = \xi^-(x) - \xi^-(x-1).$$

Proof. By Theorem 2.3 the symbol $\mathbf{a}^+(z)$ can be divided by $1+z$ and one has

$$\mathbf{a}^-(z) = \frac{(1+z)^r}{2^{r-1}} \mathbf{b}(z)$$

with the same $\mathbf{b}(z)$ as for $\mathbf{a}^+(z)$. Hence, the subdivision scheme converges resulting in the existence of ξ^- . Finally, (3.2) follows from [29, Prop. 4]. \square

Hence, the required differentiation process as in (2.9) in fact works.

Remark 3.2. (a) Note that \mathbf{a}^- also satisfies (2.22) which is easily seen by direct computation and Theorem 2.3.

(b) In general ξ^- does *not* satisfy (2.20). In fact, from (3.1) we obtain

$$(3.3) \quad 2a_k^- = a_k + a_{k-1},$$

which implies

$$a_{2k}^- = \frac{1}{2}(a_{2k} + a_{2k-1}) = \frac{1}{2}(\delta_k + a_{2k-1})$$

which only coincides with δ_k if $a_{2k-1} = \delta_k$. This however is only true for the scaling function corresponding to the orthonormal Haar wavelet.

Part (b) of the previous remark already gives a first hint, why we cannot use Proposition 3.1 to base a construction of interpolatory divergence-free wavelets upon it. In fact, the resulting scaling function ξ^- may not be interpolatory. Thus, we cannot use it in a tensor product construction like in (2.11). We will show that ξ^- is in fact not interpolatory.

One popular choice for the construction of wavelets are spline functions. Choosing the space of cardinal splines of *even* order $d > 0$ as V_0^+ , we can choose the corresponding fundamental spline [33] as interpolatory scaling function ξ^+ . However, then V_0^- is the space of splines of *odd* order, where the interpolation problem at the integer knots does not have a unique solution. Thus, we cannot have an interpolatory scaling function ξ^- , i.e., splines are not suited for the construction of

interpolatory divergence-free wavelets. Moreover, ξ^+ is only compactly supported for $d = 2$ (i.e., the hat function).

Next, we consider the interpolation process introduced by Deslauriers and Dubuc in [16]. It can be described as follows. Using the standard Lagrange fundamental polynomials $L_k \in \mathcal{P}_{2N+1}$ corresponding to the integers $k = -N, \dots, N+1$, defined by

$$L_k(x) := \prod_{\substack{i=-N \\ i \neq k}}^{N+1} \frac{x-i}{k-i}, \quad x \in \mathbb{R}, \quad k = -N, \dots, N+1,$$

the mask \mathbf{a} is defined by

$$(3.4) \quad \begin{aligned} a_{2k} &= \delta_k & k \in \mathbb{Z}, \\ a_{2k+1} &= L_{-k}\left(\frac{1}{2}\right) & k = -N-1, \dots, N, \\ a_{2k+1} &= 0 & k > N \text{ or } k < -N-1. \end{aligned}$$

The regularity of ξ in dependence of the parameter N is studied in [16].

Example 3.3. For $N = 2$ and $N = 5$ we display the original Deslauriers-Dubuc functions and the modified functions ξ^- in Figure 1.

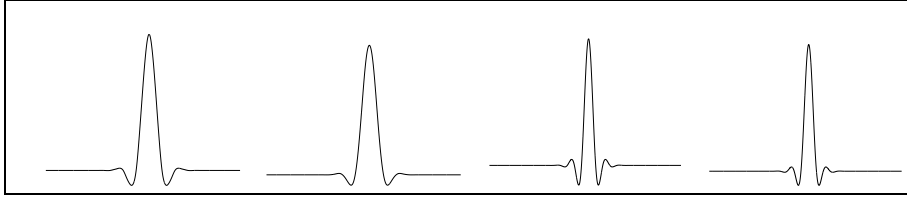


FIGURE 1. Deslauriers-Dubuc functions and modified functions φ^- for $N = 2$ (left) and $N = 5$ (right)

It is known that there is a close relationship between Deslauriers-Dubuc functions ξ_N and Daubechies' orthonormal scaling functions θ_N . In fact, ξ_N is the autocorrelation of θ_N

$$(3.5) \quad \xi_N(x) = (\theta_N * \theta_N)(x) := \int_{\mathbb{R}} \theta_N(t) \theta_N(t-x) dt, \quad x \in \mathbb{R}.$$

Moreover, it was shown in [32] that a relation like (3.5) also holds for a large class of interpolatory scaling functions arising as the limit of subdivision schemes corresponding to masks satisfying certain conditions.

Proposition 3.4. *Let $\xi^+ = \xi_N$ be a Deslauriers-Dubuc interpolatory scaling function. Then, the scaling function ξ^- defined by Proposition 3.1 is not interpolatory.*

Proof. Since $\xi^+ = \theta^+ * \theta^+$, we obtain

$$\frac{d}{dx} \xi^+ = \left(\frac{d}{dx} \theta^+ \right) * \theta^+$$

and we can apply (2.9) to the orthonormal Daubechies' scaling function θ^+ . This yields a pair of *biorthogonal* scaling functions θ^- , $\tilde{\theta}^-$, so that we obtain

$$\frac{d}{dx} \xi^+(x) = (\theta^- * \theta^+)(x) - (\theta^- * \theta^+)(x-1),$$

i.e., $\xi^- = \theta^- * \theta^+$ which is not interpolating at the integers. \square

As a consequence of Proposition 3.4 there are no Deslauriers-Dubuc interpolatory divergence-free scaling functions. Now, we describe a possible way to circumvent the above mentioned problem.

4. HERMITE INTERPOLATORY SPLINES AND MULTIWAVELETS

As we have seen in Section 3 above, it does not seem to be a good idea to try to construct interpolatory divergence-free wavelets starting from a scaling system generated by only one function ξ^+ . Thus, we now aim to construct two families of functions $\Xi^+ = \{\xi_1^+, \dots, \xi_m^+\}$ and $\Xi^- = \{\xi_1^-, \dots, \xi_m^-\}$ such that

- the derivatives of Ξ^+ are spanned in terms of Ξ^- ;
- both systems are interpolatory with respect to the integers.

This can in fact be achieved by Hermite interpolatory spline functions and leads us to multiwavelets.

Definition 4.1. The space of *cardinal splines of order d* , $d \geq 0$, with knots of multiplicity i (splines with deficiency $i - 1$) is defined by

$$S_{d,i} := \left\{ f \in C^{d-i-1}(\mathbb{R}) : f|_{[k,k+1)} \in \Pi_{d-1}, k \in \mathbb{Z} \right\}.$$

A stable basis of $S_{d,i}$ is induced by the B-splines

$$B_k^{d,i}(x) = (t_{i,k+d} - t_{i,k}) [t_{i,k}, \dots, t_{i,k+d}] (\cdot - x)_+^{d-1}, \quad k \in \mathbb{Z},$$

with knots $t_{i,k} := \lfloor \frac{k}{i} \rfloor$ (see e.g. [34]). In fact, the B-spline basis can be represented by integer translates of the i successive B-splines $B_k^{d,i}, \dots, B_{k+i-1}^{d,i}$. Splines with knots of multiplicity i can be used to interpolate not only function values but also the values of derivatives up to order $i - 1$ if the splines are sufficiently smooth, i.e., if $d \geq 2i$.

4.1. Interpolatory Hermite Splines and Differentiation. In the sequel we consider cubic Hermite splines defined as follows

$$(4.1) \quad \xi_1^+(x) := \begin{cases} 1 - 3x^2 - 2x^3, & \text{if } x \in [-1, 0]; \\ 1 - 3x^2 + 2x^3, & \text{if } x \in [0, 1]; \\ 0 & \text{otherwise,} \end{cases}$$

$$(4.2) \quad \xi_2^+(x) := \begin{cases} x + 2x^2 + x^3, & \text{if } x \in [-1, 0]; \\ x - 2x^2 + x^3, & \text{if } x \in [0, 1]; \\ 0 & \text{otherwise,} \end{cases}$$

as well as the quadratic Hermite splines

$$(4.3) \quad \xi_1^-(x) := \begin{cases} -6x - 6x^2, & \text{if } x \in [-1, 0]; \\ 0 & \text{otherwise,} \end{cases}$$

$$(4.4) \quad \xi_2^-(x) := \begin{cases} 1 + 4x + 3x^2, & \text{if } x \in [-1, 0]; \\ 1 - 4x + 3x^2, & \text{if } x \in [0, 1]; \\ 0 & \text{otherwise,} \end{cases}$$

The corresponding functions are displayed in Figure 2. These functions have been investigated in several papers. Biorthogonal multiwavelets on the interval $[0, 1]$ generated by ξ_1^+, ξ_2^+ have been introduced in [12]. It is well-known that the integer translates of ξ_1^+, ξ_2^+ generate the space of C^1 -continuous piecewise cubic functions on \mathbb{R} which interpolate function values and first derivatives at the integers. Based

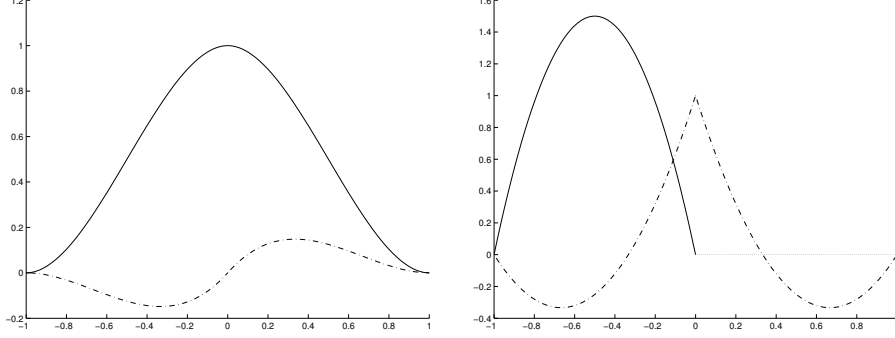


FIGURE 2. Original (left) and differentiated system of Hermite interpolatory splines.

on these functions, *biorthogonal* divergence-free multiwavelets (also on rectangular domains) have been constructed in [27, 28]. There, the system ξ_1^+ , ξ_2^+ plays a role as part of a whole family of multiwavelets that can be linked by differentiation and integration similar to (2.9).

The above functions are fundamental interpolants in the sense that

$$(4.5) \quad \frac{d^\nu}{dx^\nu} \xi_m^+(k) = \delta_{k0} \delta_{m-1,\nu}, \quad k \in \mathbb{Z}, \nu \in \{0, 1\}, m \in \{1, 2\},$$

and

$$(4.6) \quad \xi_m^-(k) = \delta_{k0} \delta_{m2}, \quad \int_{k-1}^k \xi_m^-(x) dx = \delta_{k0} \delta_{m1}, \quad k \in \mathbb{Z}, m \in \{1, 2\}.$$

Furthermore, it is easily seen that we have the following relation for the derivatives

$$(4.7) \quad \frac{d}{dx} \xi_1^+(x) = \xi_1^-(x) - \xi_1^-(x-1), \quad \frac{d}{dx} \xi_2^+(x) = \xi_2^-(x),$$

which will turn out to be crucial in the sequel.

