On Existence and Uniqueness Theorems for a Geometric Initial Value Problem in Local Isometric Embedding

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ON EXISTENCE AND UNIQUENESS THEOREMS FOR A GEOMETRIC INITIAL VALUE PROBLEM IN LOCAL ISOMETRIC EMBEDDING

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ABSTRACT. In this paper a local problem of isometric embedding is studied. Here a sufficiant regular curve - the initial curve - is complemented to a surface of a prescribed line element - an isometric embedding. While the local problem of isometric embedding is already in the focus of scientific research, solutions with a prescribed initial curve have not been considered by any other author this far. Therefore, tools are developed that are appropriate to deal with this problem, i.e. the geometric invariants of a curve, of a normal cut of a curve and of a surface and their connection. Further, the number of solutions is also discussed here.

1. INTRODUCTION

1.1. **Isometric embedding.** The problem of local isometric embedding consist of finding a solution

$$\mathbf{X} : U \to \mathbb{R}^3 \in C^1(U)$$

with $U \subset \mathbb{R}^2$ being an open set, such that the system of partial differential equations

$$(\partial_i \mathbf{X}, \partial_j \mathbf{X}) = g_{ij}$$

is fulfilled in U. Here, ∂_i denotes the partial derivative with respect to the i-th dependend variable and $(\cdot\ ,\ \cdot\ )$ is the canonical scalar product. The prescribed functions $g_{ij} : V \to \mathbb{R}$ are sufficently regular coefficients of a line element or metric, where $V \supset U$ is a probably larger open set. Further the Gaussian curvature

$$K := \frac{-\frac{1}{2}(\partial_{11} g_{22} - 2\partial_{12} g_{12} + \partial_{22} g_{11}) + g_{mk}(\Gamma^m_{11}\Gamma^k_{22} - \Gamma^m_{12}\Gamma^k_{12})}{\det(g_{ij})}. \tag{1}$$

is introduced. Here as in the following sections, the Einsteinian summation convention is used, i.e. it will automatically be summed over the same sub- and superscripts. The Christoffel symbols are denoted by

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right)$$

and g^{kl} is the to g_{ij} inverse matrix.

The problem of local isometric embedding has a long history. The first results go back to Darboux [3], where he shows the equivalency of the solvability of that problem above to the solvability of some Monge-Amperè equation. Moreover, he introduced in the hyperbolic case ($K < 0$) characteristical parameters and deduced the later-called Darboux' system.

In the 1950s Hartman and Wintner contributed some existence theorems, [8, 9, 10] for different signs of K. Later in the 1980s this works were extended by Lin [11, 12] to non-negative case and the case with a sign change of K. In the 2000s this work was extended again by Han [5, 6]. For a comprehensive compendium with more other references, see [7]. All these papers have a common property, in that they do not consider any initial value problems.

1.2. **The problem.** All functions are assumed to be sufficiently regular. Consider a line element with coefficients $g_{ij} : V \to \mathbb{R}$, having negative Gaussian curvature. This matrix, like all other matrices in this paper, is a symmetric 2×2-matrix. Further, it is assumed to be positive-definite. Moreover, $\Gamma \subset V$ is a regular curve und $\mathbf{Y} : \Gamma \to \mathbb{R}^3$ is a curve in the Euclidian 3-dimensional space.

In this paper the following problem will be formulated in a strict way: Find a non-empty open set $U \subset V$ with $\Gamma \subset U$ and a mapping $\mathbf{X} : U \to \mathbb{R}^3 \in C^1(U)$, such that

$$
\begin{aligned}
&(\partial_i \mathbf{X}, \partial_j \mathbf{X}) = g_{ij} \quad \text{in} \quad U \\
&\mathbf{X}|_\Gamma = \mathbf{Y}.
\end{aligned} \tag{2}
$$

Because the approach in this paper, even to solve the differential equations, is not the familiar one as treated in the cited works, this approach will be proven completely.

The assumption on K being negative should be weakend in the future. It was made for simplicity to obtain an existence result, covering the initial condition. The method here to treat the initial condition is up to a point so universal that it might be applied to other approaches of solving the differential equations.

Moreover, some uniqueness results are obtained, too. It will be shown that the problem has exactly two solutions, which may coincide. A sufficient condition to the data is given, such that these solutions are two different ones. A counter example for a problem having also two different solutions where this condition is not valid is given, too. Further, in a special situation the transition from one solution to the other is presented.

