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Dedicated to Charles Chui on the occasion of his 65th birthday

Abstract. We investigate biorthogonal spline wavelets on the interval. We give sufficient and necessary conditions for the reconstruction and decomposition matrices to be sparse. Furthermore, we give numerical estimates for the Riesz stability of such bases.

§1. Introduction

In [7], Dahmen, Kunoth, and Urban introduced locally supported spline wavelets for the interval with locally supported dual wavelets. These wavelet bases consist of the biorthogonal spline wavelets of Cohen, Daubechies, and Feauveau [5] as 'inner wavelets', supplemented by boundary wavelets, which are linear combinations of translates of the corresponding scaling functions for the real line. Although it has been shown that these wavelets form Riesz bases a quantitative statement about stability is hard to obtain. Much more the condition numbers of these bases seem to be undesirable high. One reason might be the technical complexity of the construction in [7]. Furthermore, the assumption that the dual boundary scaling functions and wavelets are linear combinations of translates of one scaling function is rather restrictive.

Our goal is to present a completely different approach for the construction of biorthogonal spline wavelets for the interval. As a first step, biorthogonal spline wavelets on the real line where investigated in a time domain approach, making extensive use of the spline structure of the functions [3]. Here, we want to use the ideas of [3] for the construction of wavelet bases for the interval. In this approach dual scaling functions or wavelets will not appear explicitly. Instead we will consider only the primal scaling functions and wavelets, as introduced in Section 2. In particular, this approach allows us to consider splines with nonuniform knots, too. Then we show in Section 3 which spline functions are wavelets in the

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sense that they generate a basis for a complement space to a spline space over a coarser knot level. Conditions for the existence of local and fast wavelet transforms are investigated in Section 4. How the coefficients for the reconstruction and decomposition relations can be computed is shown in Section 5. Three examples for spline wavelets with uniform knots are given in Section 6. Numerical results show, that for certain settings the condition numbers are nearly optimal.

§2. B-Splines on the Interval

Since the scaling functions and wavelets are splines we collect a few facts about splines on the interval, which we will need in the sequel.

For given knots $\boldsymbol{t} = (t_0, \ldots, t_n)$ with $t_i \le t_{i+1}$ we define the B-spline

$$B_{(\boldsymbol{t})}(x) := (t_n - t_0)\ [t_0, \ldots, t_n](\cdot - x)_+^{n-1}, \tag{1}$$

with the divided difference

$$[t_0, \ldots, t_n]f := \begin{cases} \frac{[t_1,\ldots,t_n]f - [t_0,\ldots,t_{n-1}]f}{t_n - t_0}, & \text{if } t_0 < t_n, \\ \frac{f^{(n)}(t_0)}{n!}, & \text{if } t_0 = t_n, \end{cases} \tag{2}$$

and the truncated powers $x_+^m := \chi_{[0,\infty)}(x)\ x^m$. (see e.g.[9, 15]).

Now we choose the nested knot sets

$$\mathcal{T}_j := \{t_k^j :\ k = 0, \ldots, 2^j\}, \quad j \in \mathbb{N}_0,$$

with $t_0^j = 0$, $t_{2^j}^j = 1$, $t_k^j < t_{k+1}^j$, $k = 0, \ldots, 2^{j+1}$, and $t_k^j = t_{2k}^{j+1}$, i.e., $\mathcal{T}_j \subset \mathcal{T}_{j+1}$. Furthermore, we set $t_{-k}^j := 0$ and $t_{2^j+k}^j := 1$, $k > 0$. For a given spline order d we define the *scaling functions*

$$\varphi_{j,k} := B_{(t_k^j, \ldots, t_{k+d}^j)}, \quad k = 1 - d, \ldots, 2^j - 1. \tag{3}$$

It is well known that $\Phi_j := \{\varphi_{j,k} :\ k = 1 - d, \ldots, 2^j - 1\}$ is a stable basis for the space

$$V_j := \left\{ f \in C^{d-2} :\ f\big|_{[t_k^j, t_{k+1}^j)} \in \Pi_{d-1},\ k = 0, \ldots, 2^j - 1 \right\}$$

of polynomial splines in $[0, 1]$ of order d over the knots $\mathcal{T}_j$ (see e.g. [9, 15]). These scaling functions have minimal support $\sigma_{j,k} := \operatorname{supp} \varphi_{j,k} = [t_k^j, t_{k+d}^j]$ in the sense that there is no spline of order d supported in a proper subset of $\sigma_{j,k}$. Furthermore, the $\varphi_{j,k}$ are locally linear independent, i.e., if for any open set $\Omega \subset [0, 1]$

$$\sum_{k=1-d}^{2^j-1} c_k\, \varphi_{j,k}\Big|_\Omega = 0,$$

then $c_k = 0$ if $\sigma_{j,k} \cap \Omega \neq \emptyset$.

