A new view on biorthogonal spline wavelets

Kai Bittner

Preprint Series: 2005-03

Fakultät für Mathematik und Wirtschaftswissenschaften
UNIVERSITÄT ULM

A new view on biorthogonal spline wavelets

Kai Bittner

Universität Ulm, Abteilung Numerik, 89079 Ulm, Germany
Tel. ++49 731 5023534, Fax ++49 731 5023548

Abstract

The biorthogonal wavelets introduced by Cohen, Daubechies, and Feauveau contain in particular compactly supported biorthogonal spline wavelets with compactly supported duals. We present a new approach for the construction of compactly supported spline wavelets, which is entirely based on properties of splines in the time domain. We are able to characterize a large class of such wavelets which contains the spline wavelets of Cohen, Daubechies, and Feauveau as a special case. Further, we prove a new result on the Riesz stability of such spline wavelets.

Key words: Splines, wavelets, biorthogonal bases, Riesz bases
1991 MSC: 42C40

Email address: `bittner@mathematik.uni-ulm.de` (Kai Bittner).
URL: `http://numerics.uni-ulm.de/staff/bittner` (Kai Bittner).

Preprint submitted to Appl. Comput. Harmon. Anal. *8 August 2005*

1 Introduction

The construction of wavelets is usually based on the determination of the mask coefficients of refinable functions. The orthogonal wavelets by Daubechies [13,14] where constructed in this way. Analogously, biorthogonal wavelets where introduced by Cohen, Daubechies, and Feauveau [10] starting with two refinement masks, which define two refinable functions generating a pair of biorthogonal Riesz bases. Having found the refinement masks the construction of an orthogonal wavelet basis or a pair of biorthogonal wavelet bases is straight forward.

In many applications, in particular in signal processing, only the refinement coefficients are needed in computations. On the other hand, in many wavelet methods in numerical analysis one has to compute function values, derivatives or integrals of wavelets by a fast method (see e.g. [6–8,12]). Although all this computations can be done knowing only the refinement coefficients (see [16]), such a computation might be relative expensive. Therefore, spline wavelets are often preferred in this case. B-splines and spline wavelets have the advantage of an explicit representation as piecewise polynomials so that the computation of function values, derivatives or integrals is easy to perform. Furthermore, the theory of splines has developed very far in the last decades, and optimized methods for splines are available.

For computational reasons one is interested in wavelets with compact support. Compactly supported spline wavelets have been developed independently by Cohen, Daubechies, and Feauveau [10] and by Chui and Wang [4,5]. While the first approach aims on compactly supported dual wavelets, which are usually no splines, the second approach considers semi-orthogonal wavelets, where the dual wavelets are also splines but in general not compactly supported. Compact support of the duals seems to be desirable for computational reasons. However, we want to mention here that semi-orthogonal spline wavelets may be more stable, while computational problems can often be circumvented applying the spline structure of the dual wavelets [1,2,17].

In this article we want to investigate biorthogonal compactly supported spline wavelets with compactly supported duals. An important issue is the stability of such bases. In [10], a necessary condition is given under which a wavelet generates a Riesz basis. A sufficient and necessary condition was given in [9]. While a qualitative statement about stability can be made it is not easy to measure the stability in the sense that one can determine the Riesz bounds for a wavelet basis. Furthermore, it is not clear if there are other biorthogonal spline wavelets with better stability than the ones from [10].

The situation becomes more involved, if one wants to construct wavelets on the interval or even on more complicated domains in $\mathbb{R}^n$. As in [11] Riesz bases of spline wavelets for the interval can be constructed, but due to technical difficulties it turns out to be very hard to control the stability. One difficulty here is that the original approach in [10] is based on Fourier methods. To obtain functions for the interval one has to truncate the wavelets or scaling functions on the real line. But truncation usually leads to the loss of biorthogonality and vanishing moments. The preservation of these properties leads to boundary constructions which are technically involved.

On the other hand, splines on the interval, even with non-uniform knots, are well understood. Therefore, we want to present an approach to biorthogonal spline wavelets, which is based on a construction in the time domain making use of the fact that the functions in consideration are splines. In Section 2, we will recall results on splines and spline wavelets that we will need in the following sections. Section 3 contains sufficient and necessary conditions for biorthogonal spline wavelets in order to have finitely supported decomposition sequences (Theorems 2 and 3). Then we show in Section 4 how refinement coefficients for such wavelets and the corresponding scaling functions can be obtained. The connection to the biorthogonal spline wavelets of Cohen, Daubechies, and Feauveau is shown in Section 5. Finally, in Section 6, we give a result on the Riesz bounds of biorthogonal spline wavelets (Theorem 6).

2 B-splines and spline wavelets

Since the scaling functions and wavelets will be B-splines and linear combinations of B-splines, respectively, we collect a few facts about B-splines, which we will need in the sequel.

For given knots $t_0 < t_1 < \ldots < t_n$, $n \in \mathbb{N}_0$ we define the B-spline

$$B_{(t_0,\ldots,t_n)}(x) := (t_n - t_0)\, [t_0, \ldots, t_n](\cdot - x)_+^{n-1}, \tag{1}$$

with the divided difference

$$[t_0, \ldots, t_n]f := \frac{[t_1, \ldots, t_n]f - [t_0, \ldots, t_{n-1}]f}{t_n - t_0} \tag{2}$$

and the truncated powers $x_+^m := \chi_{[0,\infty)}(x)\, x^m$. (see e.g.[15,18]).

Later, we will need the following statement about divided differences.