It is well known, that $\{\xi_1^+(\cdot - k), \xi_2^+(\cdot - k) : k \in \mathbb{Z}\}$ is a stable basis for $S_{4,2}$. Indeed, this property follows immediately from $\xi_1^+ = B_{-1}^{4,2} + B_{-2}^{4,2}$ and $\xi_2^+ = \frac{1}{3}(B_{-1}^{4,2} - B_{-2}^{4,2})$. Furthermore, from $\xi_1^- = 3B_{-2}^{3,2}$ and $\xi_2^- = 3(B_{-1}^{3,2} - B_0^{3,2} - B_0^{3,2}(\cdot + 1))$ it follows immediately that $\{\xi_1^-(\cdot - k), \xi_2^-(\cdot - k) : k \in \mathbb{Z}\}$ is a stable basis for $S_{3,2}$.

For our further investigations we will need the cubic Hermite spline interpolant defined as

$$(4.8) \quad \Lambda_j^+ f := \sum_{k \in \mathbb{Z}} \left(f(2^{-j}k) \xi_{1;j,k}^+ + 2^{-j} f'(2^{-j}k) \xi_{2;j,k}^+ \right),$$

as well as the generalized quadratic Hermite spline interpolant

$$\Lambda_j^- f := \sum_{k \in \mathbb{Z}} \left(2^j \int_{2^{-j}(k-1)}^{2^{-j}k} f(t) dt \xi_{1;j,k}^- + f(2^{-j}k) \xi_{2;j,k}^- \right),$$

where

$$\xi_{m;j,k}^\pm(x) := \xi_m^\pm(2^j x - k), \quad x \in \mathbb{R}, j, k \in \mathbb{Z},$$

(i.e., $\xi_{m;j,k}^\pm$ is normalized in L_∞), i.e., by $f_{;j,k}$ we denote the L_∞ -normalized shifted scaling of a real-valued function f . Obviously, Λ_j^\pm is a projection onto the space

$$V_j^\pm := \text{clos}_{L_2} \text{span}\{\xi_{m;j,k}^\pm : m \in \{1,2\}, k \in \mathbb{Z}\}$$

and we have for all $j \in \mathbb{Z}$

$$(4.9) \quad (\Lambda_j^\pm f)(2^{-j}k) = f(2^{-j}k),$$

$$(4.10) \quad (\Lambda_j^+ f)'(2^{-j}k) = f'(2^{-j}k),$$

$$(4.11) \quad \int_{2^{-j}(k-1)}^{2^{-j}k} \Lambda_j^- f(x) dx = \int_{2^{-j}(k-1)}^{2^{-j}k} f(x) dx, \quad k \in \mathbb{Z}.$$

It will be convenient to express the above interpolants in terms of dual functionals in the sense

$$(4.12) \quad \Lambda_j^\pm f = \sum_{k \in \mathbb{Z}} \sum_{m=1}^2 \langle f, \tilde{\xi}_{m;j,k}^\pm \rangle \xi_{m;j,k}^\pm,$$

where $\langle \cdot, \cdot \rangle$ denotes the dual pair. It is readily seen that, in the sense of distributions,

$$\tilde{\xi}_1^+ = \delta_0, \quad \text{i.e., } \langle f, \tilde{\xi}_{1;j,k}^+ \rangle = f(2^{-j}k),$$

$$\tilde{\xi}_2^+ = -\delta'_0, \quad \text{i.e., } \langle f, \tilde{\xi}_{2;j,k}^+ \rangle = 2^{-j} f'(2^{-j}k),$$

as well as

$$\tilde{\xi}_1^- = \chi_{[-1,0)}, \quad \text{i.e., } \langle f, \tilde{\xi}_{1;j,k}^- \rangle = 2^j \int_{2^{-j}(k-1)}^{2^{-j}k} f(x) dx,$$

$$\tilde{\xi}_2^- = \delta_0, \quad \text{i.e., } \langle f, \tilde{\xi}_{2;j,k}^- \rangle = f(2^{-j}k).$$

Note, that the dual functionals are (L_∞ -normalized) scaled and shifted versions of $\tilde{\xi}_m^\pm$, in the sense that

$$(4.13) \quad \langle f, \tilde{\xi}_{m;j,k}^\pm \rangle = \langle f(\frac{\cdot+k}{2^j}), \tilde{\xi}_m^\pm \rangle.$$

Then, we have the following relations.

Proposition 4.2. *It holds:*

$$(a) \quad \frac{d}{dx} \tilde{\xi}_1^- = \tilde{\xi}_1^+(\cdot + 1) - \tilde{\xi}_1^+, \quad \frac{d}{dx} \tilde{\xi}_2^- = -\tilde{\xi}_2^+ \text{ in the sense of distributions.}$$

$$(b) \quad \langle f', \tilde{\xi}_{1;j,k}^- \rangle = \langle f, 2^j (\tilde{\xi}_{1;j,k}^+ - \tilde{\xi}_{1;j,k-1}^+) \rangle, \quad \langle f', \tilde{\xi}_{2;j,k}^- \rangle = \langle f, 2^j \tilde{\xi}_{2;j,k}^+ \rangle.$$

$$(c) \quad \frac{d}{dx} \Lambda_j^+ f = \Lambda_j^-(f'), \text{ if } f \in C^1(\mathbb{R}).$$

$$(d) \quad \langle \xi_{m';j,k'}^\pm, \tilde{\xi}_{m;j,k}^\pm \rangle = \delta_{m,m'} \delta_{k,k'}.$$

Proof. (a) is obvious, also the second part of (b). By definition, we have

$$\begin{aligned} \langle f', \tilde{\xi}_{1;j,k}^- \rangle &= 2^j \int_{2^{-j}(k-1)}^{2^{-j}k} f'(x) dx = 2^j (f(2^{-j}k) - f(2^{-j}(k-1))) \\ &= \langle f, 2^j (\tilde{\xi}_{1;j,k}^+ - \tilde{\xi}_{1;j,k-1}^+) \rangle, \end{aligned}$$

which proves (b). Finally, for $f \in C^1(\mathbb{R})$, we have by (a), (b) and (4.7)

$$\begin{aligned} \frac{d}{dx} \Lambda_j^+ f &= \sum_{k \in \mathbb{Z}} \left(\langle f, \tilde{\xi}_{1;j,k}^+ \rangle 2^j (\xi_{1;j,k}^- - \xi_{1;j,k+1}^-) + \langle f, \tilde{\xi}_{2;j,k}^+ \rangle 2^j \xi_{2;j,k}^- \right) \\ &= \sum_{k \in \mathbb{Z}} \left(\langle f, 2^j (\tilde{\xi}_{1;j,k}^+ - \tilde{\xi}_{1;j,k-1}^+) \rangle \xi_{1;j,k}^- + \langle f, 2^j \tilde{\xi}_{2;j,k}^+ \rangle \tilde{\xi}_{2;j,k}^- \right) \\ &= \sum_{k \in \mathbb{Z}} \sum_{m=1}^2 \langle f', \tilde{\xi}_{m;j,k}^- \rangle \tilde{\xi}_{i;j,k}^- = \Lambda_j^- f'. \end{aligned}$$

Finally (d) follows from (4.5) and (4.6). \square

Often, one collects the two generating functions in one vector

$$\boldsymbol{\xi}^+ := \begin{pmatrix} \xi_1^+ \\ \xi_2^+ \end{pmatrix}, \quad \boldsymbol{\xi}^- := \begin{pmatrix} \xi_1^- \\ \xi_2^- \end{pmatrix}.$$

Such a vector is called a *multigenerator* if the integer translates form an L_2 -stable basis and $\boldsymbol{\xi}$ satisfies a *refinement* equation

$$(4.14) \quad \boldsymbol{\xi}(x) = \sum_{k \in \mathbb{Z}} \mathbf{A}_k \boldsymbol{\xi}(2x - k), \quad x \in \mathbb{R} \text{ a.e.},$$

with mask $\mathbf{A} := \{\mathbf{A}_k\}_{k \in \mathbb{Z}}$, $\mathbf{A}_k \in \mathbb{R}^{2 \times 2}$.

Applying the projectors Λ_1^+ to $\xi_i^+(\frac{\cdot}{2}) \in S_{4,2}$, $i = 1, 2$, it follows that $\boldsymbol{\xi}^+$ satisfies the refinement equation (4.14) with mask matrices

$$\mathbf{A}_{-1}^+ = \begin{pmatrix} \frac{1}{2} & \frac{3}{4} \\ -\frac{1}{8} & -\frac{1}{8} \end{pmatrix}, \quad \mathbf{A}_0^+ = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad \mathbf{A}_1^+ = \begin{pmatrix} \frac{1}{2} & -\frac{3}{4} \\ \frac{1}{8} & -\frac{1}{8} \end{pmatrix},$$

(see also [12, 23]) and the application of Λ_1^- to $\xi_i^-(\frac{\cdot}{2}) \in S_{3,2}$ yields that $\boldsymbol{\xi}^-$ is refinable with mask matrices

$$\mathbf{A}_{-2}^- = \begin{pmatrix} 0 & 0 \\ -\frac{1}{4} & 0 \end{pmatrix}, \quad \mathbf{A}_{-1}^- = \begin{pmatrix} 0 & 0 \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}, \quad \mathbf{A}_0^- = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & 1 \end{pmatrix}, \quad \mathbf{A}_1^- = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ -\frac{1}{4} & -\frac{1}{4} \end{pmatrix}.$$

Next, we define interpolating wavelets

$$(4.15) \quad \begin{aligned} \eta_m^+ &:= \xi_m^+(2 \cdot -1), \quad m = 1, 2, \\ \eta_1^- &:= \xi_1^-(2 \cdot -1) - \xi_1^-(2 \cdot -2), \\ \eta_2^- &:= \xi_2^-(2 \cdot -1). \end{aligned}$$

The following interpolation relations follow from (4.5) and (4.6)

$$\begin{aligned} \frac{d^\nu}{dx^\nu} \eta_m^+(\frac{k}{2}) &= 2^\nu \delta_{k1} \delta_{m-1,\nu}, \quad \nu \in \{0, 1\}, \\ \eta_m^-(\frac{k}{2}) &= \delta_{k1} \delta_{m2}, \\ \int_{k/2}^{(k+1)/2} \eta_m^-(x) dx &= \frac{1}{2} \delta_{m1} (\delta_{k0} - \delta_{k1}), \quad k \in \mathbb{Z}, \quad i \in \{1, 2\}. \end{aligned}$$

Note that we obtain the following crucial property

$$(4.16) \quad \frac{d}{dx} \eta_m^+(x) = 2 \eta_m^-(x), \quad m = 1, 2.$$

By the interpolation properties one sees immediately that the wavelet spaces

$$W_j^\pm := \text{clos}_{L_2} \text{span}\{\eta_{m;j,k}^\pm : m \in \{1, 2\}, k \in \mathbb{Z}\}$$

satisfy $V_j^\pm \cap W_j^\pm = \{0\}$ and $V_j^\pm \cup W_j^\pm = V_{j+1}^\pm$. In particular, we have

$$(4.17) \quad \begin{aligned} g(2^{-j}k) &= g'(2^{-j}k) = 0, & \text{if } g \in W_j^+, \\ g(2^{-j}k) &= \int_{2^{-j}k}^{2^{-j}(k+1)} g(x) dx = 0, & \text{if } g \in W_j^-, \end{aligned}$$

i.e. $\langle g, \tilde{\xi}_{\mu;j,k}^\pm \rangle = 0$ and $\Lambda_j^\pm g = 0$ if $g \in W_j^\pm$.