Obviously the spaces V_j are nested, i.e., $V_j \subset V_{j+1}$. This implies that each $\varphi_{j+1,k}$ is a linear combination of elements Φ_j, i.e.,

$$\varphi_{j,k} = \sum_{\ell=1-d}^{2^{j+1}-1} a^j_{k,\ell}\, \varphi_{j+1,\ell}. \tag{4}$$

Next we look for wavelet spaces W_j with $V_{j+1} = V_j + W_j$ and $V_j \cap W_j = \{0\}$. This spaces shall be given by a basis of *wavelets* $\Psi_j := \{\psi_{j,k} : \ k = 0, \dots, 2^j - 1\}$. Since $\psi_{j,k} \in V_{j+1}$ we know that there are coefficients $b^j_{k,\ell}$, so that

$$\psi_{j,k} = \sum_{\ell=1-d}^{2^{j+1}-1} b^j_{k,\ell}\, \varphi_{j+1,\ell}. \tag{5}$$

Obviously, Ψ_j is a basis for a wavelet space W_j if and only if $\Phi_j \cup \Psi_j$ is a basis for V_{j+1}, what is in turn equivalent to the existence of coefficients $c^j_{k,\ell}$ and $d^j_{k,\ell}$ which satisfy the decomposition relation

$$\varphi_{j+1,k} = \sum_{\ell=1-d}^{2^j-1} c^j_{k,\ell}\, \varphi_{j,\ell} + \sum_{\ell=0}^{2^j-1} d^j_{k,\ell}\, \psi_{j,\ell}. \tag{6}$$

§3. Truncated Power Representation and the existence of a Wavelet Basis

Another basis for the spline space V_j is the truncated power basis

$$\big\{(\cdot - t^j_k)^{d-1}_+ : \ k = 1, \dots, 2^j - 1\big\} \cup \{(\cdot)^k : \ k = 0, \dots, d-1\}.$$

Therefore, the scaling functions and wavelets can be written as

$$\varphi_{j,k}(x) \quad = \quad \sum_{\ell=1-d}^{0} \alpha^j_{k,\ell}\, x^\ell + \sum_{\ell=1}^{2_j-1} \alpha^j_{k,\ell}\, (x - t^j_\ell)^{d-1}_+ \tag{7}$$

$$\psi_{j,k}(x) \quad = \quad \sum_{\ell=1-d}^{0} \beta^j_{k,\ell}\, x^\ell + \sum_{\ell=1}^{2^{j+1}-1} \beta^j_{k,\ell}\, (x - t^{j+1}_\ell)^{d-1}_+. \tag{8}$$

Theorem 1. *Let the scaling functions $\varphi_{j,k}$ and the wavelets $\psi_{j,k}$ have the truncated power representations (7) and (8). For any $j \in \mathbb{N}_0$ there are uniquely determined coefficients $c^j_{k,\ell}$ and $d^j_{k,\ell}$ satisfying the decomposition relation (6) if and only if the matrix*

$$\boldsymbol{K}_j := \Big(\beta^j_{k,2\ell+1}\Big)_{\ell,k=0}^{2^j-1} \tag{9}$$

is invertible.