1.3. **Organisation of the paper.** This paper is organised as follows. In section 2 the way to solve the differential equation will be proven. In the next section 3, the invariants of normal cuts will be established and a simple existence theorem will be proven. In section 4, these invariants will be connected with the second fundamental form of a surface and lead to proper initial conditions. In the last section, the problem (2) will be formulated in a strict way and the existence of exactly two solutions will be proven. Moreover, the number of solutions is treated in that section in a quite complete way.

Compactness theorems will be proven by the author in a further paper. That is, under special assumptions to the data, one can state a minimal size of the the open set U.

2. An equivalent system for the differential equation

The following theorem is the starting point of the further considerations

Theorem 1. (Fundamental theorem of surface theory) *Let $U \subset \mathbb{R}^2$ be a simply connected open set. Let g_{ij} be a symmetric positive-definite matrix of class $C^2(U)$ and b_{ij} be a symmetric matrix of class $C^1(U)$. Let further the conditions of integrability, namely the Gauss*

$$K = \frac{\det(b_{ij})}{\det(g_{kl})}$$

and Codazzi-Mainardi equations

$$\partial_k b_{ij} - \Gamma^m_{ik} b_{mj} = \partial_j b_{ik} - \Gamma^m_{ij} b_{mk}$$

be fulfilled. Then there exists up to isometric transformations (translations and rotations) exactly one regular surface $\mathbf{X} \in C^3(U)$ *with normal* $\mathbf{N} \in C^2(U)$*, such that*

$$(\partial_i \mathbf{X}, \partial_j \mathbf{X}) = g_{ij}, \quad (\partial_i \mathbf{X}, \partial_j \mathbf{N}) = -b_{ij} \quad in \quad U$$

holds true.

For this theorem, many references are valid, see [1, pp. 138], [2, pp. 237], [14, Vol. IV, pp. 61] or [15, pp. 146]. From that point the question of how to solve the conditions of integrability arises.

These equations will be solved by the tool of characteristic parameter transformation. This technique goes back to Darboux [3]. If there is a posibility to use a different, more modern technique, the tools of section 3 can be applied in that case.

Nevertheless, the following is written down for the convenience of the reader. Remark that b_{ij} denotes the matrix arising in the conditions of integrability, often called matrix of the second fundamental form of a surface. For the rest of the paper it is assumed to have a negative determinant; also let $\beta := \sqrt{-\det(b_{ij})} > 0$ be a shortcut.

Definition 1. *Let* $W \subset \mathbb{R}^2$ *be an open set. A diffeomorphic mapping*

$$z = (z^1(w), z^2(w)) : W \to U \in C^1(W, U)$$

which introduces characteristic parameters in the matrix b_{ij} *with* $\det(b_{ij}) < 0$*, i.e. one has the relations of characteristicy*

$$b_{ij}(z) \partial_1 z^i \partial_1 z^j = 0 = b_{ij}(z) \partial_2 z^i \partial_2 z^j \quad in \quad W$$

will be called a conjugate-characteristic mapping with respect to b_{ij} *or shortly a* conjugate-characteristic mapping. *Set further*

$$B := b_{ij}(z) z^i_u z^j_v.$$

Lemma 2. *Let* z *be a conjugate-characteristic mapping and let* $\sigma = \operatorname{sgn} J_z$ *denote the sign of the Jacobian* $J_z = z^1_u z^2_v - z^1_v z^2_u$ *of* z*. Then one has in* W *the equations*

$$\left(\beta\, b^{kl}\right)\big|_{z=z(w)} = \frac{z^k_u z^l_v + z^k_v z^l_u}{|J_z|},$$

which are equivalent to relations of characteristicy.

The proof is elementary and left to the reader.