In order to compute the wavelet expansion of a function, we are also interested in dual wavelets.

Theorem 4.3. *The functionals $\tilde{\eta}_{m;j,k}^\pm$ defined by*

$$\langle f, \tilde{\eta}_{m;j,k}^\pm \rangle := \langle f - \Lambda_j^\pm f, \tilde{\xi}_{m;j+1,2k+1}^\pm \rangle$$

satisfy the biorthogonality conditions

$$(4.18) \quad \langle \eta_{m';j',k'}^\pm, \tilde{\eta}_{m;j,k}^\pm \rangle = \delta_{m,m'} \delta_{j,j'} \delta_{k,k'}$$

as well as

$$(4.19) \quad \langle \eta_{m';j,k'}^\pm, \tilde{\xi}_{m;j,k}^\pm \rangle = \langle \xi_{m';j,k'}^\pm, \tilde{\eta}_{m;j,k}^\pm \rangle = 0$$

for all $m, m' = 1, 2$ and $j, j', k, k' \in \mathbb{Z}$. Furthermore, $\tilde{\eta}_{m;j,k}^\pm$ is a shifted translate of $\tilde{\eta}_m^\pm$ in the sense that

$$(4.20) \quad \langle f, \tilde{\eta}_{j,k}^\pm \rangle = \langle f(\cdot + \frac{k}{2^j}), \tilde{\eta}_j^\pm \rangle,$$

where

$$\begin{aligned} \tilde{\eta}_1^+ &:= \delta_{\frac{1}{2}} - \frac{1}{2}\delta_0 - \frac{1}{2}\delta_1 + \frac{1}{8}\delta'_0 - \frac{1}{8}\delta'_1, \\ \tilde{\eta}_2^+ &:= \frac{3}{4}\delta_0 - \frac{3}{4}\delta_1 - \frac{1}{2}\delta'_{\frac{1}{2}} - \frac{1}{8}\delta'_0 - \frac{1}{8}\delta'_1, \\ \tilde{\eta}_1^- &:= \chi_{[0, \frac{1}{2}]} - \chi_{[\frac{1}{2}, 1]} - \frac{1}{4}\delta_0 + \frac{1}{4}\delta_1, \\ \tilde{\eta}_2^- &:= \delta_{\frac{1}{2}} + \frac{1}{4}\delta_0 + \frac{1}{4}\delta_1 - \frac{3}{2}\chi_{[0, 1]}. \end{aligned}$$

Proof. First, we show the biorthogonality relation (4.18). The equalities in (4.17) yield $\Lambda_j^\pm \eta_{m;j,k}^\pm = 0$, i.e.,

$$\langle \eta_{m';j',k'}^\pm, \tilde{\eta}_{m;j,k}^\pm \rangle = \langle \eta_{m';j',k'}^\pm - \Lambda_j^\pm \eta_{m';j',k'}^\pm, \tilde{\xi}_{m;j+1,2k+1}^\pm \rangle = \langle \eta_{m';j',k'}^\pm, \tilde{\xi}_{m;j+1,2k+1}^\pm \rangle.$$

Now (4.18) follows from the definition of $\tilde{\eta}_m^\pm$ in (4.15) and biorthogonality for the scaling functions in Proposition 4.2 (d). The equalities in (4.19) follow immediately from the definitions of $\eta_{m;j,k}^\pm$ and $\tilde{\eta}_{m;j,k}^\pm$.

Furthermore, (4.13) gives

$$\langle f, \tilde{\eta}_{m;j,k}^\pm \rangle = \langle f(\cdot + \frac{k}{2^j}) - \Lambda_j^\pm f(\cdot + \frac{k}{2^j}), \tilde{\xi}_{m;1,1}^\pm \rangle.$$

With (4.12), we obtain the representations of the particular dual functionals by

$$\langle f, \tilde{\eta}_m^\pm \rangle = \langle f, \tilde{\xi}_{m;1,1}^\pm \rangle - \sum_{k \in \mathbb{Z}} \langle f, \tilde{\xi}_{m;0,k}^\pm \rangle \langle \xi_{m;0,k}^\pm, \tilde{\xi}_{m;1,1}^\pm \rangle.$$

□

The dual functionals lead to a fast decomposition algorithm. For a given function $f = \sum_{m,k} c_{m;j+1,k}^\pm \xi_{m;j+1,k}^\pm \in V_{j+1}^\pm$ we want to compute the coefficients $c_{m;j,k}^\pm$ and $d_{m;j,k}^\pm$ of the expansion

$$(4.21) \quad f = \sum_{m,k} c_{m;j,k}^\pm \xi_{m;j,k}^\pm + \sum_{m,k} d_{m;j,k}^\pm \eta_{m;j,k}^\pm.$$

Applying $\tilde{\xi}_{m;j,k}^\pm$ to f we obtain with (4.19) that $c_{m;j,k}^\pm = \langle f, \tilde{\xi}_{m;j,k}^\pm \rangle$, i.e.,

$$\begin{aligned} c_{1;j,k}^+ &= c_{1;j+1,2k}^+, & c_{2;j,k}^+ &= 2c_{2;j+1,2k}^+, \\ c_{2;j,k}^- &= c_{2;j+1,2k}^-, & c_{1;j,k}^- &= \frac{1}{2}(c_{1;j+1,2k-1}^- + c_{1;j+1,2k}^-). \end{aligned}$$

Analogously, the wavelet coefficients can be obtained by $d_{m;j,k}^\pm = \langle f, \tilde{\eta}_{m;j,k}^\pm \rangle$, i.e.,

$$\begin{aligned} d_{1;j,k}^+ &:= c_{1;j+1,2k+1}^+ - \frac{1}{2}c_{1;j+1,2k}^+ - \frac{1}{2}c_{1;j+1,2k+2}^+ - \frac{1}{4}c_{2;j+1,2k}^+ + \frac{1}{4}c_{2;j+1,2k+2}^+, \\ d_{2;j,k}^+ &:= \frac{3}{4}c_{1;j+1,2k}^+ - \frac{3}{4}c_{1;j+1,2k+2}^+ + c_{2;j+1,2k+1}^+ + \frac{1}{4}c_{2;j+1,2k}^+ + \frac{1}{4}c_{2;j+1,2k+2}^+, \\ d_{1;j,k}^- &:= \frac{1}{2}c_{1;j+1,2k+1}^- - \frac{1}{2}c_{1;j+1,2k+2}^- - \frac{1}{4}c_{2;j+1,2k}^- + \frac{1}{4}c_{2;j+1,2k+2}^-, \\ d_{2;j,k}^- &:= c_{2;j+1,2k+1}^- + \frac{1}{4}c_{2;j+1,2k}^- + \frac{1}{4}c_{2;j+1,2k+2}^- - \frac{3}{4}c_{1;j+1,2k+1}^- - \frac{3}{4}c_{1;j+1,2k+2}^-. \end{aligned}$$

Finally, relation (4.16) implies an analogous relation

$$\frac{d}{dx} \tilde{\eta}_m^- = -\frac{1}{2} \tilde{\eta}_m^+$$

for the duals in the sense of distributions, i.e.,

$$(4.22) \quad \langle f', \tilde{\eta}_{m;j,k}^- \rangle = -2^{j-1} \langle f, \tilde{\eta}_{m;j,k}^+ \rangle.$$

4.2. Characterization of Function Spaces. For any function $f \in C^1(\mathbb{R})$ ($f \in C(\mathbb{R})$) we can compute the coefficients $\langle f, \tilde{\eta}_{m;j,k}^+ \rangle$ ($\langle f, \tilde{\eta}_{m;j,k}^- \rangle$). An interesting question is in which sense the expansion $\sum_{j,k} \langle f, \tilde{\eta}_{m;j,k}^\pm \rangle \eta_{m;j,k}^\pm$ is to be understood. In this section we show that Hermite interpolatory wavelets form unconditional bases for certain Besov spaces, which contain the space of continuous (differentiable) functions.

In the sequel we will use the following standard definition of Besov spaces. For $0 < p, q \leq \infty$ and $s > 0$ we define a semi-norm

$$(4.23) \quad |f|_{B_q^s(L_p(\Omega))} := \begin{cases} \left(\sum_{j \in \mathbb{Z}} 2^{jq_s} \omega_m(f, 2^{-j}, L_p(\Omega))^q \right)^{\frac{1}{q}}, & \text{if } 1 \leq q < \infty, \\ \sup_{j \in \mathbb{Z}} 2^{js} \omega_d(f, 2^{-j}, L_p(\Omega)), & \text{if } q = \infty, \end{cases}$$

with $m := [s] + 1$, where $\omega_m(f, t, L_p(\Omega))$ is the m -th order modulus of smoothness. The Besov space $B_{p,q}^s(\Omega)$ is the set of all $f \in L_p(\Omega)$ such that $|f|_{B_q^s(L_p(\Omega))}$ is finite. A norm is defined by

$$\|f\|_{B_q^s(L_p(\Omega))} := \|f\|_{L_p(\Omega)} + |f|_{B_q^s(L_p(\Omega))}.$$

There are numerous equivalent norms in the literature. Here, we only need that an equivalent norm is obtained if m in (4.23) is replaced by any integer $d > s$ (see e.g. [10]).

Furthermore, we need the following discrete norms for coefficient sequences. For the sequences $\boldsymbol{\beta} = (\beta_k)_{k \in \mathbb{Z}^n}$, $\boldsymbol{\alpha} = (\alpha_{jk})_{j \geq j_0, k \in \mathbb{Z}^n}$, and $\boldsymbol{\alpha}_j = (\alpha_{jk})_{k \in \mathbb{Z}^n}$, we define

$$\|\boldsymbol{\beta}\|_{\ell_p} := \begin{cases} \left(\sum_k |\beta_k|^p \right)^{\frac{1}{p}}, & \text{if } 0 < p < \infty, \\ \sup_k |\beta_k|, & \text{if } p = \infty, \end{cases}$$

$$\|\boldsymbol{\alpha}\|_{\ell_{pq}^s} := \begin{cases} \left(\sum_j (2^{j(s-\frac{n}{p})} \|\boldsymbol{\alpha}_j\|_{\ell_p})^q \right)^{\frac{1}{q}}, & \text{if } 0 < q < \infty, \\ \sup_j 2^{js} \|\boldsymbol{\alpha}_j\|_{\ell_p}, & \text{if } q = \infty, \end{cases}$$

for $p \in (0, \infty]$.

In [18], Besov spaces on the real line are characterized in terms of the wavelet expansion with respect to interpolatory wavelets (see Theorem 2.4). Here we need more general results for the characterization of Besov spaces on \mathbb{R}^n with Hermite interpolatory wavelets. Our characterization is essentially based on results reported in [10].

The Besov norm of a multiscale representation can be estimated by the corresponding discrete norm of the coefficients.