Proof: Assume that the wavelets $\psi_{j,k}$ are chosen such that $\boldsymbol{K}_j$ is invertible. By (7) and (8) we obtain

$$\begin{aligned} g_{j,k} &:= \varphi_{j+1,k} - \sum_{\ell=0}^{2^j-1} d^j_{k,\ell}\,\psi_{j,\ell} \qquad (10) \\ &= \sum_{n=1-d}^{0} \Big(\alpha^{j+1}_{k,n} - \sum_{\ell=0}^{2^j-1} d^j_{k,\ell}\beta^j_{\ell,n}\Big)x^n \\ &\quad + \sum_{n=1}^{2^{j+1}-1} \Big(\alpha^{j+1}_{k,n} - \sum_{\ell=0}^{2^j-1} d^j_{k,\ell}\beta^j_{\ell,n}\Big)(x-t^{j+1}_n)^{d-1}_+. \qquad (11) \end{aligned}$$

If we choose

$$\boldsymbol{d}^j_k := \Big(d^j_{k,\ell}\Big)_{\ell=0}^{2^j-1} = \boldsymbol{K}_j^{-1}\,\tilde{\boldsymbol{\alpha}}^{j+1}_k, \qquad (12)$$

where $\tilde{\boldsymbol{\alpha}}^{j+1}_k := \big(\alpha^{j+1}_{k,2\ell+1}\big)_{\ell=0}^{2^j-1}$, then

$$\alpha^{j+1}_{k,2n+1} - \sum_{\ell=0}^{2^j-1} d^j_{k,\ell}\beta^j_{\ell,2n+1} = 0, \quad n = 0,\dots,2^j-1 \qquad (13)$$

i.e. $g_{j,k} \in V_j$. Thus, there are uniquely determined coefficients $c^j_{k,\ell}$ so that

$$g_{j,k} = \sum_{\ell=1-d}^{2^j-1} c^j_{k,\ell}\,\varphi_{j,\ell}, \qquad (14)$$

i.e., the decomposition relation (6) is satisfied. The invertibility of K_j implies that any other choice of the $d^j_{k,\ell}$ will not satisfy (13) and we cannot represent $g_{j,k}$ by the scaling functions $\varphi_{j,k}$. Thus, (12) is the only choice for the coefficients $d^j_{k,\ell}$ which satisfies (6).

On the other hand, if there are coefficients $c^j_{k,\ell}$ and $d^j_{k,\ell}$ satisfying (6) for some $j \in \mathbb{N}_0$, then we conclude immediately that $g_{j,k}$ from (10) is contained in V_j. This implies (13) or equivalently

$$\boldsymbol{K}_j\,\boldsymbol{d}^j_k = \tilde{\boldsymbol{\alpha}}^{j+1}_k, \quad k = 0,\dots,2^j-1.$$

Since any spline with knots $\mathcal{T}_{j+1}$ is a linear combination of the B-splines $\varphi_{j+1,k}$ we conclude that the vectors $\tilde{\boldsymbol{\alpha}}^{j+1}_k$ span $\mathbb{R}^{2^j}$, i.e., we have shown that the linear system

$$\boldsymbol{K}_j\,\boldsymbol{x} = \boldsymbol{y},$$

has for every $\boldsymbol{y} \in \mathbb{R}^{2^j}$ a solution $\boldsymbol{x} \in \mathbb{R}^{2^j}$, i.e., $\boldsymbol{K}_j$ is invertible. □

§4. Fast Reconstruction and Decomposition

Let $f \in V_{j+1}$ be given by coefficients v_k^j and w_k^j which satisfy

$$f = \sum_{k=1-d}^{2^j-1} v_k^j \, \varphi_{j,k} + \sum_{k=0}^{2^j-1} w_k^j \, \psi_{j,k}. \tag{15}$$

The task of *reconstruction* is to determine coefficients v_k^{j+1} so that

$$f = \sum_{k=1-d}^{2^{j+1}-1} v_k^{j+1} \, \varphi_{j,k}. \tag{16}$$

From (4) and (5) it follows immediately that

$$v_\ell^{j+1} = \sum_{k=1-d}^{2^j-1} v_k^j \, a_{k,\ell}^j + \sum_{k=0}^{2^j-1} w_k^j \, b_{k,\ell}^j. \tag{17}$$

In order to have a fast reconstruction algorithm we demand that the number of indices k with $a_{k,\ell}^j \neq 0$ and $b_{k,\ell}^j \neq 0$ is bounded by a constant independent of ℓ and j. Then, reconstruction can be done by $\mathcal{O}(2^j)$ operations.