Lemma 1 *For $t_0 < t_1 < \ldots < t_n$ we have*

$$[t_0, \ldots, t_n]f := \sum_{k=0}^{n} \frac{f(t_k)}{\prod_{\ell \neq k}(t_k - t_\ell)}.$$

PROOF. For $n = 0$ the statement is obvious. For $n > 0$ we show by induction

$$
\begin{aligned}
[t_0, \ldots, t_n]f &= \frac{1}{t_n - t_0} \left(\sum_{k=1}^{n} \frac{f(t_k)}{\prod_{\ell \neq 0,k}(t_k - t_\ell)} - \sum_{k=0}^{n-1} \frac{f(t_k)}{\prod_{\ell \neq k,n}(t_k - t_\ell)} \right) \\
&= \frac{1}{t_n - t_0} \left(\frac{f(t_n)}{\prod_{\ell \neq 0,n}(t_n - t_\ell)} \right. \\
&\qquad \left. + \sum_{k=1}^{n-1} \frac{\big((t_k - t_0) - (t_k - t_n)\big) f(t_k)}{\prod_{\ell \neq k}(t_k - t_\ell)} - \frac{f(t_0)}{\prod_{\ell \neq 0,n}(t_0 - t_\ell)} \right) \\
&= \sum_{k=0}^{n} \frac{f(t_k)}{\prod_{\ell \neq k}(t_k - t_\ell)}. \qquad \square
\end{aligned}
$$

If the knots t_k are equidistant, the corresponding B-spline will be a scaled translate of the cardinal B-spline

$$N_d(x) := B_{(0,\ldots,d)}(x) = \frac{1}{(d-1)!} \sum_{k=0}^{d} (-1)^k \binom{d}{k} (x - k)_+^{d-1}. \tag{3}$$

The cardinal B-spline satisfies the refinement equation

$$N_d = \sum_{k=0}^{d} a_k \, N_d(2 \cdot - k)$$

with refinement coefficients

$$a_k = 2^{1-d} \binom{d}{k}. \tag{4}$$

Further, the integer translates of N_d are locally linear independent, i.e., if for any open set $\Omega \subset \mathbb{R}$

$$\sum_{k \in \mathbb{Z}} c_k \, N_d(\cdot - k)\Big|_{\Omega} = 0,$$

then $c_k = 0$ if $\mathrm{supp} N_d(\cdot - k) \cap \Omega \neq \emptyset$ (see e.g. [3]).

It is well known that for a given spline of order d the spaces

$$
\begin{aligned}
V_j &:= \mathrm{clos}_{L_2} \mathrm{span} \Big\{ N_d(2^j \cdot - k) : \; k \in \mathbb{Z} \Big\} \\
&= \Big\{ f \in L_2(\mathbb{R}) \cap C^{d-2}(\mathbb{R}) : \; f\Big|_{2^{-j}[k,k+1]} \in \Pi_{d-1}, \; k \in \mathbb{Z} \Big\}
\end{aligned}
$$

form a multi resolution analysis (MRA). Here, Π_{d-1} denotes the space of polynomials of degree less than d. Therefore, we can introduce a spline wavelet

$$\psi = \sum_{k\in\mathbb{Z}} b_k\, N_d(2\cdot -k) \in V_1 \tag{5}$$

so that

$$W_j := \mathrm{clos}_{L_2}\mathrm{span}\Big\{\psi(2^j\cdot -k):\ k\in\mathbb{Z}\Big\}$$

satisfies $V_{j+1} = V_j + W_j$ and $V_j \cap W_j = \{0\}$. This implies the decomposition relation

$$N_d(2x-\ell) = \sum_{k\in\mathbb{Z}} c_{\ell-2k}\, N_d(x-k) + d_{\ell-2k}\, \psi(x-k) \tag{6}$$

The sequences $\boldsymbol{a} = (a_k)_{k\in\mathbb{Z}}$, $\boldsymbol{b} = (b_k)_{k\in\mathbb{Z}}$ and $\boldsymbol{c} = (c_k)_{k\in\mathbb{Z}}$, $\boldsymbol{d} = (d_k)_{k\in\mathbb{Z}}$ play an important role for the reconstruction and decomposition algorithm (synthesis and analysis), respectively (see e.g. [3, Sect. 5.4]). Therefore, it is desirable that these sequences have finite support, i.e., there are only finitely many nonzero coefficients. Note that the local linear independence of B-splines implies that the finite support of $\boldsymbol{b}$ is equivalent to a compact support of the wavelet ψ. Furthermore, we know already that $\boldsymbol{a}$ is finitely supported.

In the sequel we look for *compactly supported spline wavelets with finitely supported decomposition sequences*, i.e., we want the sequences $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$ to be finitely supported. As an abbreviation we will use the term *biorthogonal spline wavelets* in the remainder of the article, if a misunderstanding is out of question.

3 Compactly supported spline wavelets with finitely supported decomposition sequences

An example for a class of biorthogonal spline wavelets is given in [10]. Looking at the graphs of these spline wavelets with order 1 and 2 it seems that there is only one non-integer knot at $\frac{1}{2}$. These observation is confirmed by the following theorem.

Theorem 2 *Let ψ be a spline wavelet with support in $[0,r]$, $r\in\mathbb{N}$ which satisfies (6) with finitely supported sequences $\mathbf{c}$ and $\mathbf{d}$. Then, ψ is a spline with knots*

$$t_k = \begin{cases} k, & \text{if } k = 0,\dots,s-1,\\ k-\frac{1}{2}, & \text{if } k = s\\ k-1, & \text{if } k = s+1,\dots,r+1, \end{cases}$$