Lemma 3. (Geometric Transformation Lemma) *Let* $z : W \to U \in C^2(W)$ *be a diffeomorphic parameter transformation. Let further denote* $\overline{g}_{ij}$, $\overline{\Gamma}^k_{ij}$, $\overline{b}_{ij}$ *the via* z *transformed symbols, i.e.*

$$\overline{g}_{ij} = g_{kl}(z) \partial_i z^k \partial_j z^l, \ \overline{b}_{ij} = b_{kl}(z) \partial_i z^k \partial_j z^l \ \text{and}\ \overline{\Gamma}^k_{ij} = \frac{1}{2} \overline{g}^{kl} \left(\partial_i \overline{g}_{jl} + \partial_j \overline{g}_{il} - \partial_l \overline{g}_{ij} \right).$$

Then the following two assertions are valid:

(1)
$$\overline{\Gamma}^m_{kl} \partial_m z^p = \Gamma^p_{ij} \partial_k z^i \partial_l z^j + \partial^p_{kl}.$$

(2) *The conditions of integrability are fulfilled in the overlined parameter frame iff they are fulfilled in the original one.*

Moreover, if z is a conjugate-characteristic mapping, the Codazzi-Mainardi equations can be rewritten in the form

$$\partial_u \overline{K} + 4\overline{\Gamma}^2_{12}\overline{K} = 0 \quad \textit{and} \quad \partial_v \overline{K} + 4\overline{\Gamma}^1_{12}\overline{K} = 0,$$

if one defines $\overline{K} = K(z)$.

The first part of the proof is standard and can be found in many differential geometry books, see [1, 2, 4, 14, 15] for instance. The second one is a simple calculation involving the coditions of integrability together with $\overline{b}_{12} = \overline{b}_{21} = \pm\sqrt{-\overline{K}\det(\overline{g}_{ij})}$ and $\overline{b}_{11} = \overline{b}_{22} = 0$. For this, take $\partial_k \sqrt{\det(g_{ij})} = \Gamma^l_{kl}$ in any parameter frame into account.

Lemma 4. *Let $z \in C^2(W)$ be a conjugate-characteristic mapping. Then z is a solution of the System*

$$\mathcal{D}_u\left(\mathcal{D}_v z^k\right) + \mathcal{D}_v\left(\mathcal{D}_u z^k\right) + \Gamma^k_{ij}\left(\mathcal{D}_u z^i \mathcal{D}_v z^j + \mathcal{D}_v z^i \mathcal{D}_u z^j\right) = 0 \tag{3}$$

in W with the Darboux derivative $\mathcal{D}_i = \sqrt{-K(z)}\partial_i$. The converse is also true: Let $z \in C^2(W)$ be a diffeomorphic solution of (3), then z is a conjugate-characteristic mapping of the inverse of

$$b^{ij}(z) = \frac{1}{\sqrt{-K(z)\det(g_{ij}(z))}} \left.\frac{z^k_u z^l_v + z^k_v z^l_u}{|J_z|}\right|_{w=z^{-1}(z)}.$$

Moreover, the b_{ij}'s as defined above fulfill the conditions of integrability.

Proof.

(1) In virtue of lemma 3, the conditions of integrability have to be rewritten in the cojugate-characteristic parameter frame. After taking point 2.) of the main part of the cited lemma into account, equation (3) is reached.

(2) In order to prove the converse, define b_{ij} as stated in the assumption. A short calculation reveals $\det(b_{ij}) = K\det(g_{ij})$. Taking lemma 2 into account, z is indeed a cojugate-characteristic mapping with respect to b_{ij}. By part (1) of this proof and lemma 3, one easily checks the validity of the missing Codazzi-Mainardi equations.

□

As a conclusion of this part it is to state: The solvability of the isometric embedding problem is equivalent to finding a diffeomorphic solution $z \in C^2(W)$ of equation (3).

3. Invariants of normal cuts and a existence theorem for this objects

In this section normal cuts of curves are discussed. Well-known in differential geometry of curves are the notions of curvature and winding and the system of Frenet. These concepts will be generalised in this scetion. By standard results, all curves will be written in arc-length parametrisation, see for instance [1, 2, 4, 15].

Definition 5. *Let $\mathbf{Y} = \mathbf{Y}(t) : [0,T] \to \mathbb{R}^3 \in C^3[0,T]$ be a regular curve with $\mathbf{P} := \mathbf{Y}' \in S^2$. Let further $\mathbf{N} = \mathbf{N}(t) : [0,T] \to S^2 \in C^2[0,T]$ be a unit normal vector with $(\mathbf{P},\mathbf{N}) = 0$. Denoting the vector $\mathbf{Q} := -\mathbf{P}\wedge\mathbf{N}$ then the tupel $\{\mathbf{Y},\mathbf{N}\}$*