Lemma 4.4. *If $\phi \in B_\infty^\sigma(L_p(\mathbb{R}^n))$ is compactly supported, $0 < p, q \leq \infty$, and $0 < s < \sigma$, then*

$$(4.24) \quad \left\| \sum_{k \in \mathbb{Z}^n} \beta_k \phi(2^j \cdot -k) \right\|_{B_q^s(L_p(\mathbb{R}^n))} \lesssim 2^{(s-\frac{n}{p})j} \|\boldsymbol{\beta}\|_{\ell_p},$$

$$(4.25) \quad \left\| \sum_{j=j_0}^{\infty} \sum_{k \in \mathbb{Z}^n} \alpha_{j,k} \phi(2^j \cdot -k) \right\|_{B_q^s(L_p(\mathbb{R}^n))} \lesssim \|\boldsymbol{\alpha}\|_{\ell_{pq}^s}.$$

Proof. Obviously, it is sufficient to show the statement for finite right-hand sides. Let us first assume that $1 \leq p < \infty$. For $v_j := \sum_{k \in \mathbb{Z}^n} \alpha_{j,k} \phi(2^j \cdot -k)$ we have by scaling and the compact support of ϕ

$$\|v_j\|_{L_p(\mathbb{R}^n)} = 2^{-\frac{jn}{p}} \left\| \sum_{k \in \mathbb{Z}^n} \alpha_{j,k} \phi(\cdot -k) \right\|_{L_p(\mathbb{R}^n)} \lesssim 2^{-\frac{jn}{p}} \|\boldsymbol{\alpha}_j\|_{\ell_p} \|\phi\|_{L_p(\mathbb{R}^n)}.$$

Now, (4.24) follows from [10, Thm. 30.2]. Analogously we conclude for $M \in \mathbb{N}$, $M > \sigma$ that

$$\begin{aligned} \omega_M(v_j, t, L_p(\mathbb{R}^n)) &\lesssim 2^{-\frac{jn}{p}} \|\boldsymbol{\alpha}_j\|_{\ell_p} \omega_M(\phi, 2^j t, L_p(\mathbb{R}^n)) \\ &\lesssim 2^{j(\sigma-\frac{n}{p})} t^\sigma \|\boldsymbol{\alpha}_j\|_{\ell_p} \|\phi\|_{B_\infty^\sigma(L_p(\mathbb{R}^n))}. \end{aligned}$$

Since $\omega_M(f, t, L_p(\mathbb{R}^n)) \lesssim \|f\|_{L_p(\mathbb{R}^n)}$ we conclude for $0 < q < \infty$ that

$$\begin{aligned} \left| \sum_{j=j_0}^{\infty} v_j \right|_{B_q^s(L_p(\mathbb{R}^n))} &\lesssim \left(\sum_{r \in \mathbb{Z}} \left(2^{rs} \sum_{j=j_0}^{\infty} \omega_M(v_j, 2^{-r}, L_p(\mathbb{R}^n)) \right)^q \right)^{\frac{1}{q}} \\ &\lesssim \left(\sum_{r \in \mathbb{Z}} \left(2^{rs} \sum_{j=r}^{\infty} \|v_j\|_{L_p(\mathbb{R}^n)} \right)^q \right)^{\frac{1}{q}} + \left(\sum_{r=j_0}^{\infty} \left(2^{r(s-\sigma)} \sum_{j=j_0}^r 2^{j(\sigma-\frac{n}{p})} \|\boldsymbol{\alpha}_j\|_{\ell_p} \right)^q \right)^{\frac{1}{q}}. \end{aligned}$$

Furthermore, for any $q \in (0, \infty)$, we have

$$\left\| \sum_{j=j_0}^{\infty} v_j \right\|_{L_p(\mathbb{R}^n)} \leq \sum_{j=j_0}^{\infty} 2^{js} \|v_j\|_{L_p(\mathbb{R}^n)} \leq \left(\sum_{r=j_0}^{\infty} \left(2^{rs} \sum_{j=j_0}^r \|v_j\|_{L_p(\mathbb{R}^n)} \right)^q \right)^{\frac{1}{q}}.$$

Thus, substituting $r \rightarrow -r$ and $j \rightarrow -j$ in the second sum we obtain

$$\begin{aligned} & \left\| \sum_{j=j_0}^{\infty} v_j \right\|_{B_q^s(L_p(\mathbb{R}^n))} \\ & \lesssim \left(\sum_{r \in \mathbb{Z}} \left(2^{rs} \sum_{j=r}^{\infty} 2^{-\frac{jn}{p}} \|\alpha_j\|_{\ell_p} \right)^q \right)^{\frac{1}{q}} + \left(\sum_{r \in \mathbb{Z}} \left(2^{r(\sigma-s)} \sum_{j=r}^{\infty} 2^{j(\frac{n}{p}-\sigma)} \|\alpha_{-j}\|_{\ell_p} \right)^q \right)^{\frac{1}{q}}. \end{aligned}$$

Now, (4.25) follows immediately from the discrete Hardy inequality [17, Chap. 2, Lemma 3.4].

For $p = \infty$ or $q = \infty$ the proof is analogous, and for $p < 1$ the use of $\|x + y\|_p^p \leq \|x\|_p^p + \|y\|_p^p$ instead of Minkowski's inequality leads to

$$\left\| \sum_{j=j_0}^{\infty} v_j \right\|_{B_q^s(L_p(\mathbb{R}^n))} \lesssim \left(\sum_{r \in \mathbb{Z}} \left(2^{prs} \sum_{j=j_0}^r \omega_M(v_j, 2^{-r}, L_p(\mathbb{R}^n))^p + \sum_{j=r}^{\infty} \|v_j\|_{L_p(\mathbb{R}^n)}^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}.$$

Then, the estimate can be shown as before. \square

Applying Lemma 4.4 to our interpolatory wavelets we obtain by the facts $\xi_m^+ \in B_{\infty}^{2+1/p}(L_p(\mathbb{R}))$ and $\xi_m^- \in B_{\infty}^{1+1/p}(L_p(\mathbb{R}))$ the following corollary.

Corollary 4.5. *Let $0 < s < 2 + \frac{1}{p}$ and $0 < p, q \leq \infty$. Then*

$$\left\| \sum_{m=1}^2 \left(\sum_{k \in \mathbb{Z}} \beta_{m,k} \xi_{m;j_0,k}^+ + \sum_{j=j_0}^{\infty} \sum_{k \in \mathbb{Z}} \alpha_{m;j,k} \eta_{m;j,k}^+ \right) \right\|_{B_q^s(L_p(\mathbb{R}))} \lesssim \sum_{m=1}^2 \left(\|\beta_m\|_{\ell_p} + \|\alpha_m\|_{\ell_{\frac{p}{q}}} \right).$$

If $0 < s < 1 + \frac{1}{p}$ the estimate holds also if ξ_m^+ and η_m^+ are replaced by ξ_m^- and η_m^- , respectively. \square

In order to estimate the Besov norm of a function by a discrete norm of its wavelet coefficients we will employ the vanishing moments of the dual functionals, which enable us to estimate the wavelet coefficients by the error of a locally best approximation

$$E_d(f, W_{\tau}^{\mu}(\sigma)) := \inf_{P \in \Pi_{d-1}} \|f - P\|_{W_{\tau}^{\mu}(\sigma)}$$

by a polynomial P of degree less than d , in the norm of the Sobolev space $W_{\tau}^{\mu}(\sigma)$ on some $\sigma \subset \mathbb{R}^n$. These errors depend in turn on the Besov regularity of f .

Lemma 4.6. *Let $\frac{n}{p} - \frac{n}{\tau} + \mu < s < d$, $s > 0$, $\mu \in \mathbb{N}_0$, $0 < p, q \leq \infty$, $1 \leq \tau \leq \infty$. Furthermore, let $\sigma_{jk} \subset 2^{-j}(\sigma + k)$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, with a cube $\sigma \subset \mathbb{R}^n$. Then, we have*

$$\sum_{j \in \mathbb{Z}} \left(2^{j(s - \frac{n}{p} + \frac{n}{\tau} - |\mu|)} \left(\sum_{k \in \mathbb{Z}^n} E_d(f, W_{\tau}^{\mu}(\sigma_{jk}))^p \right)^{\frac{1}{p}} \right)^q \lesssim |f|_{B_q^s(L_p(\mathbb{R}^n))}^q.$$

Proof. If $\tau < p$ we can apply Hölder's inequality to obtain

$$2^{j(\frac{n}{\tau} - \frac{n}{p})} \|f^{(\alpha)}\|_{L_{\tau}(\sigma_{jk})} \leq \|f^{(\alpha)}\|_{L_p(\sigma_{jk})}, \quad |\alpha| \leq \mu,$$

i.e., it is sufficient to prove the lemma for $p \leq \tau$.

For $P \in \Pi_{d-1}$ and $T_{jk}f(x) = f(2^{-j}(x+k))$ we have, by the embedding $B_1^\mu(L_\tau(\sigma)) \subset W_\tau^\mu(\sigma)$ (see [4, Sect. 6.2]), that

$$\begin{aligned} \|f - P\|_{W_\tau^\mu(\sigma_{jk})} &= 2^{j(\mu - \frac{n}{\tau})} \|T_{jk}(f - P)\|_{W_\tau^\mu(\sigma)} \\ &\leq C 2^{j(\mu - \frac{n}{\tau})} \|T_{jk}(f - P)\|_{B_1^\mu(L_\tau(\sigma))} \\ &\leq C 2^{j\mu} \left(\|f - P\|_{L_\tau(\sigma_{jk})} + 2^{-\frac{jn}{\tau}} \sum_{r=0}^{\infty} 2^{r\mu} \omega_d(T_{jk}f, 2^{-r}, L_\tau(\sigma)) \right), \end{aligned}$$

with a constant C independent of j and k . Let the side length of the cube σ be 2^ν . Applying the Whitney-type inequality $E_d(f, L_\tau(\sigma_{jk})) \lesssim \omega_d(f, 2^{\nu-j}, L_\tau(\sigma_{jk}))$ (see e.g. [10, (25.24)]) and the identity $\omega_d(T_{jk}f, 2^{-r}, L_\tau(\sigma)) = 2^{\frac{jn}{\tau}} \omega_d(f, 2^{-j-r}, L_\tau(\sigma_{jk}))$, we obtain

$$(4.26) \quad E_d(f, W_\tau^\mu(\sigma_{jk})) \lesssim \sum_{r=j}^{\infty} 2^{r\mu} \omega_d(f, 2^{\nu-r}, L_\tau(\sigma_{jk})).$$

For $\mu = 0$ the estimate follows immediately from the Whitney-type estimate.