for some $s\in\mathbb{Z}$, $1\le s\le r$

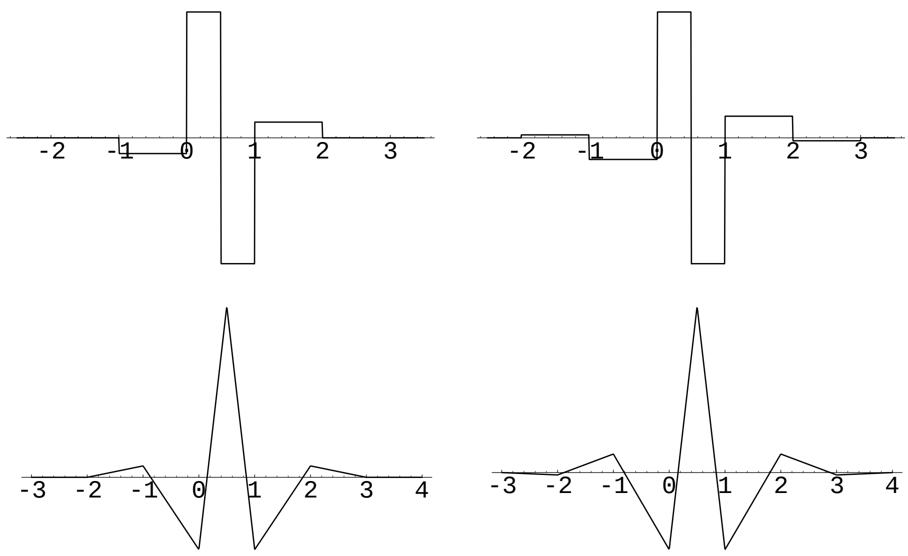

Fig. 1. Spline wavelets of order 1 and 2.

PROOF. From (3), we know that

$$N_d(2x-\ell) = \frac{2^{d-1}}{(d-1)!}\sum_{k=0}^{d}(-1)^{k-\ell}\binom{d}{k-\ell}\left(x-\tfrac{k}{2}\right)_+^{d-1}. \tag{7}$$

Furthermore, $\psi \in V_1$ is a compactly supported spline with knots in $\frac{1}{2}\mathbb{Z}$, i.e.,

$$\psi(x) = \sum_{k=0}^{2r}\alpha_k\left(x-\tfrac{k}{2}\right)_+^{d-1}. \tag{8}$$

Since the functions $N_d(x-k)$ have only integer knots we can write (6) as

$$\begin{aligned} N_d(2x-\ell) &= \sum_k \beta_k(x-k)_+^{d-1} + \sum_{k\in\mathbb{Z}} d_{\ell-2k}\sum_{n=1}^{r}\alpha_{2n-1}\left(x-n-k+\tfrac{1}{2}\right)_+^{d-1} \\ &= \sum_k \beta_k(x-k)_+^{d-1} + \sum_{k\in\mathbb{Z}}\sum_{n=1}^{r} d_{\ell+2(n-k)}\alpha_{2n-1}\left(x-k+\tfrac{1}{2}\right)_+^{d-1} \end{aligned} \tag{9}$$

with some coefficients β_k which depend on $\boldsymbol{c}$ and $\boldsymbol{d}$. By comparison of coefficients in (7) and (9) we conclude that

$$\frac{2^{d-1}}{(d-1)!}\binom{d}{k}(-1)^k = \sum_{n=1}^{r} d_{2n-k-1}\alpha_{2n-1}, \quad k\in\mathbb{Z}. \tag{10}$$

By z-transform we obtain

$$\frac{2^{d-1}}{(d-1)!}\sum_{k\in\mathbb{Z}}\binom{d}{k}(-z)^k = \sum_{k\in\mathbb{Z}}\sum_{n=1}^{r} d_{2n-1-k}\alpha_{2n-1}z^k,$$

i.e.,
$$\frac{2^{d-1}}{(d-1)!}(1-z)^d = D(\tfrac{1}{z})\, A(z),$$
where $D(z) = \sum_{n\in\mathbb{Z}} d_{n+1} z^n$ and $A(z) = \sum_{n=1}^{r} \alpha_{2n-1} z^{2n}$ are Laurent polynomials. From the equality above it follows that $A(z)$ can have zeros only at $z = 1$ or $z = 0$. On the other hand, since $A(z)$ is even we know that $A(1) = 0$ implies $A(-1) = 0$, i.e., if $z \neq 0$ then $A(z) \neq 0$. Hence, $A(z) = \alpha_{2s-1} z^{2s}$ or equivalently $\alpha_{2n-1} = \alpha_{2s-1}\delta_{ns}$ for some $s \in \mathbb{Z}$, $1 \le s \le r$. Therefore,

$$\psi(x) = \alpha_{2s-1}\left(x - s + \tfrac{1}{2}\right)_+^{d-1} + \sum_{k=0}^{r} \alpha_{2k}\,(x-k)_+^{d-1}, \tag{11}$$

which proves the theorem. □

Theorem 2 states that it is necessary that the wavelet is a spline with one and only one knot in $\mathbb{Z} + \frac{1}{2}$. From the proof one concludes by some simple arguments immediately that this condition is sufficient, too.

Theorem 3 *Let $\psi(x)$ defined in (11) be a spline wavelet with compact support. Then (6) holds with finitely supported sequences $\boldsymbol{c}$ and $\boldsymbol{d}$.*

PROOF. If ψ is a wavelet, then there exist (not necessarily finitely supported) sequences which satisfy the decomposition relation (6) and (10) holds. Since $\psi(x)$ satisfies (11) we know $\alpha_{2n-1} = \alpha_{2s-1}\delta_{ns}$ and (10) becomes

$$d_k = -\frac{2^{d-1}}{\alpha_{2s-1}(d-1)!}(-1)^k \binom{d}{2s-k-1}, \tag{12}$$

i.e., $d_k = 0$ if $k < 2s-1-d$ or $k > 2s-1$. Now we conclude from (6) with $\ell = 0$ that
$$\sum_k c_{-2k} N_d(x-k) = N_d(2x) - \sum_k d_{-2k}\psi(x-k) \tag{13}$$
is a compactly supported spline with knots in $\mathbb{Z}$. By local linear independence it follows immediately that there can be only a finite number of non-vanishing coefficients c_{-2k}. For $\ell = 1$ in (6) one obtains analogously the statement for the coefficients c_{1-2k}. □