will be called a normal cut *and the tripel* $\{\mathbf{P},\mathbf{Q},\mathbf{N}\}$ *will be called the* trihedron of the normal cut $\{\mathbf{Y},\mathbf{N}\}$. *Moreover, the vector*

$$\mathbf{K}=\mathbf{K}(t)=(\mathbf{K}^{(1)}(t),\mathbf{K}^{(2)}(t),\mathbf{K}^{(3)}(t)):[0,T]\to\mathbb{R}^3\in C^1[0,T]$$

will be called the curvature vector of the normal cut $\{\mathbf{Y},\mathbf{N}\}$ *with the components*

$$\begin{array}{rclclcl} -\mathbf{K}^{(1)} &=& -(\mathbf{Q},\mathbf{N}') &=& (\mathbf{Q}',\mathbf{N}) &=& \det(\mathbf{N},\mathbf{N}',\mathbf{P}) \\ \mathbf{K}^{(2)} &=& -(\mathbf{P},\mathbf{N}') &=& (\mathbf{P}',\mathbf{N}) &=& -\det(\mathbf{P},\mathbf{P}',\mathbf{Q}) \\ -\mathbf{K}^{(3)} &=& (\mathbf{P}',\mathbf{Q}) &=& -(\mathbf{P},\mathbf{Q}') &=& \det(\mathbf{P},\mathbf{P}',\mathbf{N}) \end{array} \tag{4}$$

which are called

$$\begin{array}{rl} -\mathbf{K}^{(1)} & \textit{(geodesic) torsion,} \\ \mathbf{K}^{(2)} & \textit{normal curvature,} \\ -\mathbf{K}^{(3)} & \textit{geodesic curvature.} \end{array}$$

Remark 6. *Let* $\mathbf{Y}$ *be a regular curve with* $\mathbf{P}'\neq 0$. *The mapping* $\mathbf{N}(t):=\mathbf{P}(t)\wedge\frac{\mathbf{P}'(t)}{|\mathbf{P}'(t)|}$ *fulfills* $(\mathbf{P},\mathbf{N})\equiv 0$ *and* $\{\mathbf{Y},\mathbf{N}\}$ *is a normal cut. In that case one has*

$$\begin{array}{rclcl} -\mathbf{K}^{(1)} &=& -(\mathbf{Q}',\mathbf{N}) &=& \dfrac{\det(\mathbf{P},\mathbf{P}',\mathbf{P}'')}{(\mathbf{P}',\mathbf{P}')} \\ \mathbf{K}^{(2)} &=& -\det(\mathbf{P},\mathbf{P}',\mathbf{Q}) &=& 0 \\ -\mathbf{K}^{(3)} &=& (\mathbf{P}',\mathbf{Q}) &=& |\mathbf{P}'|. \end{array} \tag{5}$$

These formulae are well-known in differential geometry of curves, see again [1, 2, 4, 15]. *By these comments, a curvature vector is defined in a natural way.*

Lemma 7. *Let* $\mathbf{K}$ *be a curvature vector. Then there is up to isometric transformations exactly one normal cut* $\{\mathbf{Y},\mathbf{N}\}$ *having* $\mathbf{K}$ *as its curvature vector.*

Proof. (1) Let $\{\mathbf{P},\mathbf{Q},\mathbf{N}\}$ the trihedron of a normal cut. By devoloping its first derivatives as a linear combination of the trihedron, one reaches at a Frenet-like system

$$\begin{array}{rclcl} \mathbf{P}' &=& &-& \mathbf{K}^{(3)}\mathbf{Q} + \mathbf{K}^{(2)}\mathbf{N} \\ \mathbf{Q}' &=& \mathbf{K}^{(3)}\mathbf{P} & & - \mathbf{K}^{(1)}\mathbf{N} \\ \mathbf{N}' &=& -\mathbf{K}^{(2)}\mathbf{P} &+& \mathbf{K}^{(1)}\mathbf{Q}. \end{array} \tag{6}$$