From [10, Theorem 30.5] we conclude

$$\begin{aligned} E_d(f, W_\tau^\mu(\sigma_{jk})) &\lesssim \sum_{r=j}^{\infty} 2^{r\mu} \sum_{\ell=r-\nu}^{\infty} 2^{n\ell(\frac{1}{p} - \frac{1}{\tau})} \omega_d(f, 2^{-\ell}, L_p(\sigma_{jk})) \\ &= \sum_{\ell=j-\nu}^{\infty} \underbrace{\sum_{r=j}^{\ell+\nu} 2^{r\mu} 2^{n\ell(\frac{1}{p} - \frac{1}{\tau})}}_{\lesssim 2^{\ell\mu}} \omega_d(f, 2^{-\ell}, L_p(\sigma_{jk})). \end{aligned}$$

Together with (4.26) for $\tau = p$ we have

$$(4.27) \quad E_d(f, W_\tau^\mu(\sigma_{jk})) \lesssim \sum_{r=j-\nu}^{\infty} 2^{r(\mu + \frac{n}{p} - \frac{n}{\tau})} \omega_d(f, 2^{-r}, L_p(\sigma_{jk})).$$

For $1 \leq p < \infty$ we conclude by Minkowski's inequality and [10, (26.14)]

$$\begin{aligned} \left(\sum_{k \in \mathbb{Z}^n} E_d(f, W_\tau^\mu(\sigma_{jk}))^p \right)^{\frac{1}{p}} &\lesssim \sum_{r=j-\nu}^{\infty} 2^{r(\mu + \frac{n}{p} - \frac{n}{\tau})} \left(\sum_{k \in \mathbb{Z}} \omega_d(f, 2^{-r}, L_p(\sigma_{jk}))^p \right)^{\frac{1}{p}} \\ (4.28) \quad &\lesssim \underbrace{\sum_{r=j-\nu}^{\infty} 2^{r(\mu + \frac{n}{p} - \frac{n}{\tau})} \omega_d(f, 2^{-r}, L_p(\mathbb{R}))}_{=: b_{j-\nu}}. \end{aligned}$$

Now, by the discrete Hardy inequality [17, Chap. 2, Lemma 3.4] we obtain (for $0 < q < \infty$)

$$\begin{aligned} \sum_{j \in \mathbb{Z}} \left(2^{j(s - \frac{n}{p} + \frac{n}{\tau} - \mu)} \left(\sum_{k \in \mathbb{Z}^n} E_d(f, W_\tau^\mu(\sigma_{jk}))^p \right)^{\frac{1}{p}} \right)^q &\lesssim \sum_{j \in \mathbb{Z}} \left(2^{j(s - \frac{n}{p} + \frac{n}{\tau} - \mu)} b_j \right)^q \\ &\lesssim \sum_{j \in \mathbb{Z}} \left(2^{js} \omega_d(f, 2^{-j}, L_p(\mathbb{R})) \right)^q \\ &\lesssim |f|_{B_{\frac{q}{q-1}}^s(L_p(\mathbb{R}))}^q. \end{aligned}$$

For $p = \infty$ or $q = \infty$ the estimate works analogously.

For $p < 1$ Minkowski's inequality is replaced by $\|x + y\|_p^p \leq \|x\|_p^p + \|y\|_p^p$ and we have

$$\sum_{k \in \mathbb{Z}^n} E_d(f, W_\tau^\mu(\sigma_{jk}))^p \lesssim 2^{-jp} \sum_{r=j-\nu}^{\infty} 2^{r(n+p\mu-\frac{pn}{\tau})} \omega_d(f, 2^{-r}, L_p(\mathbb{R}))^p$$

instead of (4.28). Thus,

$$\begin{aligned} & \sum_{j \in \mathbb{Z}} \left(2^{j(s-\frac{n}{p})} \left(\sum_{k \in \mathbb{Z}^n} E_d(f, W_\tau^\mu(\sigma_{jk}))^p \right)^{\frac{1}{p}} \right)^q \\ & \lesssim \sum_{j=j_0}^{\infty} \left(2^{j(ps-n-p\mu+\frac{np}{\tau})} \sum_{r=j-\nu}^{\infty} 2^{r(n+p\mu-\frac{np}{\tau})} \omega_d(f, 2^{-r}, L_p(\mathbb{R}))^p \right)^{\frac{q}{p}} \end{aligned}$$

and the final inequality follows again by the discrete Hardy inequality. \square

We will also need an estimate for the coefficients of the scaling functions on the coarse level. This estimate will be based on the following lemma.

Lemma 4.7. *Let $\frac{1}{p} - \frac{1}{\tau} + \mu < s$, $\mu \in \mathbb{N}_0$, $0 < p, q \leq \infty$, $1 \leq \tau \leq \infty$. For $\sigma_k \subset \sigma + k$, $k \in \mathbb{Z}^n$, where $\sigma \subset \mathbb{R}^n$ is a n -dimensional cube, we have*

$$\left(\sum_{k \in \mathbb{Z}^n} \|f\|_{W_\tau^\mu(\sigma_k)}^p \right)^{\frac{1}{p}} \lesssim \|f\|_{B_q^s(L_p(\mathbb{R}^n))}.$$

Proof. Again by Hölders inequality we conclude that it is sufficient to prove the result for $p \leq \tau$. For $s > \alpha > \frac{1}{p} - \frac{1}{\tau} + \mu$ we know that $B_q^\alpha(L_p(\sigma_k))$ is embedded in the Sobolev space $W_\tau^\mu(\sigma_k)$, i.e.,

$$\|f\|_{W_\tau^\mu(\sigma_k)}^p \lesssim \|f\|_{L_p(\sigma_k)} + |f|_{B_q^\alpha(L_p(\sigma_k))}.$$

Now the statement follows as in the proof of Lemma 4.6. \square

For our Hermite spline wavelets we obtain from Lemma 4.6 and Lemma 4.7 the following estimate.

Corollary 4.8. *Let $f \in B_q^s(L_p(\mathbb{R}))$, $1 + \frac{1}{p} < s < 4$, $0 < p, q \leq \infty$. Then the coefficient sequences $\alpha_m := (\langle f, \tilde{\eta}_{m;j,k}^+ \rangle)_{j \geq j_0, k \in \mathbb{Z}}$, $\beta_m := (\langle f, \tilde{\xi}_{m;j_0,k}^+ \rangle)_{k \in \mathbb{Z}}$, $j_0 \in \mathbb{Z}$, $m = 1, 2$, satisfy*

$$\sum_{m=1}^2 \left(\|\beta_m\|_{\ell_p} + \|\alpha_m\|_{\ell_{pq}^s} \right) \lesssim \|f\|_{B_q^s(L_p(\mathbb{R}))}.$$

If $\frac{1}{p} < s < 3$ the estimate holds also if $\tilde{\xi}_m^+$ and $\tilde{\eta}_m^+$ are replaced by $\tilde{\xi}_m^-$ and $\tilde{\eta}_m^-$.

Proof. Obviously

$$|\langle f, \tilde{\xi}_{m;j_0,k}^+ \rangle| \lesssim \|f^{(m-1)}\|_{L_\infty(2^{-j_0}[k, k+1])}$$

and since $\Pi_3 \subset V_j^+$ we know $\tilde{\eta}_{m;j,k}^+(P) = 0$, $P \in \Pi_3$. Thus,

$$\begin{aligned} |(f, \tilde{\eta}_{m;j,k}^+)| &= \inf_{P \in \Pi_3} |\langle f - P, \tilde{\eta}_{i;j,k}^+ \rangle| \lesssim E_4(f, L_\infty(\sigma_{jk})) + 2^{-j} E_3(f', L_\infty(\sigma_{jk})) \\ &\lesssim E_4(f, W_\infty^1(\sigma_{jk})), \end{aligned}$$

with $\sigma_{jk} = 2^{-j}[k, k+1]$. Now, the statement follows immediately from Lemma 4.6 and Lemma 4.7.

For $\tilde{\xi}_{m;j_0,k}^-$ and $\tilde{\eta}_{m;j,k}^-$ we obtain analogously

$$|\langle f, \tilde{\xi}_{1;j_0,k}^- \rangle| \lesssim \|f\|_{L_1(2^{-j_0}[k-1,k])} \quad \text{and} \quad |\langle f, \tilde{\xi}_{2;j_0,k}^- \rangle| \lesssim \|f\|_{L_\infty(2^{-j_0}[k-1,k])}$$

as well as

$$|\langle f, \tilde{\eta}_{m;j,k}^- \rangle| \lesssim E_3(f, L_\infty(\sigma_{jk})) + 2^j E_3(f, L_1(\sigma_{jk})).$$

Next, Hölder's inequality yields $\|f - P\|_{L_1(\sigma_{jk})} \lesssim 2^{-j} \|f - P\|_{L_\infty(\sigma_{jk})}$, i.e.,

$$|\langle f, \tilde{\eta}_{m;j,k}^- \rangle| \lesssim E_3(f, L_\infty(\sigma_{jk})).$$

Now the statement follows as before from Lemma 4.6 and Lemma 4.7. \square

Finally, putting all pieces together, we obtain that these wavelets form unconditional bases for a certain range of Besov spaces.

Theorem 4.9. *Let $1 + \frac{1}{p} < s < \min(4, 2 + \frac{1}{p})$, $0 < p, q \leq \infty$. Then every $f \in B_q^s(L_p(\mathbb{R}))$ has the expansion*

$$(4.29) \quad f = \sum_{m=1}^2 \left(\sum_{k \in \mathbb{Z}} \beta_{m,k} \xi_{j_0,k}^+ + \sum_{j=j_0}^{\infty} \sum_{k \in \mathbb{Z}} \alpha_{m;j,k} \eta_{m;j,k}^+ \right)$$

with uniquely determined coefficients $\beta_{m,k} = \langle f, \tilde{\xi}_{m;j_0,k}^+ \rangle$ and $\alpha_{m;j,k} = \langle f, \tilde{\eta}_{m;j,k}^+ \rangle$. Furthermore,

$$\|f\|_{B_q^s(L_p(\mathbb{R}))} \sim \sum_{m=1}^2 \left(\|\beta_m\|_{\ell_p} + \|\alpha_m\|_{\ell_{pq}^s} \right).$$

If ξ_m^+ and η_m^+ are replaced by ξ_m^- and η_m^- , then the statement holds for $\frac{1}{p} < s < \min(3, 1 + \frac{1}{p})$.

Proof. The unconditional convergence of (4.29) is shown with the estimates in Corollaries 4.5 and 4.8 and the biorthogonality relations in Prop 4.2(d), (4.18), and (4.19). Then the norm equivalence follows immediately from Corollaries 4.5 and 4.8. \square

Note that the bounds for s are relatively narrow. In particular, we can not choose the same value of s for the wavelets η_m^- and η_m^+ . This will cause some difficulties for the divergence-free wavelets constructed in Sect. 4.3 below, which can be avoided by considering anisotropic Besov spaces as done later in Sect. 4.4. However, wavelets of higher regularity might be desirable in some cases.

4.3. Divergence-free Interpolatory Hermite Splines. We now describe a construction of divergence-free interpolatory Hermite spline wavelets based on (4.7). We first consider the interpolatory tensor-product MRA V_j^I similar to (2.11). Note that these spaces are generated by scaled translates of

$$\varphi_m^I(x_1, \dots, x_n) := \prod_{\nu=1}^n \xi_{m,\nu}^I(x_\nu), \quad m = (m_1, \dots, m_n)^T \in \{1, 2\}^n,$$

where

$$\xi_{\mu,\nu}^I := \begin{cases} \xi_\mu^+, & \text{if } \nu \in I, \\ \xi_\mu^-, & \text{otherwise,} \end{cases} \quad \mu = 1, 2.$$

This means that V_j^I is generated by a multigenerator of dimension 2^n . The corresponding wavelets are

$$\psi_{e,m}^I(x_1, \dots, x_n) := \prod_{\nu=1}^n \vartheta_{e_\nu, m_\nu, \nu}^I(x_\nu), \quad e \in E_n^*, \quad m = (m_1, \dots, m_n)^T \in \{1, 2\}^n,$$

where

$$\vartheta_{\ell, \mu, \nu}^I := \begin{cases} \xi_{\mu, \nu}^I, & \text{if } \ell = 0, \\ \eta_{\mu, \nu}^I & \text{otherwise,} \end{cases}$$

and $\eta_{i, \nu}^I$ is defined in a similar fashion as $\xi_{i, \nu}^I$. Analogously, we define $\tilde{\varphi}_m^I$ and $\tilde{\psi}_{e,m}^I$ as tensor product functionals. Thus, we obtain

$$|E_n^*| 2^n = (2^n - 1) 2^n$$

wavelets, i.e., 12 wavelets in 2D and 56 wavelets in 3D. We denote by

$$\Lambda_j^I := \bigotimes_{\nu=1}^n \Lambda_{\nu; j}, \quad \Lambda_{\nu; j} := \begin{cases} \Lambda_j^+, & \text{if } \nu \in I, \\ \Lambda_j^-, & \text{if } \nu \notin I, \end{cases}$$

the corresponding Hermite spline interpolants. Before we proceed, we collect some properties of these interpolants.