Local linear independence implies that $a_{k,\ell}^j$ and $b_{k,\ell}^j$ are different from zero only if $\sigma_{j+1,\ell} \subset \sigma_{j,k}$ and $\sigma_{j+1,\ell} \subset \operatorname{supp} \psi_{j,k}$, respectively. Since $\sigma_{j,k} = [t_k^j, t_{k+d}^j] = [t_{2k}^{j+1}, t_{2(k+d)}^{j+1}]$ one sees immediately that $a_{k,\ell}^j$ can only not vanish if $\ell - d \leq 2k \leq \ell$. Analogously, if the wavelet ψ_j^k is supported in $[t_{2k-L}^{j+1}, t_{2k+d+L}^{j+1}]$ with L independent of j and k, then $b_{k,\ell}^j$ is non-zero only if $\ell - L \leq 2k \leq \ell + L$. Thus, imposing suitable support conditions on the wavelets will yield a fast reconstruction algorithm.

On the other hand for *decomposition* we have to determine coefficients v_k^j and w_k^j from the v_k^{j+1} satisfying (15) and (16), respectively. Applying (6) one obtains

$$v_\ell^j = \sum_{k=1-d}^{2^j-1} v_k^{j+1} \, c_{k,\ell}^j \quad \text{and} \quad w_\ell^j = \sum_{k=1-d}^{2^j-1} v_k^{j+1} \, d_{k,\ell}^j. \tag{18}$$

Thus, we have a fast decomposition algorithm of order $\mathcal{O}(2^j)$, if the number of indices k with $c_{k,\ell}^j \neq 0$ and $d_{k,\ell}^j \neq 0$ is bounded by a constant independent of ℓ and j.

It turns out, that such a sparse structure of the coefficients depends mainly on properties of the matrices $\boldsymbol{K}_j$.

Theorem 2. *Let the spline wavelets $\psi_{j,k}$ of order d be supported in $[t^{j+1}_{2k-L}, t^{j+1}_{2k+2+L}]$. If the matrices $\boldsymbol{K}_j$ given in (9) have uniformly banded inverses, i.e., there is a constant B so that the entries of $\boldsymbol{K}_j^{-1} = (\gamma_{k,\ell})$ satisfy $\gamma_{k,\ell} = 0$ if $|k-\ell| > B$, then $c^j_{k,\ell} = 0$ and $d^j_{k,\ell} = 0$ for $k \notin \big[2(\ell-B)-L-1, 2(\ell+B)+L+1-d\big]$ and $k \notin \big[2(\ell-B)+1-d, 2(\ell-B)+1\big]$, respectively.*

Proof: From the definition of the B-splines in (1) we infer that $\alpha^j_{k,\ell} = 0$ if $\ell \notin [k, \ldots, k+d]$. Since (12) means

$$d^j_{k,\ell} = \sum_{n=k-B}^{k+B} \gamma_{\ell,n}\alpha^{j+1}_{k,2n+1},$$

i.e., $d^j_{k,\ell} = 0$ if $k \notin \big[2(\ell-B)+1-d, 2(\ell-B)+1\big]$.

Now it follows that the spline $g_{j,k}$ defined in (11) is supported in $[t^{j+1}_{k-2B-L-1}, t^{j+1}_{k+2B+L+1+d}]$. Using local linear independence in (14) we conclude that $c^j_{k,\ell} = 0$ if $2\ell \notin \big[k-2B-L-1, k+2B+L+1-d\big]$, i.e., $k \notin \big[2\ell-2B-L-1, 2\ell+2B+L+1-d\big]$. $\square$

Remark 1. *The conditions in Theorem 2 are not necessary for the existence of a fast decomposition method (see e.g. [1, 2, 13, 14]). However, sometimes it is desired that the number of* non-vanishing *output coefficients v^j_k and w^j_k is of the same order as the number of* non-vanishing *input coefficients v^{j+1}_k, as for example in a local decomposition scheme (as used e.g. in [4, 8]). To achieve this goal the number of non-vanishing coefficients $c^j_{k,\ell}$ and $d^j_{k,\ell}$ has indeed to be bounded for every ℓ and j. Then the computational complexity depends linearly on the number of nonzero coefficients v^{j+1}_k, which can be in some cases much smaller than 2^{j+1}.*

In general, it is not obvious for which wavelets the matrices $\boldsymbol{K}_j^{-1}$ are uniformly banded. In particular, the support restrictions on the wavelets imply that the $\boldsymbol{K}_j$ are banded themselves so that one could expect full inverse matrices in many cases. The uniform bandedness of $\boldsymbol{K}_j^{-1}$ is straight forward if $\boldsymbol{K}_j$ is a diagonal or block diagonal matrix. To obtain such matrices one has to choose the wavelets as splines which have only certain points from $\mathcal{T}_{j+1} \setminus \mathcal{T}_j$ as knots.