4 Computation of the Decomposition and Reconstruction sequences

Let us assume we have chosen a spline wavelet and the coefficients α_k in the representation (11) are known. From (4) and (12) we know already the sequences a_k and d_k. In order to determine the coefficients b_k we apply (7) and (8) to (5) and obtain

$$\begin{aligned}\sum_{k=0}^{2r} \alpha_k \left(x-\tfrac{k}{2}\right)_+^{d-1} &= \frac{2^{d-1}}{(d-1)!} \sum_{k\in\mathbb{Z}} b_k \sum_{\ell=0}^{d} (-1)^\ell \tbinom{d}{\ell} \left(x-\tfrac{k+\ell}{2}\right)_+^{d-1} \\ &= \frac{2^{d-1}}{(d-1)!} \sum_{k=0}^{2r} \sum_{\ell=0}^{d} b_{k-\ell} (-1)^\ell \tbinom{d}{\ell} \left(x-\tfrac{k}{2}\right)_+^{d-1}.\end{aligned}$$

That is

$$\frac{(d-1)!}{2^{d-1}} \alpha_k = \sum_{\ell=0}^{d} b_{k-\ell} (-1)^\ell \tbinom{d}{\ell}, \quad k=0,\dots,2r. \tag{14}$$

or equivalently $\frac{(d-1)!}{2^{d-1}} \boldsymbol{\alpha} = \Delta^d \boldsymbol{b}$, where Δ is the difference operator defined by $(\Delta \boldsymbol{x})_k := x_k - x_{k-1}$.

That means $\boldsymbol{b}$ can be computed as follows. Set $\boldsymbol{b}^{(0)} := \frac{(d-1)!}{2^{d-1}} \boldsymbol{\alpha}$ and compute $b_k^{(m)}$ by the recursion

$$b_{-1}^{(m)} = 0, \quad b_k^{(m)} := b_k^{(m-1)} + b_{k-1}^{(m)}, \qquad k=0,\dots,2r+1-m, \quad m=1,\dots,d,$$

i.e., $b_k^{(m-1)} = \Delta b_k^{(m)}$, $k=0,\dots,2r+1-m$, and thus $b_k^{(d)} = b_k$, $k=0,\dots,2r+1-d$. The remaining coefficients b_k are zero, what can be shown by the local linear independence of the B-splines or directly from (14).

Analogously, applying (3), (7), and (8) to (6) yields

$$\begin{aligned}&\frac{2^{d-1}}{(d-1)!} \sum_{k=0}^{d} (-1)^k \tbinom{d}{k} (x-\tfrac{k+\ell}{2})_+^{d-1} \\ &\quad = \frac{1}{(d-1)!} \sum_{k\in\mathbb{Z}} c_{\ell-2k} \sum_{m=0}^{d} (-1)^m \tbinom{d}{m} (x-k-m)_+^{d-1} \\ &\qquad\qquad + \sum_{k\in\mathbb{Z}} d_{\ell-2k} \sum_{m=0}^{2r} \alpha_m \ (x-k-\tfrac{m}{2})_+^{d-1},\end{aligned}$$

By comparison of coefficients of x_+^{d-1} we obtain

$$\begin{aligned}\sum_{k\in\mathbb{Z}} c_{\ell+2k} (-1)^k \tbinom{d}{k} &= 2^{d-1} (-1)^\ell \tbinom{d}{-\ell} - (d-1)! \sum_{k\in\mathbb{Z}} d_{\ell+2k}\ \alpha_{2k} \\ &= \frac{2^{d-1}(-1)^\ell}{\alpha_{2s-1}} \left(\tbinom{d}{-\ell} \alpha_{2s-1} - \sum_{k\in\mathbb{Z}} \tbinom{d}{2(s-k)-1-\ell} \alpha_{2k} \right) \\ &= \frac{2^{d-1}(-1)^\ell}{\alpha_{2s-1}} \sum_{k\in\mathbb{Z}} \tbinom{d}{2s-k-1-\ell} \alpha_k,\end{aligned}$$

i.e.,

$$\sum_{k\in\mathbb{Z}} c_{\ell+2k}(-1)^k\binom{d}{k} = \frac{2^{d-1}(-1)^\ell}{\alpha_{2s-1}}\sum_{k\in\mathbb{Z}}\binom{d}{k}\alpha_{2s-k-1-\ell}.$$

Using z-transform the equality reads as

$$\begin{aligned}(1-z^{-2})^d\sum_k c_k z^k &= (1-z^{-1})^d(1+z^{-1})^d\sum_k c_k z^k\\ &= \frac{2^{d-1}}{\alpha_{2s-1}}(1-z^{-1})^d\sum_k \alpha_{2s-k-1}(-z)^k,\end{aligned}$$

i.e.,

$$\sum_{\ell=0}^{d-1} c_{k+\ell}\binom{d}{\ell} = \frac{2^{d-1}}{\alpha_{2s-1}}\alpha_{2s-k-1}(-1)^k. \tag{15}$$

With (4) we obtain

$$\sum_\ell c_{k+\ell}a_\ell = \frac{\alpha_{2s-k-1}}{\alpha_{2s-1}}(-1)^k.$$

For $k=-2m$ this becomes

$$\sum_\ell c_{\ell-2m}a_\ell = \delta_{m0}, \tag{16}$$

Further, (14) and (15) imply

$$b_k = \frac{\alpha_{2s-1}(d-1)!}{4^{d-1}}(-1)^{1-k}c_{2s-k-1},$$

while (12) means that

$$d_k = \frac{4^{d-1}}{\alpha_{2s-1}(d-1)!}(-1)^{1-k}a_{2s-k-1}.$$

That is, given the coefficients α_k we are able to determine $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$.

If the refinement equation

$$\tilde\varphi(x) = \sum_{k\in\mathbb{Z}} c_k\tilde\varphi(2x-k)$$

has a unique solution $\tilde\varphi\in L^2$, then (16) implies that $\tilde\varphi$ satisfies the biorthogonality condition $\langle N_d(\cdot-k),\tilde\varphi(\cdot-\ell)\rangle = \delta_{kl}$. Furthermore, the wavelets $\psi = \sum_k b_k N_d(2\cdot-k)$ and $\tilde\psi = \sum_k d_k\tilde\varphi(2\cdot-k)$ generate a pair of biorthogonal wavelet bases (see e.g. [19]).