Let $\mathbf{Z}_i=(\mathbf{P}_i,\mathbf{Q}_i,\mathbf{N}_i)$ be the vector of the i-th component of the trihedron for $i=1,2,3$. Then the system above can be rewritten in the form

$$\mathbf{Z}_i'=\mathbf{K}\wedge\mathbf{Z}_i. \tag{7}$$

(2) Now, consider the unique solutions $\mathbf{Z}_i$ of the system (7) to the initial values $\mathbf{Z}_i(0)=e_i$ with the canonical unit vectors e_i. These solutions provide an orthonormal system of vectors, taking into account

$$\frac{d}{dt}(\mathbf{Z}_i,\mathbf{Z}_j)=(\mathbf{K}\wedge\mathbf{Z}_i,\mathbf{Z}_j)+(\mathbf{Z}_i,\mathbf{K}\wedge\mathbf{Z}_j)=0,$$

and therefore

$$(\mathbf{Z}_i(t),\mathbf{Z}_j(t))=(\mathbf{Z}_i(0),\mathbf{Z}_j(0))=\delta_{ij}.$$

By continuity, the equation

$$\det(\mathbf{Z}_1(t),\mathbf{Z}_2(t),\mathbf{Z}_3(t))=\det(\mathbf{Z}_1(0),\mathbf{Z}_2(0),\mathbf{Z}_3(0))=1$$

holds true. So $\mathbf{Z}_i$ form an orthonormal system of vectors for each $t\in[0,T]$ and so the vectors $\mathbf{P},\mathbf{Q},\mathbf{N}$ do likewise, and are therefore a trihedron of the normal cut $\{\mathbf{Y},\mathbf{N}\}$ with $\mathbf{Y} = \int_0^t \mathbf{P}\,dt$. Evaluating each equation (6), one has proven that the normal cut really has the curvature vector $\mathbf{K}$.

(3) Finally, the uniqueness has to be shown. Take two trihedrons of normal cuts, namely $\{\mathbf{P},\mathbf{Q},\mathbf{N}\}$ and $\{\overline{\mathbf{P}},\overline{\mathbf{Q}},\overline{\mathbf{N}}\}$ which have the same curvature vector. To prove the assumption it is sufficient to find a isometric transformation $\mathcal{P}$ such that $\mathbf{Y} = \mathcal{P}\circ\overline{\mathbf{Y}}$. Taking $\det(\overline{\mathbf{P}}(0),\overline{\mathbf{Q}}(0),\overline{\mathbf{N}}(0)) = 1 = \det(\mathbf{P}(0),\mathbf{Q}(0),\mathbf{N}(0))$ into account and define

$$\tilde{\mathbf{P}}(t) := \mathcal{P}\circ\overline{\mathbf{P}}(t), \quad \tilde{\mathbf{Q}}(t) := \mathcal{P}\circ\overline{\mathbf{Q}}(t), \quad \tilde{\mathbf{N}}(t) := \mathcal{P}\circ\overline{\mathbf{N}}(t)$$

with the orthogonal matrix

$$\mathcal{P} = (\mathbf{P}(0),\mathbf{Q}(0),\mathbf{N}(0))\circ(\overline{\mathbf{P}}(0),\overline{\mathbf{Q}}(0),\overline{\mathbf{N}}(0))^{-1}.$$

An elementary calculation reveals that the normal cut $\{\tilde{\mathbf{Y}},\tilde{\mathbf{N}}\}$ has the same curvature vector as $\{\mathbf{Y},\mathbf{N}\}$ and by the assymption as $\{\overline{\mathbf{Y}},\overline{\mathbf{N}}\}$. Therefore the vectors of the trihedron $\{\tilde{\mathbf{P}},\tilde{\mathbf{Q}},\tilde{\mathbf{N}}\}$ and of $\{\overline{\mathbf{P}},\overline{\mathbf{Q}},\overline{\mathbf{N}}\}$ are solutions of the same linear system (6) with the same initial values and have to coincide. Therefore, an isometric transformation was found if one takes into account that the integration to $\tilde{\mathbf{Y}}$, resp. $\overline{\mathbf{Y}}$ is unique up to a constant.

□

Lemma 8. *Let* $\{\mathbf{Y},\mathbf{N}\}$, $\{\mathbf{Y},\overline{\mathbf{N}}\}$ *be two normal cuts with trihedrons* $\{\mathbf{P},\mathbf{Q},\mathbf{N}\}$ *and* $\{\mathbf{P},\overline{\mathbf{Q}},\overline{\mathbf{N}}\}$. *Then there is an angle* $\phi = \phi(t) : [0,T] \to\in C^1[0,T]$ *such that*

$$\begin{array}{ccccc} \overline{\mathbf{Q}} & = & \cos\phi\,\mathbf{Q} & - & \sin\phi\,\mathbf{N}, \\ \overline{\mathbf{N}} & = & \sin\phi\,\mathbf{Q} & + & \cos\phi\,\mathbf{N}. \end{array}$$