If we want $\boldsymbol{K}_j$ to be diagonal the wavelet $\psi_{j,k}$ has to be a spline with all knots in $\mathcal{T}_j$ with the only exception of t^{j+1}_{2k+1}. That is, every wavelet is associated with one and only one 'new' knot. An analogous setting where obtained for classical spline wavelet bases on the real line in [3]. A more general setting can be obtained by permitting $\boldsymbol{K}_j$ to be a block diagonal matrix with block size less than or equal to B and invertible blocks. Then $\boldsymbol{K}_j^{-1}$ will have the same block structure as $\boldsymbol{K}_j$ and both matrices are banded. We will consider examples for both settings in Section 6.

§5. Computation of Reconstruction and Decomposition Sequences

In order to perform discrete wavelet transforms (decomposition and reconstruction as described in (18) and 17)) we need to know the coefficients $a^j_{k,\ell}$, $b^j_{k,\ell}$, $c^j_{k,\ell}$, and $d^j_{k,\ell}$, explicitly.

The coefficients $a^j_{k,\ell}$ can be obtained by the Oslo Algorithm [6, 12]. This algorithm determines the representation of the n-th order B-splines $B_{(\tau_k,\ldots,\tau_{k+n})}$ in terms of B-splines $B_{(t_\ell,\ldots,t_{\ell+n})}$ over a finer knot set $\{t_\ell\} \supset \{\tau_k\}$, i.e., one obtains the coefficients $u^n_{k,\ell}$ such that

$$B_{(\tau_k,\ldots,\tau_{k+n})} := \sum_\ell u^n_{k,\ell} B_{(t_\ell,\ldots,t_{\ell+n})}.$$

Namely, this coefficients are computed by the recursion

$$u^n_{k,\ell} = (t_{\ell+n-1} - \tau_k)\hat{u}^{n-1}_{k,\ell} + (\tau_{k+n} t_{\ell+n-1})\hat{u}^{n-1}_{k+1,\ell},$$

where $u^1_{k,\ell} = \chi_{[\tau_k,\tau_{k+1})}(t_\ell)$ and

$$\hat{u}^n_{k,\ell} = \begin{cases} u^n_{k,\ell}/(\tau_{k+n} - \tau_k), & \text{if } \tau_{k+n} > \tau_k, \\ 0 & \text{otherwise.} \end{cases}$$

The result so far suggest that the wavelets $\psi_{j,k}$ need not to be given as a linear combination of B-splines $\varphi_{j+1,k}$ in the beginning A minimal requirement is that we are able to evaluate function values or even derivatives of the wavelet $\psi_{j,k}$ then the coefficients $b^j_{k,\ell}$ can be obtained by a projection onto V_j as e.g. spline interpolation or the quasi interpolation operator of de Boor and Fix [10]. On the other we can assume that the wavelet was constructed according to Theorem 2 as a linear combination of B-splines over some subset of $\mathcal{T}_{j+1}$. Then the computation of the coefficients $b^j_{k,\ell}$ could again be based on the Oslo Algorithm.

The coefficients $d^j_{k,\ell}$ are given by (12). To apply (12) we need to know the truncated power representation (7) and (8) of the scaling functions and wavelets. These functions are splines given as linear combination of B-splines. Let a spline $s = \sum_k \alpha^d_k \varphi_{j,k}$ of order d be given by the expansion coefficients α^d_k. The $d-1$-th derivative is given by

$$s^{(d-1)} = (d-1)! \sum_k \alpha^1_k B_{(t^j_k, t^j_{k+1})} = (d-1)! \sum_k \alpha^0_k \chi_{[t^j_k,1]},$$

where

$$\alpha^n_k = \begin{cases} \alpha^1_k - \alpha^1_{k-1}, & \text{if } n = 0, \\ 0, & \text{if } n > 0 \text{ and } t^j_k = t^j_{k+n}, \\ \frac{\alpha^{n-1}_k - \alpha^{n-1}_{k-1}}{t^j_{k+n} - t^j_k} & \text{otherwise} \end{cases}$$

(cf. [9, 15]). Obviously, $s = p + \sum_k \alpha_k^0(\cdot - t_k^j)_+^{d-1}$, where p is a polynomial of degree less than $d-1$. Thus, we are able to determine the coefficients $\alpha_{k,2\ell+1}^j$ and $\beta_{k,2\ell+1}^j$ in (7) and (8) and use them to calculate the coefficients $d_{k,\ell}^j$ by (12).