5 Biorthogonal spline wavelets with minimal support

Up to now, we have only described all spline wavelets with finitely supported reconstruction and decomposition sequences. However, there are other properties desired in applications as vanishing moments, symmetry, or small support.

Assume ψ is compactly supported in $[0, r]$ and has $\tilde{d}$ vanishing moments, i.e.,

$$\int_{\mathbb{R}} x^m \psi(x)\, dx = 0, \quad m = 0, \dots, \tilde{d} - 1.$$

Let f be the uniquely determined solution of $f^{(\tilde{d})} = \psi$ and $f^{(k)}(0) = 0$, $k = 0, \dots, \tilde{d} - 1$. Obviously, f is a spline of order $d + \tilde{d}$ with $f(x) = 0$ for $x \leq 0$. The vanishing moments of ψ imply for $x \geq r$ that

$$f(x) = \int_0^x f'(t)\, dt = \int_0^x (x - t) f''(t)\, dt = \frac{1}{(\tilde{d} - 1)!} \int_0^x (x - t)^{\tilde{d}-1} f^{(\tilde{d})}(t)\, dt = 0,$$

i.e., ψ is the $\tilde{d}$-th derivative of a spline of order $d + \tilde{d}$ with support in $[0, r]$. Conversely, if ψ is the $\tilde{d}$-th derivative of a compactly supported spline of order $d + \tilde{d}$ one shows by partial integration that ψ has $\tilde{d}$ vanishing moments.

Now it turns out, that we can fix ψ by demanding symmetry and minimal support.

Theorem 4 *Let $d + \tilde{d} = 2n$ be even. If ψ is a symmetric or antisymmetric spline wavelet of order d, with $\tilde{d}$ vanishing moments and minimal support $[0, r]$, which satisfies* (6) *with finitely supported sequences $\boldsymbol{c}$ and $\boldsymbol{d}$, then $\psi(x) = C\, \psi^{d,\tilde{d}}(x)$ with some $C \neq 0$ and*

$$\psi^{d,\tilde{d}}(x) := B^{(\tilde{d})}_{(0,1,\dots,n-1,n-\frac{1}{2},n,n+1,\dots,2n-1)}(x) \tag{17}$$

$$= \frac{2n - 1}{(d - 1)!} \left(\frac{(-16)^n}{2n \binom{2n}{n}} (x - n + \tfrac{1}{2})_+^{d-1} + \sum_{k=0}^{2n-1} \frac{(-1)^k \binom{2n-1}{k}}{(n - \frac{1}{2} - k)} (x - k)_+^{d-1} \right). \tag{18}$$

PROOF. If ψ is a symmetric or antisymmetric spline then the corresponding knot sequence must be symmetric, too. Together with Theorem 2 this implies that ψ has the knots $0, 1, \dots, r = 2\nu - 1$ and $\nu + \frac{1}{2}$ for some $\nu \in \mathbb{N}$. The vanishing moments imply that ψ is the $\tilde{d}$-th derivative of a compactly supported spline f of order $2n = \tilde{d} + d$ with $2\nu + 1$ knots. Now it is well known that a compactly supported spline of order $2n$ has at least $2n + 1$ knots, i.e., in order to have minimal support f must have the knots $0, 1, \dots, n - 1, n - \frac{1}{2}, n, n + 1, \dots, 2n - 1$. But $B_{(0,1,\dots,n-1,n-\frac{1}{2},n,n+1,\dots,2n-1)}$ is (up to a constant factor) the only compactly supported spline of order $2n$ having these knots (see e.g. [18, § 4.2]), i.e., equality (17) is proven.

Now Lemma 2 applied to (1) implies

$$
\begin{aligned}
&B_{(0,1,\ldots,n-1,n-\frac{1}{2},n,n+1,\ldots,2n-1)}(x)\\
&\quad = (2n-1)\Biggl(\sum_{k=0}^{2n-1} \frac{(k-x)_+^{2n-1}}{(k-n+\frac{1}{2})\prod_{\ell=0}^{k-1}(k-\ell)\ \prod_{\ell=k+1}^{2n-1}(k-\ell)}\\
&\qquad\qquad + \frac{2^{2n}\ (n-\frac{1}{2}-x)_+^{2n-1}}{\prod_{\ell=0}^{2n-1}(2n-1-2\ell)}\Biggr)\\
&\quad = (2n-1)\Biggl(\sum_{k=0}^{2n-1}(-1)^{2n-1}\frac{(-1)^k\ (k-x)_+^{2n-1}}{(k-n+\frac{1}{2})k!(2n-1-k)!}\\
&\qquad\qquad + \frac{(-1)^n\ 2^{2n}\ (n-\frac{1}{2}-x)_+^{2n-1}}{\left(\prod_{\ell=1}^{n}(2\ell-1)\right)^2}\Biggr)\\
&\quad = \frac{2n-1}{(2n-1)!}\Biggl(-\sum_{k=0}^{2n-1}\frac{(-1)^k\binom{2n-1}{k}}{(k-n+\frac{1}{2})}\ (k-x)_+^{2n-1}\\
&\qquad\qquad + \frac{(-1)^n\ 4^{2n}}{2n\ \binom{2n}{n}}\ (n-\tfrac{1}{2}-x)_+^{2n-1}\Biggr).
\end{aligned}
$$

Finally, applying the symmetry of $B_{(0,1,\ldots,n-1,n-\frac{1}{2},n,n+1,\ldots,2n-1)}$ and taking the $\tilde{d}$-th derivative yields equality (18). $\square$

Note that the biorthogonal spline wavelet of order d with $\tilde{d}$ vanishing moments found by Cohen, Daubechies, and Feauveau in [10, Sect. 6A] has support length $d+\tilde{d}-1$ and is symmetric or antisymmetric. Hence, it is up to a constant factor and some integer shift identical with $\psi^{d,\tilde{d}}$. In particular, from [10] we know that $\psi^{d,\tilde{d}}$ has the Fourier transform

$$
\begin{aligned}
\hat{\psi}^{d,\tilde{d}}(\xi) &= \int_{\mathbb{R}} \psi^{d,\tilde{d}}(x)\, e^{-ix\xi}\, dx\\
&= C\, e^{-i(n-\frac{1}{2})\xi}\, (\tfrac{4}{\xi})^d \sin^{2n}(\tfrac{\xi}{4}) \sum_{\ell=0}^{n-1} \binom{n-1+\ell}{\ell} \cos^{2\ell}(\tfrac{\xi}{4})
\end{aligned}
\tag{19}
$$

with some constant $C \neq 0$.