Proposition 4.10. *The following properties hold for sufficiently smooth functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$*

- (a) $\langle \Lambda_j^I f, \tilde{\varphi}_{m; j, k}^I \rangle = \langle f, \tilde{\varphi}_{m; j, k}^I \rangle$, for $k \in \mathbb{Z}^n$, $m \in \{1, 2\}^n$;
- (b) $\frac{\partial}{\partial x_i} \Lambda_j^I f = \Lambda_j^{I \setminus \{i\}} \left(\frac{\partial}{\partial x_i} f \right)$, if $i \in I$;
- (c) $\frac{\partial}{\partial x_i} \psi_{e, m}^I = 2 \psi_{e, m}^{I \setminus \{i\}}$, if $i \in I$, $e_i = 1$.

Proof. (a) follows directly from (4.9–4.11), whereas (b) follows from Proposition 4.2 (c). As for (c), by $i \in I$ and $e_i = 1$ we have

$$\vartheta_{e_i, m_i, i}^I = \vartheta_{1, m_i, i}^I = \eta_{m_i}^+$$

so that the claim follows from (4.16). \square

Note that (a) describes the (Hermite) interpolation properties of Λ_j^I , similarly to (4.9–4.11) in the univariate setting. In particular, for $m_i = 1$, $i \in I$, and $m_i = 2$, $i \notin I$, statement (a) becomes $\Lambda_j^I f(2^{-j}k) = f(2^{-j}k)$.

The next step is again a vector-valued MRA for the component i similar to (2.13), i.e.,

$$\varphi_{m, i} := \varphi_m^{\{i\}} \delta_i, \quad m \in \{1, 2\}^n, \quad 1 \leq i \leq n,$$

and the corresponding interpolants

$$\Lambda_{i; j} := \Lambda_j^{\{i\}} \delta_i.$$

Putting these together, we obtain an interpolatory MRA $\{\mathbf{V}_j\}_{j \geq j_0}$ in $\mathcal{C}(\mathbb{R}^n)$ generated by

$$\{\varphi_{m, i} : m \in \{1, 2\}^n, 1 \leq i \leq n\}$$

with the associated interpolant $\Lambda_j := \sum_{i=1}^n \Lambda_{i; j}$. Hence, we have $n 2^n$ generators. As an immediate consequence of Proposition 4.10 we obtain

Proposition 4.11. *For a vector field $\mathbf{f} \in \mathbf{C}(\operatorname{div}; \mathbb{R}^n)$ we have*

$$\operatorname{div}(\Lambda_j \mathbf{f}) = \Lambda_j^\emptyset(\operatorname{div} \mathbf{f}),$$

where $\Lambda_j^\emptyset := \Lambda_j^- \otimes \cdots \otimes \Lambda_j^-$.

Proof. With 4.10, (d), we obtain

$$\operatorname{div}(\Lambda_j \mathbf{f}) = \sum_{i=1}^n \frac{\partial}{\partial x_i} \Lambda_j^{\{i\}} f_i = \sum_{i=1}^n \Lambda_j^\emptyset \left(\frac{\partial}{\partial x_i} f_i \right) = \Lambda_j^\emptyset(\operatorname{div} \mathbf{f}).$$

□

The Hermite interpolant Λ_j is used to interpolate given sample data of a flow field. The above proposition ensures that the interpolant of an incompressible flow field remains divergence-free.

Now, we construct divergence-free interpolatory multiwavelets. Let us first briefly sketch the idea, which is similar to (2.15) above. Let $\mathbf{W}_j := \mathbf{V}_{j+1} \ominus \mathbf{V}_j$ denote the vector-valued interpolatory wavelet spaces. The road map is to decompose \mathbf{W}_j as

$$\mathbf{W}_j^\nabla := \mathbf{W}_j \cap \mathbf{N}(\operatorname{div}; \Omega), \quad \mathbf{W}_j^\Delta := \mathbf{W}_j \ominus \mathbf{W}_j^\nabla$$

in a stable way. To this end, we define

$$\psi_{e,m,i} := \psi_{e,m}^{\{i\}} \delta_i$$

so that

$$\mathbf{W}_j = \operatorname{closspan} \{ \psi_{e,m,i,j,k} : e \in E_n^*, m \in \{1, 2\}^n, i = 1, \dots, n, k \in \mathbb{Z}^n \}.$$

This means, we have

$$|E_n^*| 2^n n = n 2^n (2^n - 1)$$

vector-wavelets, i.e., 24 in 2D and 168 in 3D. From (4.16) we conclude

$$\frac{\partial}{\partial x_i} \psi_{e,m,i} = 2 \psi_{e,m}^\emptyset, \quad e_i = 1.$$

For $e \in E_n^*$ and $i \neq i_e$, we define

$$(4.30) \quad \psi_{e,m,i}^\nabla := \psi_{e,m}^{\{i\}} \delta_i - \frac{1}{2} \frac{\partial}{\partial x_i} \psi_{e,m}^{\{i,i_e\}} \delta_{i_e}.$$

Again, it is readily seen that due to $e_{i_e} = 1$, we have

$$(4.31) \quad \operatorname{div} \psi_{e,m,i}^\nabla = \frac{\partial}{\partial x_i} \psi_{e,m}^{\{i\}} - \frac{1}{2} \frac{\partial}{\partial x_{i_e}} \frac{\partial}{\partial x_i} \psi_{e,m}^{\{i,i_e\}} = \frac{\partial}{\partial x_i} \psi_{e,m}^{\{i\}} - \frac{\partial}{\partial x_i} \psi_{e,m}^{\{i\}} = 0.$$

In order to obtain the desired decomposition, we define

$$(4.32) \quad \psi_{e,m}^\Delta := \psi_{e,m}^{\{i_e\}} \delta_{i_e}.$$

The dual functionals for the divergence-free wavelets turn out to be the standard ones, i.e.,

$$(4.33) \quad \tilde{\psi}_{e,m,i}^\nabla := \tilde{\psi}_{e,m}^{\{i\}} \delta_i, \quad i \neq i_e;$$

Furthermore, it is easy to verify that

$$\tilde{\psi}_{e,m}^\Delta := -\frac{1}{2} \mathbf{grad} \tilde{\psi}_{e,m}^\emptyset,$$

where the derivatives are meant in the sense of distributions (cf. (4.22)), i.e.,

$$(4.34) \quad \langle \mathbf{f}, \tilde{\psi}_{e,m}^\Delta \rangle = -\frac{1}{2} \langle \mathbf{f}, \mathbf{grad} \tilde{\psi}_{e,m}^\emptyset \rangle = \frac{1}{2} \langle \operatorname{div} \mathbf{f}, \tilde{\psi}_{e,m}^\emptyset \rangle.$$

4.4. **Characterization of Spaces of Vector Fields.** We will consider the spaces

$$\check{B}_q^s(\mathbf{L}_p(\mathbb{R}^n)) := \left\{ \mathbf{f} \in \mathbf{B}_q^s(\mathbf{L}_p(\mathbb{R}^n)) : \frac{\partial}{\partial x_i} f_i \in B_q^s(L_p(\mathbb{R}^n)), i = 1, \dots, n \right\}$$

with the semi-norm

$$|\mathbf{f}|_{\check{B}_q^s(\mathbf{L}_p(\mathbb{R}^n))} := \max_i \left| \frac{\partial}{\partial x_i} f_i \right|_{B_q^s(L_p(\mathbb{R}^n))}$$

and the norm $\|\mathbf{f}\|_{\check{B}_q^s(\mathbf{L}_p(\mathbb{R}^n))} := \|\mathbf{f}\|_{\mathbf{B}_q^s(\mathbf{L}_p(\mathbb{R}^n))} + |\mathbf{f}|_{\check{B}_q^s(\mathbf{L}_p(\mathbb{R}^n))}$. Note that equivalent norms can be defined by replacing the maximum by any p -norm for \mathbb{R}^n .

The components of a vector field $\mathbf{f} \in \check{B}_q^s(\mathbf{L}_p(\mathbb{R}^n))$ are from certain anisotropic Besov spaces (see [5]). In particular, we have for $\mathbf{f} \in \check{B}_q^s(\mathbf{L}_p(\mathbb{R}^n))$ that $\operatorname{div} \mathbf{f} \in B_q^s(L_p(\mathbb{R}^n))$.

Theorem 4.12. *Let $0 < s < 1 + \frac{1}{p}$ and $0 < p, q \leq \infty$. Then*

$$\begin{aligned} & \left\| \sum_{m,i,k} \beta_{m,i,k} \varphi_{m,i;j_0,k} \right. \\ & \quad \left. + \sum_{j \geq j_0} \sum_{e,m,k} \left(\sum_{i \neq i_e} \alpha_{e,m,i;j,k}^\nabla \psi_{e,m,i;j,k}^\nabla + \alpha_{e,m;j,k}^\Delta \psi_{e,m;j,k}^\Delta \right) \right\|_{\check{B}_q^s(\mathbf{L}_p(\mathbb{R}^n))} \\ & \lesssim \sum_{m,i} \|\beta_{m,i}\|_{\ell_p} + \sum_{e,m} \left(\|\alpha_{e,m}^\Delta\|_{\ell_{pq}^{s+1}} + \sum_{i \neq i_e} \|\alpha_{e,m,i}^\nabla\|_{\ell_{pq}^{s+1}} \right). \end{aligned}$$

Proof. Let

$$\mathbf{g} := \sum_{j \geq j_0} \sum_{e,m,k} \left(\sum_{i \neq i_e} \alpha_{e,m,i;j,k}^\nabla \psi_{e,m,i;j,k}^\nabla + \alpha_{e,m;j,k}^\Delta \psi_{e,m;j,k}^\Delta \right).$$

Let us choose a fixed but arbitrary ν . From the construction of the wavelets we know that the ν -th component has the form

$$g_\nu = \sum_{\kappa} \sum_{j \geq j_0} \sum_{k \in \mathbb{Z}^n} \alpha_{\kappa,j,k} \Phi_{\nu,\kappa}(2^j \cdot -k),$$

where κ represents any possible choice of e, m, i and Δ or ∇ . Further,

$$\Phi_{\nu,\kappa} = \bigotimes_{\ell=1}^{\infty} \phi_{\nu,\kappa,\ell}$$

with compactly supported Hermite splines $\phi_{\nu,\kappa,\ell} \in S_{3,2}$ for $\nu \neq \ell$ and $\phi_{\nu,\kappa,\nu} \in S_{4,2}$. Then

$$\frac{\partial}{\partial x_\nu} g_\nu = \sum_{\kappa} \sum_{j \geq j_0} \sum_{k \in \mathbb{Z}^n} 2^j \alpha_{\kappa,j,k} \left(\frac{\partial}{\partial x_\nu} \Phi_{\nu,\kappa} \right) (2^j \cdot -k),$$

where $\frac{\partial}{\partial x_\nu} \Phi_{\nu,\kappa}$ is a tensor product of compactly supported splines from $S_{3,2}$ and therefore contained in $B_\infty^{1+\frac{1}{p}}(L_p(\mathbb{R}^n))$. Now, Lemma 4.4 yields

$$\left\| \frac{\partial}{\partial x_\nu} g_\nu \right\|_{B_q^s(L_p(\mathbb{R}^n))} \lesssim \sum_{\kappa} \|\alpha_{\kappa}\|_{\ell_{pq}^{s+1}}.$$