It remains to determine the coefficients $c_{k,\ell}^j$. Having got explicit representations of wavelets and scaling functions as well as the coefficients $d_{k,\ell}^j$, we have to satisfy (14) for $g_{j,k} \in V_j$ defined in (10). Again this can be done by interpolation or some other projection onto V_j.

§6. Construction of Stable Wavelet Bases in the Uniform Setting

Up to now, we have a lot of freedom for the choice of the knot sequences as well as the wavelets. Many of our ideas may even work for more general knot refinements, where we can permit variable numbers of new knots in any interval (t_k^j, t_{k+1}^j) or even multiple knots. Here we have refrained from this general setting in order to avoid technical difficulties.

On the other hand, classical wavlets, i.e., translates and dilates of a single function lead to much more efficient algorithms. However, for the interval the construction of a wavelet basis which consists of scaled shifts of only one function does not work. In order to keep the construction as simple as possible, one starts with a compactly supported wavelet function ψ and considers its sclaled shifts $\psi_{j,k} = 2^{j/2}\psi(2^j \cdot -k)\big|_{[0,1]}$ as *inner wavelets*, as long as $\operatorname{supp}\psi(2^j \cdot -k) \in [0,1]$. What remains to do, is to find suitable boundary wavelets in order to obtain a basis for $L^2([0,1])$ with certain desired properties. Such properties include e.g. vanishing moments, small support, and stability.

The goal of this section is to construct such boundary wavelets based on the results of the previous sections. Therefore, we will assume in the sequel that the knot sets $\mathcal{T}_j$ are uniform, i.e., $t_k^j = 2^{-j}k$, $k = 0, \ldots, 2^j$. Then the *inner scaling functions* from (3) are scaled shifts of the cardinal B-spline $N_d = B_{(0,\ldots,d)}$, i.e. $\varphi_{j,k} = 2^{j/2}N_d(2^j \cdot -k)$, $k = 0, \ldots, 2^j - d$. For $2^j \geq d$ we have at least one inner function. The remaining B-splines from (3) are the left and right boundary functions which can be obtained by scaling and reflection of a few prototype splines, namely $\varphi_{j,k}(x) = 2^{j/2}\phi_k(2^j x)$, $k = 1-d, \ldots, -1$ and $\varphi_{j,k}(x) = \varphi_{j,2^j-k-d}(1-x)$, $k = 2^j - d+1, \ldots, 2^j - 1$, where $\phi_k := B_{\theta_k,\ldots,\theta_{k+d}}$, $\theta_k = \max(0,k)$.

For the wavelets we aim for a similar setting, i.e., we choose inner wavelets $\psi_{j,k} = \psi(2^j \cdot -k)$ which are entirely supported in $[0,1]$ and try to construct boundary wavelets, such that the assumptions of Theorem 1 are satisfied. In particular, we will choose the inner wavelets as spline wavelets of order d with $\tilde{d}$ vanishing moments, so that $d + \tilde{d} = 2n$ is even) as introduced by Cohen, Daubechies and Feauveau in [5]. In [3, Theorem 4] it has been shown, that these wavelets are (up to a constant factor) the only ones with support length $d+\tilde{d}-1$, which are symmetric or antisymmetric. Furthermore, these wavelets are given as

$$\psi^{d,\tilde{d}} = B^{\tilde{d}}_{(1-n,2-n,\ldots,0,\frac{1}{2},1,\ldots,n)}\,, \qquad n = \tfrac{d+\tilde{d}}{2}.$$

i.e., $\frac{1}{2}$ is the only non-integer knot of the spline $\psi^{d,\tilde{d}}$. The inner wavelets are therefore $\psi_{j,k} = \psi^{d,\tilde{d}}(2^j\cdot-k)$, $k = n-1,\ldots,2^j-n$. Note, that this implies immediately that the 'inner' rows of K_j (for $k = n-1,\ldots,2^j-n$) are zero out of the diagonal. The remaining rows correspond to the boundary wavelets and have to be chosen in order to satisfy the assumptions of Theorems 1 and 2.