6 Stability of Biorthogonal Spline Wavelets

The stability of a wavelet basis is crucial in applications. In particular, we want the wavelet system

$$\mathbf{\Psi} = \left\{ 2^{\frac{j}{2}} \psi(2^j \cdot -k) : \; j,k \in \mathbb{Z} \right\}$$

to be a Riesz basis for $L_2(\mathbb{R})$, i.e.

$$A \sum_{j,k\in\mathbb{Z}} |c_{jk}|^2 \leq \left\| \sum_{j,k\in\mathbb{Z}} c_{jk} 2^{\frac{j}{2}} \psi(2^j \cdot -k) \right\|^2 \leq B \sum_{j,k\in\mathbb{Z}} |c_{jk}|^2 \tag{20}$$

for some constants $0 < A \leq B < \infty$. In [10] it was shown that $\psi^{d,\tilde{d}}$ generates a Riesz basis if $\tilde{d} > 4.1653\, d + 2.5826$. Furthermore, in [9] a necessary and sufficient criterion for Riesz stability was established, which permits to prove Riesz stability also for some smaller $\tilde{d}$.

However, in applications it is usually not sufficient to know that there are positive constants A and B satisfying inequality (20), but also that these constant are close to each other. Therefore, we introduce the condition number

$$\kappa_\psi := \min \left\{ \tfrac{B}{A} : \; A \text{ and } B \text{ satisfy } (20) \right\}.$$

Obviously, $\kappa_\psi \geq 1$ for any wavelet, while $\kappa_\psi = 1$ if and only Ψ is an orthogonal basis (cf. Parseval's identity). Therefore, κ_ψ is a measure of stability, in the sense how good (20) approximates Parseval's identity.

Next we will show that for biorthogonal spline wavelets there is a lower bound for the condition number depending only on the spline order. We start with an auxiliary result showing that any biorthogonal spline wavelet can be obtained by lifting the classical ones from [10].

Lemma 5 *Let ψ be given by (11). Then there exists a $\gamma \in \mathbb{R}$ and a $k \in \mathbb{Z}$ so that*

$$\gamma\, \psi(x-k) = \psi^{d,d}(x) + \sum_{\ell\in\mathbb{Z}} \alpha_\ell\, N_d(x-\ell). \tag{21}$$

PROOF. With ψ given by (11) and $\psi^{d,d}$ given by (18) we choose $k = s - d$ and $\gamma = \frac{(2d-1)(-16)^d}{(d-1)!2d\binom{2d}{d}\alpha_{2s-1}}$. Then it is obvious that $\gamma\,\psi(\cdot - k) - \psi^{d,d} \in V_0$, what proves the lemma.

Theorem 6 *Assume ψ given by (21) has at least one vanishing moment. The corresponding condition number satisfies*

$$\kappa_\psi \geq 4^{d-1}$$

PROOF. Note that shifting the mother wavelet ψ does not change the basis, while multiplying γ changes the Riesz bounds A and B by the same factor γ^2, i.e., the condition number is the same. Therefore, we can assume without loss of generality that $\gamma = 1$ and $k = 0$ in (21). If the wavelet system is a Riesz basis with Riesz bounds A and B, then it is obvious, that the single scale basis

$$\mathbf{\Psi}_0 = \Big\{\psi(\cdot - k) : \ k \in \mathbb{Z}\Big\}$$

is also a Riesz basis, with Riesz bounds A and B. Since $\mathbf{\Psi}_0$ consists of shifts of a single function we know that

$$A \leq \sum_{k\in\mathbb{Z}} \left|\hat{\psi}(\xi - 2\pi k)\right|^2 \leq B$$

(see e.g. [3, Theorem 3.24]). The Poisson summation formula implies

$$\sum_{k\in\mathbb{Z}} \left|\hat{\psi}(\xi - 2\pi k)\right|^2 = \sum_{k\in\mathbb{Z}} \langle \psi, \psi(\cdot + k)\rangle e^{ik\xi}.$$

For $\xi = 0$ we obtain by (21)

$$\begin{aligned} A \leq & \sum_{k\in\mathbb{Z}} \langle \psi, \psi(\cdot + k)\rangle \\ = & \sum_{k\in\mathbb{Z}} \langle \psi^{d,d}, \psi^{d,d}(\cdot + k)\rangle + \sum_{k\in\mathbb{Z}} \langle \psi^{d,d}, s(\cdot + k)\rangle + \sum_{k\in\mathbb{Z}} \underbrace{\langle s, \psi(\cdot + k)\rangle}_{\langle s(\cdot - k), \psi\rangle} \\ = & \sum_{k\in\mathbb{Z}} \langle \psi^{d,d}, \psi^{d,d}(\cdot + k)\rangle + \left\langle \psi^{d,d}, \sum_{k\in\mathbb{Z}} s(\cdot + k)\right\rangle + \left\langle \sum_{k\in\mathbb{Z}} s(\cdot + k), \psi \right\rangle, \end{aligned}$$

where $s = \sum_\ell \alpha_\ell N_d(\cdot - \ell)$. Since the B-splines form a partition of unity we have

$$\sum_{k\in\mathbb{Z}} s(\cdot + k) = \sum_{\ell\in\mathbb{Z}} \alpha_\ell \sum_{k\in\mathbb{Z}} N_d(\cdot + k - \ell) = \sum_{\ell\in\mathbb{Z}} \alpha_\ell,$$

and the vanishing moment of ψ and $\psi^{d,d}$ imply

$$A \leq \sum_{k\in\mathbb{Z}} \langle \psi^{d,d}, \psi^{d,d}(\cdot + k)\rangle = \sum_{k\in\mathbb{Z}} \left|\hat{\psi}^{d,d}(2\pi k)\right|^2$$