Hence,

$$\|\mathbf{g}\|_{\check{B}_q^s(\mathbf{L}_p(\mathbb{R}^n))} \lesssim \sum_{e,m} \left(\|\alpha_{e,m}^\Delta\|_{\ell_{pq}^{s+1}} + \sum_{i \neq i_e} \|\alpha_{e,m,i}^\nabla\|_{\ell_{pq}^{s+1}} \right).$$

The estimate

$$\left\| \sum_{m,i,k} \beta_{m,i,k} \varphi_{m,i;j_0,k} \right\|_{\dot{B}_q^s(\mathbf{L}_p(\mathbb{R}^n))} \lesssim \sum_{m,i} \|\beta_{m,i}\|_{\ell_p}$$

follows analogously. \square

Theorem 4.13. *Let $\frac{n}{p} < s < 3$, $0 < p, q \leq \infty$.*

Then the sequences $\beta_{m,i} := (\langle \mathbf{f}, \tilde{\varphi}_{m,i;j_0,k} \rangle)_{k \in \mathbb{Z}}$, $\alpha_{e,m,i}^\nabla := (\langle \mathbf{f}, \tilde{\psi}_{e,m,i;j,k}^\nabla \rangle)_{j \geq j_0, k \in \mathbb{Z}}$ and $\alpha_{e,m}^\Delta := (\langle \mathbf{f}, \tilde{\psi}_{e,m;j,k}^\Delta \rangle)_{j \geq j_0, k \in \mathbb{Z}}$, $j_0 \in \mathbb{Z}$, satisfy

$$\sum_{m,i} \|\beta_{m,i}\|_{\ell_p} + \sum_{e,m} \left(\|\alpha_{e,m}^\Delta\|_{\ell_{pq}^{s+1}} + \sum_{i \neq i_e} \|\alpha_{e,m,i}^\nabla\|_{\ell_{pq}^{s+1}} \right) \lesssim \|\mathbf{f}\|_{\dot{B}_q^s(\mathbf{L}_p(\mathbb{R}^n))}.$$

Proof. If $e_i = 1$ we have by (4.22) that

$$\begin{aligned} \alpha_{e,m,i;j,k}^\nabla &= \left\langle \mathbf{f}, \tilde{\psi}_{e,m,i;j,k}^\nabla \right\rangle = \left\langle f_i, \tilde{\psi}_{e,m,i;j,k}^{\{i\}} \right\rangle = 2^{-j-1} \left\langle \frac{\partial}{\partial x_i} f_i, \tilde{\psi}_{e,m,i;j,k}^\emptyset \right\rangle \\ &\lesssim 2^{-j-1} E_3 \left(\frac{\partial}{\partial x_i} f_i, L_\infty(\sigma_{jk}) \right), \end{aligned}$$

where $\sigma_{jk} = 2^{-j}([0, 1]^d + k)$. Now the application of Lemma 4.6 yields

$$(4.35) \quad \|\alpha_{e,m,i}^\nabla\|_{\ell_{pq}^{s+1}} \lesssim \left| \frac{\partial f_i}{\partial x_i} \right|_{B_q^s(\mathbf{L}_p(\mathbb{R}^n))} \lesssim \|\mathbf{f}\|_{\dot{B}_q^s(\mathbf{L}_p(\mathbb{R}^n))}.$$

The same estimate follows for $e_i = 0$ and $m_i = 2$ by Proposition 4.2.

On the other hand, for $e_i = 0$ and $m_i = 1$, we assume without loss of generality that $i = 1$ and then

$$(4.36) \quad \left\langle \mathbf{f}, \tilde{\psi}_{e,m,1;j,k}^\nabla \right\rangle = \left\langle f_1, \tilde{\psi}_{e,m,1;j,k}^{\{1\}} \right\rangle = \left\langle f_1, \delta_{2^{-j}k_1} \otimes \vartheta_{e_2,m_2}^- \otimes \dots \otimes \vartheta_{e_n,m_n}^- \right\rangle.$$

Now, set

$$X := \left\{ g \in C^1(\mathbb{R}^n) : g|_{x_1=2^{-j}k_1} \in \Pi_3 \right\}.$$

Obviously,

$$\left\langle g, \tilde{\psi}_{e,m,1;j,k}^{\{1\}} \right\rangle = \left\langle g|_{x_1=2^{-j}k_1}, \vartheta_{e_2,m_2}^- \otimes \dots \otimes \vartheta_{e_n,m_n}^- \right\rangle = 0, \quad g \in X.$$

Hence, we have with Proposition 4.2

$$\begin{aligned} \left| \left\langle f_1, \tilde{\psi}_{e,m,1;j,k}^{\{1\}} \right\rangle \right| &= \inf_{g \in X} \left| \left\langle f_1 - g, \tilde{\psi}_{e,m,1;j,k}^{\{1\}} \right\rangle \right| \\ &\leq \inf_{g \in X} \left(2^{-j} \left| \left\langle \frac{\partial}{\partial x_1} (f_1 - g), \tilde{\psi}_{e,m,1;j,k}^\emptyset \right\rangle \right| + \left| \left\langle f_1 - g, \tilde{\psi}_{e,m,1;j,k-\delta_1}^{\{1\}} \right\rangle \right| \right) \\ &\lesssim \inf_{g \in X} \left(2^{-j} \left\| \frac{\partial}{\partial x_1} (f_1 - g) \right\|_{L_\infty(\sigma_{jk})} + \left\| (f_1 - g)|_{x_1=2^{-j}(k-\delta_1)} \right\|_{L_\infty(\mathbb{R}^{n-1})} \right). \end{aligned}$$

Note that $\frac{\partial}{\partial x_1} g$ can be chosen independently of $g|_{x_1=2^{-j}(k-\delta_1)}$. In particular, for $(f_1 - g)|_{x_1=2^{-j}(k-\delta_1)} = 0$ and $\frac{\partial}{\partial x_1} g \in \Pi_3$ one obtains

$$\left| \left\langle f_1, \tilde{\psi}_{e,m,1;j,k}^{\{1\}} \right\rangle \right| \lesssim 2^{-j} E_3 \left(\frac{\partial}{\partial x_1} f_1, L_\infty(\sigma_{jk}) \right),$$

which leads again to estimate (4.35).

For $\alpha_{e,m}^\Delta$ the proof is done analogously.

Finally, one shows by similar arguments and Lemma 4.7 that

$$\|\beta_{m,i}\|_{\ell_p} \lesssim \begin{cases} \|f_i\|_{B_q^s(L_p(\mathbb{R}))}, & \text{if } m_i = 1, \\ \left\| \frac{\partial}{\partial x_i} f_i \right\|_{B_q^s(L_p(\mathbb{R}))}, & \text{if } m_i = 2. \end{cases}$$

□

Applying biorthogonality we obtain the following result.

Theorem 4.14. *Let $\frac{n}{p} < s < \min(3, 1 + \frac{1}{p})$, $0 < p, q \leq \infty$. Then every $\mathbf{f} \in \check{B}_q^s(L_p(\mathbb{R}^n))$ has the expansion*

$$(4.37) \quad \mathbf{f} = \sum_{m,i,k} \beta_{m,i;k} \varphi_{m,i;j_0,k} + \sum_{j \geq j_0} \sum_{e,m,k} \left(\sum_{i \neq i_e} \alpha_{e,m,i;j,k}^\nabla \psi_{e,m,i;j,k}^\nabla + \alpha_{e,m;j,k}^\Delta \psi_{e,m;j,k}^\Delta \right)$$

with uniquely determined coefficients, and

$$\|f\|_{\check{B}_q^s(L_p(\mathbb{R}^n))} \sim \sum_{m,i} \|\beta_{m,i}\|_{\ell_p} + \sum_{e,m} \left(\|\alpha_{e,m}^\Delta\|_{\ell_{pq}^{s+1}} + \sum_{i \neq i_e} \|\alpha_{e,m,i}^\nabla\|_{\ell_{pq}^{s+1}} \right).$$

Note that we need the ℓ_{pq}^{s+1} -norm and not the ℓ_{pq}^s -norm in the statements above. This is due to the normalization of the wavelets in $\check{B}_q^s(L_p(\mathbb{R}^n))$.

The limited regularity of the wavelets and their dual functionals permits only a small range of smoothness parameters s . In particular, for higher spatial dimensions n there are no admissible values for s if p becomes small. However, in many cases one might only need a one-sided estimate. Therefore, we stated the inequalities in Theorems 4.12 and 4.13 separately.

By the construction of the divergence-free wavelets we know that any linear combination of the wavelets $\psi_{e,m,i;j,k}^\nabla$, $e \in E^*$, $m \in \{1, 2\}^n$, $i \in i_e$, $j, k \in \mathbb{Z}$ is divergence-free (cf. (4.31)). On the other hand, we know from (4.34) that for any divergence-free vector field \mathbf{f} the coefficients $\alpha_{e,m;j,k}^\Delta := \langle \mathbf{f}, \check{\psi}_{e,m;j,k}^\Delta \rangle$ are zero. Thus, we know that a vector field \mathbf{f} is divergence-free if and only if in the expansion (4.37) the coefficients $\alpha_{e,m;j,k}^\Delta$ are zero and the coarse part $\sum_{m,i,k} \beta_{m,i;k} \varphi_{m,i;j_0,k}$ is divergence-free. For the single-scale wavelet bases we deduce the following result

Proposition 4.15. *The functions*

$$2^{\frac{nj}{2}} \psi_{e,m,i;j,k}^\nabla, \quad i \neq i_e, \quad 2^{\frac{nj}{2}} \psi_{e,m;j,k}^\Delta, \quad e \in E_n^*, \quad m \in \{1, 2\}^n, \quad k \in \mathbb{Z}^n$$

generate a Riesz basis for \mathbf{W}_j and $2^{\frac{nj}{2}} \psi_{e,m,i;0,k}^\nabla$, $i \neq i_e$, $e \in E_n^*$, $m \in \{1, 2\}^n$, $k \in \mathbb{Z}^n$, constitute a Riesz basis for \mathbf{W}_j^∇ , with Riesz bounds independent of j .

Proof. It is well-known that it is sufficient to show the statement for $j = 0$. From the construction in Sect. 4.3 it is obvious that the functions $\psi_{e,m,i;0,k}^\nabla$ and $\psi_{e,m;0,k}^\Delta$ are complete in W_0 .