Moreover, the curvature vectors $\mathbf{K}$ *and* $\overline{\mathbf{K}}$ *transform to*

$$\begin{array}{cccccc} \overline{\mathbf{K}}^{(1)} & = & & \mathbf{K}^{(1)} + \phi', & & \\ \overline{\mathbf{K}}^{(2)} & = & & \cos\phi\,\mathbf{K}^{(2)} & + & \sin\phi\,\mathbf{K}^{(3)}, \\ \overline{\mathbf{K}}^{(3)} & = & - & \sin\phi\,\mathbf{K}^{(2)} & + & \cos\phi\,\mathbf{K}^{(3)}. \end{array}$$

The determination of the angle is elementary and the transformation of the curvature vectors follows directly from the definition.

Lemma 8 allows one to translate the data $\mathbf{Y}$ in terms of its natural curvature vector to a normal cut induced by the normal of a surface if the angle ϕ can be evaluated in a unique way.

Further, lemma 8 and remark 6 allow one to define the curvature κ of a curve $\mathbf{Y}$ with $\mathbf{P}' \neq 0$, via the equation

$$\kappa := \sqrt{\left(\mathbf{K}^{(2)}\right)^2 + \left(\mathbf{K}^{(3)}\right)^2} = |\mathbf{P}'| > 0.$$

From now on to the end of the section consider a curve $\mathbf{Y}$ with $\kappa > 0$ laying on a regular surface $\mathbf{X} \supset \mathbf{Y}$ with normal $\mathbf{N}$. Then clearly $\{\mathbf{Y},\mathbf{N}\}$ form a normal cut. Therefore one has for $\mathbf{N}$ the linear system of three equations

$$\begin{array}{rcl} (\mathbf{P}\wedge\mathbf{P}')\cdot\ \mathbf{N} & = & -\mathbf{K}^{(3)} \\ \mathbf{P}'\cdot\ \mathbf{N} & = & \mathbf{K}^{(2)} \\ \mathbf{P}\cdot\ \mathbf{N} & = & 0. \end{array} \tag{8}$$

The determinant of the coefficient matrix is $(\mathbf{P}\wedge\mathbf{P}')^2 = \kappa^2 > 0$. Therefore, $\mathbf{N}$ is uniquely determined by the right-handed side.

Lemma 9. *Let* $\mathbf{X} : U \to \mathbb{R}^3 \in C^3(U)$ *be a regular surface with normal* $\mathbf{N}$ *with line element* $g_{ij} = (\partial_i \mathbf{X}, \partial_j \mathbf{X})$ *and second fundamental form* $b_{ij} = (\partial_{ij}\mathbf{X}, \mathbf{N})$*. Let further be* $\Gamma \subset U$ *a regular curve with parametrisation* $\zeta = \zeta(t) = (\zeta^1(t), \zeta^2(t)) : [0,T] \to \mathbb{R}^3 \in C^2[0,T]$ *subject to the condition*

$$g_{ij}(\zeta)\dot\zeta^i\dot\zeta^j \equiv 1.$$

Then $\{\mathbf{X}(\zeta), \mathbf{N}(\zeta)\}$ *is a normal cut having the curvature vector*

$$\begin{aligned} -\mathbf{K}^{(1)} &= -\dot\zeta^i b_{ij} g^{jk} e_{kl}\dot\zeta^l \\ \mathbf{K}^{(2)} &= \dot\zeta^i b_{ij}\dot\zeta^j \\ -\mathbf{K}^{(3)} &= -\dot\zeta^l e_{lk}(\ddot\zeta^k + \dot\zeta^i\dot\zeta^j\Gamma^k_{ij}) \end{aligned} \tag{9}$$

with $e_{12} = \sqrt{\det(g_{ij})} = -e_{21}$*,* $e_{11} = 0 = e_{22}$*. In particular,* $\mathbf{K}^{(3)}$ *depends only on* Γ *and* g_{ij}*.*