Analogously, we obtain for $\xi = \pi$ that

$$\begin{aligned} B \geq & \sum_{k\in\mathbb{Z}} (-1)^k \langle \psi^{d,d}, \psi^{d,d}(\cdot + k)\rangle \\ & + 2\mathrm{Re}\left\langle \psi^{d,d}, \sum_{k\in\mathbb{Z}} (-1)^k s(\cdot + k)\right\rangle + \sum_{k\in\mathbb{Z}} (-1)^k \big\langle s, s(\cdot + k)\big\rangle. \end{aligned}$$

Since $u(x) = \sum_{k\in\mathbb{Z}}(-1)^k s(\cdot{+}k)$ is a spline which satisfies $u(x) = -u(x{+}1)$, we have $u^{(d)} = C_u \sum_k (-1)^k \delta_k$ (in the sense of distributions). By partial integration and (17) we obtain

$$\langle \psi^{d,d}, u\rangle = C_u \sum_{\ell} (-1)^{k+d} B_{(0,1,\ldots,d-1,d-\frac{1}{2},d,d+1,\ldots,2d-1)}(k) = 0,$$

where the last equality follows immediately from the symmetry of the B-spline $B_{(0,1,\ldots,n-1,n-\frac{1}{2},n,n+1,\ldots,2n-1)}$. Furthermore, the Poisson summation formula yields

$$\sum_{k\in\mathbb{Z}} (-1)^k \Big\langle s, s(\cdot + k)\Big\rangle = \sum_{k\in\mathbb{Z}} |\hat{s}(\pi + 2\pi k)|^2 \geq 0.$$

Hence,

$$B \geq \sum_{k\in\mathbb{Z}} (-1)^k \langle \psi^{d,d}, \psi^{d,d}(\cdot + k)\rangle = \sum_{k\in\mathbb{Z}} \left|\hat{\psi}^{d,d}(\pi + 2\pi k)\right|^2$$

Therefore, we have shown

$$\kappa_\psi \geq \frac{B}{A} \geq \frac{\sum_{k\in\mathbb{Z}} (-1)^k \langle \psi^{d,d}, \psi^{d,d}(\cdot + k)\rangle}{\sum_{k\in\mathbb{Z}} \langle \psi^{d,d}, \psi^{d,d}(\cdot + k)\rangle} = \frac{\sum_{k\in\mathbb{Z}} \left|\hat{\psi}^{d,d}(\pi + 2\pi k)\right|^2}{\sum_{k\in\mathbb{Z}} \left|\hat{\psi}^{d,d}(2\pi k)\right|^2}.$$

With (19) we obtain

$$\begin{aligned}
\kappa_\psi &\geq \frac{\sum_{k\in\mathbb{Z}} (\frac{\pi}{4} + \frac{\pi k}{2})^{-2d} \ \sin^{4d}(\frac{\pi}{4} + \frac{\pi k}{2}) \ \left(\sum_{\ell=0}^{d-1} \binom{d-1+\ell}{d} \ \cos^{2\ell}(\frac{\pi}{4} + \frac{\pi k}{2})\right)^2}{\sum_{k\in\mathbb{Z}} (\frac{\pi k}{2})^{-2d} \ \sin^{4d}(\frac{\pi k}{2}) \ \left(\sum_{\ell=0}^{d-1} \binom{d-1+\ell}{d} \ \cos^{2\ell}(\frac{\pi k}{2})\right)^2} \\
&= \frac{2\,\eta_{2d}(\frac{\pi}{4}) \ \sin^{4d}(\frac{\pi}{4}) \ \left(\sum_{\ell=0}^{d-1} \binom{d-1+\ell}{d} \ \cos^{2\ell}(\frac{\pi}{4})\right)^2}{\eta_{2d}(\frac{\pi}{2})} \\
&= \frac{2\,\eta_{2d}(\frac{\pi}{4}) \ 2^{-2d} \ \left(\sum_{\ell=0}^{d-1} \binom{d-1+\ell}{d} \ 2^{-\ell}\right)^2}{\eta_{2d}(\frac{\pi}{2})},
\end{aligned} \tag{22}$$

where

$$\eta_{2d}(\xi) := \sum_{k\in\mathbb{Z}} (\xi + \pi k)^{-2d}.$$

Note, that

$$\begin{aligned}
\eta_{2d}(\tfrac{\pi}{4}) &= \sum_{k=0}^{\infty} (\pi k + \tfrac{\pi}{4})^{-2d} + \sum_{k=-\infty}^{-1} (\pi k + \tfrac{\pi}{4})^{-2d} \\
&= \sum_{k=0}^{\infty} (\pi k + \tfrac{\pi}{4})^{-2d} + \sum_{k=1}^{\infty} (\pi k - \tfrac{\pi}{4})^{-2d} \\
&= \sum_{k=0}^{\infty} (\tfrac{\pi}{2} k + \tfrac{\pi}{4})^{-2d} = 2^{2d} \sum_{k=0}^{\infty} (\pi k + \tfrac{\pi}{2})^{-2d} = 2^{2d-1} \eta_{2d}(\tfrac{\pi}{2}).
\end{aligned} \tag{23}$$