The boundary wavelets shall be scaled versions of some prototypes, i.e., $\psi_{j,k}(x) = 2^{j/2}\psi_k(2^j x)$, $k = 0,\ldots,n-2$ and $\psi_{j,k}(x) = \psi_{j,2^j-k-1}(1-x)$, $k = 2^j-n+1,\ldots,2^j-1$, $j \geq j_0$. The minimal level j_0 is chosen such that $2^{j_0} < 2n-1$, i.e., at each level there is at least one inner wavelet, and the boundary wavelets belong either to the left or to the right boundary. The boundary wavelets shall have vanishing moments, too, i.e.,

$$\int_0^\infty x^\ell\,\psi_k(x)\,dx = 0, \quad \ell = 0,\ldots,\tilde{d}_k-1.$$

By partial integration one shows, that this is equivalent to

$$\psi_k = \Psi_k^{(\tilde{d}_k)}, \qquad \Psi_k^{(\ell)}(0) = 0, \quad \ell = 0,\ldots,\tilde{d}_k-1, \tag{19}$$

where Ψ_k is a spline of order $d+\tilde{d}_k$ with compact support in $[0,\infty)$. That means Psi_k can have a knot of multiplicity not exceeding d in 0.

In the sequel we investigate the stability for different choices of the Ψ_k, i.e., for different boundary functions. By stability we mean the existence of Riesz bounds $A, B > 0$ so that

$$A\sum_{j=j_0}^{\infty}\sum_{k=0}^{2^j-1}|w_{j,k}|^2 \leq \left\|\sum_{j=j_0}^{\infty}\sum_{k=0}^{2^j-1} w_{j,k}\psi_{j,k}\right\|^2_{L^2(\mathbb{R})} \leq B\sum_{j=j_0}^{\infty}\sum_{k=0}^{2^j-1}|w_{j,k}|^2. \tag{20}$$

We consider the condition number

$$\kappa := \min\left\{\tfrac{B}{A} :\ A \text{ and } B \text{ satisfy (20)}\right\}.$$

as a measure of stability, in the sense that κ close to 1 means good stability, while an increasing κ means that the stability becomes worse.

It has already shown in [3, Theorem 6], that a biorthogonal spline wavelet basis for $L^2(\mathbb{R})$ generated by a compactly supported spline wavelet of order d with a compactly supported dual has a condition number greater than 4^{d-1}. Since the inner wavelets for our construction are of the same type, we cannot achieve a smaller condition number for any choice of boundary wavelets. Therefore, our goal is to achieve a condition number close to 4^{d-1}.

Type A. Let all wavelets have $\tilde{d}$ vanishing moments. We assume that ψ_k has only one integer knot, namely $k+\frac{1}{2}$. Thus, $\boldsymbol{K}_j$ will be diagonal and the number of nonzero coefficients $d^j_{k,\ell}$ will be minimal. In order to have a small support we choose $\psi_k = B^{(\tilde{d})}_{\boldsymbol{\tau}^k}$ as the $\tilde{d}$-th derivative of a B-spline of order $2n$ with knots

$$\boldsymbol{\tau}^k = \begin{cases} (\underbrace{0,\dots,0}_{d},\dots,k,k+\frac{1}{2},k+1,\dots,2n-d), & k=0,\dots,n-d-1,\\ (\underbrace{0,\dots,0}_{n-k},\dots,k,k+\frac{1}{2},k+1,\dots,n+k), & k=n-d,\dots,n-2, \end{cases}$$

i.e., the remaining integer knots are chosen in order to minimize the support, while $B^{(\ell)}_{\boldsymbol{\tau}^k}(0) = 0$, $\ell = 0,\dots,\tilde{d}-1$ so that we have $\tilde{d}$ vanishing moments.