Proof. Define $\mathbf{Y}(t) = \mathbf{X}(\zeta(t))$ and $\mathbf{M}(t) = \mathbf{N}(\zeta(t))$. Then one has proved the equations

$$\begin{aligned} \mathbf{M}' &= \partial_i\mathbf{N}\dot\zeta^i = -\dot\zeta^i b_{ij} g^{jk}\partial_k\mathbf{X}, \\ \mathbf{P} &= \partial_i\mathbf{X}\dot\zeta^i, \\ \mathbf{P}' &= (\ddot\zeta^k + \dot\zeta^i\dot\zeta^j\Gamma^k_{ij})\partial_k\mathbf{X} + \dot\zeta^i\dot\zeta^j b_{ij}\mathbf{N}. \end{aligned}$$

In particular the equation

$$\mathbf{P}^2 = g_{ij}(\zeta)\dot\zeta^i\dot\zeta^j \equiv 1$$

holds true and therefore $\{\mathbf{X}(\zeta), \mathbf{N}(\zeta)\}$ is a normal cut. Moreover, one has in virtue of definition 4 the equations

$$\begin{aligned} -\mathbf{K}^{(1)} &= -\dot\zeta^i b_{ij} g^{jk}\dot\zeta^l \det(\mathbf{N}, \partial_k\mathbf{X}, \partial_l\mathbf{X}) &&= -\dot\zeta^i b_{ij} g^{jk} e_{kl}\dot\zeta^l \\ \mathbf{K}^{(2)} &= \dot\zeta^i\dot\zeta^j b_{jk} g^{kl}(\partial_i\mathbf{X}, \partial_l\mathbf{X}) &&= \dot\zeta^i b_{ij}\dot\zeta^j \\ -\mathbf{K}^{(3)} &= -\dot\zeta^l(\ddot\zeta^k + \dot\zeta^i\dot\zeta^j\Gamma^k_{ij})\det(\partial_l\mathbf{X}, \partial_k\mathbf{X}, \mathbf{N}) &&= -\dot\zeta^l e_{lk}(\ddot\zeta^k + \dot\zeta^i\dot\zeta^j\Gamma^k_{ij}). \end{aligned}$$

□

The fact that the geodesic curvature $\mathbf{K}^{(3)}$ depends only on Γ and g_{ij} leads to an angle ϕ defined in exactly two ways, in such a way that lemma 8 can be applied to transform the data $\mathbf{Y}$ into a normal cut belonging to the surface.

In other words, given a line element g_{ij}, a curve Γ and a curve $\mathbf{Y}$ with $\kappa^2 > \left(\mathbf{K}^{(3)}\right)^2$, there are only two posibilities to define the surface normal $\mathbf{N}$, namely as the solution of the linear system (8) with the right-handed side

$$\left(\mathbf{K}^{(3)}, \pm\sqrt{\kappa^2 - \left(\mathbf{K}^{(3)}\right)^2}, 0\right)^T.$$

That is why the problem will have two solutions.

Having collected all conclusions under explicit evaluation of the system (8), one has reached at

Theorem 2. *Let* $\mathbf{Y}$ *be a regular curve having* $\mathbf{K}$ *as its natural curvature vector. Let further* $g_{ij} : U \to \mathbb{R}$ *be a line element and* $\Gamma \subset U$ *be a regular curve. If one has* $\kappa^2 > \left(\mathbf{K}^{(3)}\right)^2$ *then there are exactly two functions* $\phi_\pm$ *such that the curvature vectors* $\mathbf{K}_\pm$ *tranformed via lemma 8 are curvature vectors of a normal cut* $\{\mathbf{X}(\Gamma), \mathbf{N}(\Gamma)\}$*,* *where* $\mathbf{X}$ *is a regular surface having the line element* g_{ij} *and normal* $\mathbf{N}$*. Moreover, the inequality*

$$\mathbf{K}_{\pm}^{(2)} \neq 0$$

holds true.