To show stability we consider

$$\mathbf{w} := \sum_{e,m,k} \left(d_{e,m;k}^\Delta \psi_{e,m;0,k}^\Delta + \sum_{i \neq i_e} d_{e,m,i;k}^\nabla \psi_{e,m,i;0,k}^\nabla \right) \in W_0.$$

Theorem 4.12 gives

$$\|\mathbf{w}\|_{L_2} \lesssim \|\mathbf{w}\|_{\check{B}_2^s(L_2(\mathbb{R}^n))} \lesssim \|\mathbf{d}\|_{\ell_2} := \left(\sum_{e,m,k} |d_{e,m;k}^\Delta|^2 + \sum_{i \neq i_e} |d_{e,m,i;k}^\nabla|^2 \right)^{\frac{1}{2}}.$$

Since $\frac{\partial}{\partial x_i} w_i$ is continuous we have

$$|d_{e,m,i;k}^\nabla| = |\langle \mathbf{w}, \tilde{\psi}_{e,m,i;0,k}^\nabla \rangle| \lesssim \left\| \frac{\partial}{\partial x_i} w_i \right\|_{C(\sigma+k)} \lesssim \|w_i\|_{L_2(\sigma+k)},$$

where $\text{supp } \tilde{\psi}_{e,m,i}^\nabla \subset \sigma$. Since $\tilde{\psi}_{e,m,i}^\nabla$ is compactly supported we conclude

$$\sum_{k \in \mathbb{Z}^n} |d_{e,m,i;k}^\nabla|^2 \lesssim \|w_i\|_{L_2(\mathbb{R})}^2.$$

With an analogous argument for the coefficients $d_{e,m;k}^\Delta$ we obtain finally that

$$\|\mathbf{d}\|_{\ell_2} \lesssim \|\mathbf{w}\|_{L_2(\mathbb{R})},$$

and the first part of the proposition is shown.

Furthermore, we know from (4.31) and (4.34) that \mathbf{w} is divergence-free if and only if $d_{e,m;k}^\Delta = 0$ for all $e \in E_n^*$, $m \in \{1, 2\}^n$, $k \in \mathbb{Z}^n$, which proves the second statement. \square

4.5. Quasi-interpolation. As already mentioned above, Λ^\pm interpolates function values and values of derivatives or primitives at the integers. This, however, is not realistic in applications where one is interested in the analysis of experimental data with the aid of wavelets. In fact, experimental data typically consist of measurements of the velocity field at given nodes. This means, that one only has the point values of the velocity field available and no information on the derivatives or primitives. Thus, we study approximation schemes for our divergence-free interpolating Hermite splines. Our goal is to replace $f'(k)$ and $\int_k^{k+1} f(x) dx$ by linear functionals depending only on function values $f(k)$, $k \in \mathbb{Z}$, in such a way that we still have a good approximation. This condition leads to the notion of quasi-interpolation.

Definition 4.16. Let $S \subset C(\mathbb{R}^n)$. A linear operator $Q : C^m(\mathbb{R}^n) \rightarrow S$, is called a *quasi-interpolation operator*, if

- (a) Q preserves polynomials of degree less than $d \in \mathbb{N}$, i.e., $Q(P) = P$ for $P \in \Pi_{d-1}$.
- (b) The operators Q are local and uniformly bounded, i.e., there is a constant C_Q and a compact set K so that for each $U \subset \mathbb{R}^n$

$$(4.38) \quad \|Qf\|_{L_p(U)} \leq C_Q \|f\|_{L_p(U+K)}.$$

Theorem 4.17. Let $1 \leq p, q \leq \infty$ and $m + n/p < s \leq d$. The the scaled quasi-interpolation operator Q_h defined by $Q_h f := Q(f(h \cdot))(\frac{\cdot}{h})$, with any Q from Definition 4.16, satisfies for any $f \in B_q^s(L_p(\mathbb{R}^n))$ the inequality

$$\|Q_h f - f\|_{L_p(\mathbb{R})} \lesssim h^s |f|_{B_q^s(L_p(\mathbb{R}^n))}.$$

Proof. From the assumptions of the theorem we know that $B_\infty^s(L_p(\mathbb{R}^n))$ is embedded in $C^m(\mathbb{R}^n)$, i.e., $Q_h f$ is well defined. With $\sigma_{h,k} = h([0, 1]^n + k)$ we have for any $P \in \Pi_3$ that

$$\|Q_h f - f\|_{L_p(\sigma_{h,k})}^p \leq \|Q_h f - Q_h P\|_{L_p(\sigma_{h,k})}^p + \|P - f\|_{L_p(\sigma_{h,k})}^p \leq (C_q + 1) \|P - f\|_{L_p(\sigma_{h,k}^*)}^p,$$

where $\sigma_{h,k}^* = h([0, 1]^n + k + K)$. Hence,

$$\|Q_h f - f\|_{L_p(\mathbb{R})}^p \leq \sum_{k \in \mathbb{Z}^n} (E_d(f, L_p(\sigma_{h,k}^*)))^p.$$

Now the statement follows immediately from Lemma 4.6. \square

Obviously, Λ_0^+ and Λ_0^- satisfy the conditions of Definition 4.16 with $m = 1, d = 4$ and $m = 0, d = 3$, respectively. Now, we set

$$\begin{aligned} Q^+ f(x) &:= \sum_{k \in \mathbb{Z}} \left(f(k) \xi_1^+(x - k) + q_k^+(f) \xi_2^+(x - k) \right), \\ Q^- f(x) &:= \sum_{k \in \mathbb{Z} \setminus \{0\}} \left(q_k^-(f) \xi_1^-(x - k) + f(k) \xi_2^-(x - k) \right). \end{aligned}$$

Obviously, Q^+ and Q^- are quasi-interpolation operators with exactness $d = 4$ and $d = 3$, respectively, if

$$\begin{aligned} q_k^+(p) &= p'(k), & p \in \Pi_3, \\ q_k^-(p) &= \int_{k-1}^k p(x) dx, & p \in \Pi_2, \end{aligned}$$

and the functionals q_k^\pm depend only on values of f in a neighborhood of k . Since we only know values of f at the integer points, we have to consider methods for numerical differentiation and integration with a sufficient polynomial exactness in order to define suitable functionals q_k^+ and q_k^- , respectively. Local functionals of minimal support, with sufficient polynomial exactness are

$$\begin{aligned} q_k^+(f) &= \frac{1}{6} (-f(k+2) + 6f(k+1) - 3f(k) - 2f(k-1)), \\ q_k^+(f) &= \frac{1}{6} (f(k-2) - 6f(k-1) + 3f(k) + 2f(k+1)), \\ q_k^-(f) &= \frac{1}{12} (5f(k-1) + 8f(k) - f(k+1)), \\ q_k^-(f) &= \frac{1}{12} (-f(k-2) + 8f(k-1) + 5f(k)). \end{aligned}$$

We also present the corresponding *symmetric* functionals of minimal support

$$\begin{aligned} q_k^+(f) &= \frac{1}{12} (f(k-2) - 8f(k-1) + 8f(k+1) - f(k+2)), \\ q_k^-(f) &= \frac{1}{24} (-f(k-2) + 13f(k-1) + 13f(k) - f(k+1)). \end{aligned}$$

By corresponding scaled versions of Q^\pm we are able to obtain an approximation of a smooth function f from sampled data, which has the same order of approximation as the interpolation operators Λ_j^\pm . By tensor product methods we can also construct quasi-interpolation operators replacing the multivariate Hermite interpolation operators Λ_j^I and Λ_j .

5. LIFTING SCHEME

The *lifting scheme* (yielding so called *second generation wavelets*) has become a popular standard tool to adapt wavelet bases to particular requirements, [35, 36] (see also [9] for a quite similar approach called *stable completion*). The adaptation and usage of second-generation wavelets for the numerical solution of partial differential equations by a collocation method is introduced in [24, 25, 26, 43].

For multiwavelets, the lifting scheme has been applied e.g. in [12, 20]. In [2, 21, 22], the lifting scheme has been applied for biorthogonal multiwavelets generated from Hermite splines also yielding wavelets with arbitrary order of vanishing moments.

In this section we show how a lifting scheme can be applied to interpolatory divergence-free wavelets as constructed above. Let ξ, η^{old} be an initial pair of scaling function and corresponding wavelet. Then, the simplest form of a lifting scheme on \mathbb{R} is to define a new wavelet η^{new} by

$$(5.1) \quad \eta^{\text{new}}(x) := \eta^{\text{old}}(x) + \sum_k u_k \xi(x - k),$$

where $u_k \in \mathbb{R}$ are suitably chosen coefficients. The generalization to general interpolation grids and domains is straightforward.

For the Hermite cubic splines the lifting scheme can be described by

$$\begin{pmatrix} \eta_1^{+, \text{new}} \\ \eta_2^{+, \text{new}} \end{pmatrix}(x) = \begin{pmatrix} \eta_1^{+, \text{old}} \\ \eta_2^{+, \text{old}} \end{pmatrix}(x) + \sum_k \mathbf{U}_k^+ \begin{pmatrix} \xi_1^+ \\ \xi_2^+ \end{pmatrix}(x - k)$$

with $\mathbf{U}_k^+ \in \mathbb{R}^{2 \times 2}$. We leave the choice of these matrices free in order to leave freedom to achieve desirable properties.

Having fixed the wavelets $\eta_1^{+, \text{new}}$ and $\eta_2^{+, \text{new}}$ we want to choose wavelets $\eta_1^{-, \text{new}}$ and $\eta_2^{-, \text{new}}$ which permit the construction of divergence-free wavelets. The essential relation for this construction was (4.16). Therefore, we set

$$\begin{aligned} \begin{pmatrix} \eta_1^{-, \text{new}} \\ \eta_2^{-, \text{new}} \end{pmatrix}(x) &= \frac{1}{2} \frac{d}{dx} \begin{pmatrix} \eta_1^{+, \text{new}} \\ \eta_2^{+, \text{new}} \end{pmatrix} \\ &= \begin{pmatrix} \eta_1^{-, \text{old}} \\ \eta_2^{-, \text{old}} \end{pmatrix}(x) + \frac{1}{2} \sum_k \mathbf{U}_k^+ \begin{pmatrix} \xi_1^-(x - k) - \xi_1^-(x - k - 1) \\ \xi_2^-(x - k) \end{pmatrix} \\ &= \begin{pmatrix} \eta_1^{-, \text{old}} \\ \eta_2^{-, \text{old}} \end{pmatrix}(x) + \sum_k \mathbf{U}_k^- \begin{pmatrix} \xi_1^- \\ \xi_2^- \end{pmatrix}(x - k), \end{aligned}$$

where

$$\mathbf{U}_k^- = \frac{1}{2} \left(\mathbf{U}_k^+ - \mathbf{U}_{k-1}^+ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right).$$

Since $\eta_m^{+, \text{new}}$ and $\eta_m^{-, \text{new}}$ satisfy (4.16) the construction of divergence-free wavelets can be done as in Sect. 4.3.

That means, we have to fix either the \mathbf{U}_k^+ or \mathbf{U}_k^- in order to establish certain properties of the wavelets $\eta_m^{+, \text{new}}$ and $\eta_m^{-, \text{new}}$, both. In the particular important case of vanishing moments it turns out that k vanishing moments for $\eta_m^{+, \text{new}}$ with $\text{supp } \eta_m^{+, \text{new}} \subset [-R, R]$ imply

$$\int_{\mathbb{R}} x^\ell \eta_m^{-, \text{new}}(x) dx = \begin{cases} -\frac{\ell}{2} \int_{\mathbb{R}} x^{\ell-1} \eta_m^{+, \text{new}}(x) dx = 0, & \ell = 1, \dots, k, \\ \eta_m^{+, \text{new}}(R) - \eta_m^{+, \text{new}}(-R) = 0, & \ell = 0, \end{cases}$$

i.e., $k + 1$ vanishing moments for $\eta_m^{-, \text{new}}$.

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UNIVERSITY OF ULM, DEPT. OF NUMERICAL ANALYSIS, HELMHOLTZSTR. 18, D-89069 ULM, GERMANY

E-mail address: {kai.bittner,karsten.urban}@mathematik.uni-ulm.de