Type B. Again we demand $\tilde{d}$ vanishing moments for every wavelet. and choose $\psi_k = B^{(\tilde{d})}_{\boldsymbol{\tau}^k}$ as the $\tilde{d}$-th derivative of a B-spline of order $2n$. But know we choose the knot sets

$$\boldsymbol{\tau}^k = \begin{cases} (\underbrace{0,..,0}_{d},\underbrace{1,..,k-1}_{\in\mathbb{Z}},\underbrace{k,k+\frac{1}{2},..,n-d+1}_{\in\frac{1}{2}\mathbb{Z}},\underbrace{n-d+2,..,n+k}_{\in\mathbb{Z}}), & \\ & k=0,..,n-d-1,\\ (\underbrace{0,\dots,0}_{n-k},\dots,k,k+\frac{1}{2},k+1,\dots,n+k), & k=n-d,..,n-2, \end{cases}$$

i.e., we restrict the support of ψ_k to $[0,n+k]$ (similar to the inner wavelets) and include the half-integer knots $k+\frac{1}{2},\dots,n-d+\frac{1}{2}$ so that $\boldsymbol{\tau}^k$, contains $2n+1$ knots. Then $\boldsymbol{K}_j$ is a block diagonal matrix consisting of a upper triangular block of size $n-d$, a diagonal block of size $2^j-\tilde{d}+d$, and a lower triangular block of size $n-d$. Obviously, all diagonal elements are non-zero, i.e. $\boldsymbol{K}_j$ is invertible and the inverse has bandwidth $n-d$.

Type C. Now we permit a smaller number of vanishing moments for the boundary functions. In order to improve stability we want to make the boundary wavelets from W_j orthogonal to V_j. This can be achieved, if Ψ_k from (19) is of order $2d$ (i.e. $\tilde{d}_k = d$) and $\Psi_k(\ell) = 0$, $\ell \in \mathbb{Z}$ (cf [11, Theorem 3.3]. Then the boundary wavelets will all have d vanishing moments. We choose

$$\Psi_k = \sum_{\ell=k+1-d}^{2k} h_{k,\ell} B_{(\theta_\ell/2,\dots,\theta_{2d+\ell}/2)},$$

such that Ψ_k and therefore ψ_k are determined up to a constant factor by $\Psi_k(\ell) = 0$. Then, the matrix K_j is block diagonal with block size $d+n-1$. It is not obvious if $\boldsymbol{K}_j$ is invertible in general. However, since neither the inner nor the boundary wavelets $\psi_{j,k}$ are contained in V_j it sufficies to show the linear independence of the wavelets. Let $f := \sum_{k=0}^{2^j-1} w_{j,k}\psi_{j,k} = 0$. Since $\psi_{j,k}^{(\ell)}(0) = 0$, $k > \ell$, we conclude $c_{j,k} = 0$, $k = 0, \ldots, n-2$. An analogous argument works for the right boundary. Now, the linear independence of the inner wavelet implies $c_{j,k} = 0$, $k = 0, \ldots, 2^j - 1$. Thus, Ψ_j is a basis for a complement space of V_j in V_{j+1}.

In Table 1 we have given numerical estimates of condition numbers for various choices of d and $\tilde{d}$. To obtain these estimates, we have computed the condition numbers of the Gramian matrices of the systems $\{\psi_{j,k} :\ k = 0, \ldots, 2^j - 1,\ r = j_0, \ldots, J\}$, $J = 0, \ldots, 10$. If this condition numbers do not change to fast for increasing J, we consider the result for $J = 10$ as a good estimate of the condition number, otherwise as a lower bound (indicated by a $>$).

d	$\tilde{d}$	$L^2(\mathbb{R})[3, 5]$	Type A	Type B	Type C
2	4	4.146	5.6	5.1	4.15
2	6	4.027	10.3	48	4.03
3	5	19.2	21	19.2	19.2
3	7	16.33	18	16.4	16.4
3	9	16.02	22	> 90	16.02
4	8	68.45	78	68.5	68.5
4	10	64.66	74	64.7	64.7
4	12	64.09	77	> 180	64.1
5	11	263.8	334	264	264
5	13	257.3	318	257.5	257.5
5	15	256.2	328	> 321	256.5

Tab. 1. Condition numbers for several wavelet bases

The results indicate that one can construct boundary wavelets without an essential loss of stability compared to the corresponding wavelet basis on the real line. A theoretical confirmation of these numerical results would be desirable in the future. Furthermore, for applications in numerical analysis the stability under boundary conditions and other norms has to be investigated.

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