Further,

$$\sum_{\ell=0}^{n} \binom{n+\ell}{n} 2^{-\ell} = 2^n \tag{24}$$

which can be shown by induction as follows

$$\begin{aligned}
\sum_{\ell=0}^{n} \binom{n+\ell}{n} 2^{-\ell} &= \sum_{\ell=0}^{n} \binom{n+\ell}{\ell} 2^{-\ell} = \sum_{\ell=0}^{n} \binom{n-1+\ell}{\ell} 2^{-\ell} + \sum_{\ell=1}^{n} \binom{n-1+\ell}{\ell-1} 2^{-\ell} \\
&= 2^{n-1} + \binom{2n-1}{n} 2^{-n} + \sum_{\ell=0}^{n-1} \binom{n+\ell}{\ell} 2^{-\ell-1} \\
&= 2^{n-1} + \binom{2n-1}{n} 2^{-n} - \binom{2n}{n} 2^{-n-1} + \frac{1}{2} \sum_{\ell=0}^{n} \binom{n+\ell}{\ell} 2^{-\ell}.
\end{aligned}$$

With $\binom{2n}{n} = 2\binom{2n-1}{n}$ we conclude

$$\frac{1}{2} \sum_{\ell=0}^{n} \binom{n+\ell}{n} 2^{-\ell} = 2^{n-1}$$

and (24) is shown. Plugging (23) and (24) into (22) yields $\kappa_\psi \geq 4^{d-1}$. $\square$

In Table 6 one can see numerical estimates of the condition numbers for the biorthogonal spline wavelets of Cohen, Daubechies, and Feauveau. The estimates are obtained by computing the Gramian matrix of the periodized wavelet system up to a certain level and taking the ratio of the largest and the smallest eigenvalue. Therefore, the numbers in the table are lower estimate for the condition number. However, by doing the computation for several choices of the maximum level $j_{\max}$ up to $j_{\max} = 11$ we get an idea how accurate the estimate might be, what is represented by the number of digits in the table. A dash represents choices of d and $\tilde{d}$, where we do not have a Riesz basis according to the criterion in [9].

Table 1
Condition numbers of the wavelet bases $\left\{2^{\frac{j}{2}} \psi_{d,\tilde{d}}(2^j \cdot -k) : \ k \in \mathbb{Z}, \ j = 0, \ldots, 11\right\}$

	$\tilde{d} - d$					
d	0	2	4	6	8	10
2	10	4.146	4.027	4.092	4.148	4.189
3	80	19.2	16.3336	16.0223	16.0036	16.0172
4	—	120	68.448	64.6584	64.0907	64.0067
5	—	—	330	263.78	257.299	256.225

Obviously, for a small spline order d there are spline wavelet bases with a condition number, which is very close to the lower bound found in Theorem 6. In particular, the spline wavelets of Cohen, Daubechies, and Feauveau seem to be a nearly optimal choice, if one chooses the number of vanishing moments $\tilde{d} \approx 2d$.

In the future our goal is to use ideas from this work to construct new spline wavelet bases on an interval. The bound from Theorem 6 will serve as a criterion how good the stability of such bases is.

References

[1] T. Berger and O. Strömberg. Exact reconstruction algorithms for the discrete wavelet transform using spline wavelets. *Appl. Comput. Harmon. Anal.*, 2:392–397, 1995.

[2] K. Bittner and K. Urban. Adaptive wavelet methods using semiorthogonal spline wavelets: Sparse evaluation of nonlinear functions. Preprint, Universität Ulm, 2004.

[3] C. K. Chui. *An Introduction to Wavelets.* Academic Press, Boston, 1992.

[4] C. K. Chui and J. Wang. A general framework of compactly supported splines and wavelets. *J. Approx. Theory*, 71:54–68, 1992.

[5] C. K. Chui and J. Wang. On compactly supported spline wavelets and a duality principle. *Trans. Amer. Math. Soc.*, 330:903–915, 1992.

[6] A. Cohen, W. Dahmen, and R. DeVore. Adaptive wavelet schemes for elliptic operator equations — convergence rates. *Math. Comput.*, 70(233):27–75, 2001.

[7] A. Cohen, W. Dahmen, and R. DeVore. Adaptive wavelet methods II — beyond the elliptic case. *Found. Comp. Math.*, 2(3):203–245, 2002.

[8] A. Cohen, W. Dahmen, and R. Devore. Adaptive wavelet schemes for nonlinear variational problems. *SIAM J. Numer. Anal.*, 41(5):1785–1823 (electronic), 2003.

[9] A. Cohen and I. Daubechies. A stability criterion for biorthogonal wavelet bases and their related subband coding scheme. *Duke Math. J.*, 68(2):313–335, 1992.

[10] A. Cohen, I. Daubechies, and J.-C. Feauveau. Biorthogonal bases of compactly supported wavelets. *Comm. Pure and Appl. Math.*, 45:485–560, 1992.

[11] W. Dahmen, A. Kunoth, and K. Urban. Biorthogonal Spline-wavelets on the Interval — Stability and Moment Condition. *Appl. Comp. Harm. Anal.*, 6:132–196, 1999.

[12] W. Dahmen, R. Schneider, and Y. Xu. Nonlinear functionals of wavelet expansions — adaptive reconstruction and fast evaluation. *Numer. Math.*, 86:49–101, 2000.

[13] I. Daubechies. Orthonormal bases of compactly supported wavelets. *Comm. Pure Appl. Math.*, 41(7):909–996, 1988.

[14] I. Daubechies. *Ten lectures on wavelets.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.

[15] C. de Boor. *A practical guide to splines*, volume 27 of *Applied Mathematical Sciences.* Springer-Verlag, New York, 1978.

[16] A. Kunoth. On the fast evaluation of integrals of refinable functions. In *Wavelets, images, and surface fitting (Chamonix-Mont-Blanc, 1993)*, pages 327–334. A K Peters, Wellesley, MA, 1994.

[17] E. G. Quak and N. Weyrich. Decomposition and reconstruction algorithms for spline wavelets on a bounded interval. *Appl. Comp. Harm. Anal.*, 1:217–23, 1994.

[18] L. L. Schumaker. *Spline Functions: Basic Theory.* Wiley-Interscience, New York, 1981.

[19] G. Strang and T. Nguyen. *Wavelets and filter banks.* Wellesley-Cambridge Press, Wellesley, MA, 1996.