4. Equivalent initial values for the system

From now on consider the situation of theorem 2. By this theorem one can describe the data along Γ by exactly two curvature vectors $\mathbf{K}_{\pm}$. Consider further the first two equations (9). These are linear equations in the unknown $\dot{\zeta}^i b_{ij}$ with a nonvanishing determinant of the coefficient matrix. Therefore, these equations are equivalent to

$$\dot{\zeta}^i b_{ij} = \dot{\zeta}^i (\mathbf{K}_{\pm}^{(2)} g_{ij} - \mathbf{K}_{\pm}^{(1)} e_{ij}). \tag{10}$$

Remark 10. *Together with theorem 2 and lemma 9 one has realised an isometric embedding with curve* $\mathbf{Y}$ *iff the equations* (10) *are fulfilled.*

Moreover, taking the linear dependence of the three fundamental forms into account

$$p_{ij} - 2H b_{ij} + K g_{ij} = 0,$$

with $p_{ij} = (\partial_i \mathbf{N}, \partial_j \mathbf{N})$ and the mean curvature

$$H = b_{ij} g^{ij}.$$

This can be found in [13], for instance. If one multiplies the linear dependency condition with $\dot{\zeta}^i \dot{\zeta}^j$, after a summation one reaches at

$$(\mathbf{N}')^2 - 2H(\mathbf{P}, \mathbf{N}') + K = 0$$

or equivalently

$$b_{ij} g^{ij} = \frac{K + \left(\mathbf{K}_{\pm}^{(1)}\right)^2 + \left(\mathbf{K}_{\pm}^{(2)}\right)^2}{2\mathbf{K}_{\pm}^{(2)}}.$$

This is the third linear equation for the symmetric unknown b_{ij}. Together with the first two equations of (9) one has a linear system with non-vanishing determinant of the coefficient matrix and therefore Cramers rule gives

Lemma 11. *In the situation of theorem 2 one for the second fundamental form along* Γ *the equation*

$$b_{ij} = \frac{\dot{x}^k \dot{x}^l}{\dot{x}^p g_{pq} \dot{x}^q \mathbf{K}_{\pm}^{(2)}} \left(K e_{ki} e_{lj} + \left(\mathbf{K}_{\pm}^{(2)} g_{ki} - \mathbf{K}_{\pm}^{(1)} e_{ki} \right) \left(\mathbf{K}_{\pm}^{(2)} g_{lj} - \mathbf{K}_{\pm}^{(1)} e_{lj} \right) \right)$$

holds true.

Remark 12. *The lemma above makes the translation process quite universal. If one wants to realise a line element which complements a curve* $\mathbf{Y}$ *one has to realise the second fundamental form via the formulae given above. Only with that second fundamental form, the curve* $\mathbf{Y}$ *is in the trace of* Γ *with respect to the solution surface* $\mathbf{X}$*. Every process which solves the isometric embedding problem has to solve the conditions of integrability in a direct or indirect way. This is true because the existence of a* C^3*-surface is equivalent to a solution of the conditions of integrability.*

As an application of this remark, proper initial values for the system (3) are derived. In virtue of remark 10, expressing the equations (10) in conjugate-characteristic parameters will give proper initial values.

Theorem 3. *In the situation of lemma 9 consider a diffeomorphic solution* $z \in C^2(W)$ *of the system* (3) *with*

$$z^i\left(\frac{1}{\sqrt{2}}(t,-t)\right) = \zeta^i(t)$$

In virtue of lemma 4 the normal cut of the solution surface $\{\mathbf{X}(\Gamma), \mathbf{N}(\Gamma)\}$ *has a curvature vector* $\mathbf{K}$ *iff* z *has initial Cauchy data*

$$\partial_\nu z^i\left(\frac{1}{\sqrt{2}}(t,-t)\right) = \overline{\zeta}^i(t)$$

with the normal $\nu = \frac{1}{\sqrt{2}}(1,1)$ *and*

$$\overline{\zeta}^i(t) = \frac{1}{\sqrt{-K(\zeta(t)}}\left(-\mathbf{K}^{(1)}(t)\dot{\zeta}^i(t) + \mathbf{K}^{(2)}(t)g^{ij}e_{jk}\dot{\zeta}^k(t)\right).$$

Proof. Let $\partial_\nu z = \overline{\zeta}$ and denote as a shortcut $\mathcal{K} = K(z)$. The first step is to show the equation (10). Taking $z_t^i = \dot{\zeta}^i$ and the definition of $\overline{\zeta}$ into account, one has reached the two equations

$$e_{ij}z_\nu^i z_t^j = \frac{\mathbf{K}^{(2)}}{\sqrt{-\mathcal{K}}}, \quad g_{ij}z_\nu^i z_t^j = -\frac{\mathbf{K}^{(3)}}{\sqrt{-\mathcal{K}}}$$

after a multiplication of $g_{ij}\dot{\zeta}^j$ resp. $e_{ij}\dot{\zeta}^j$. These equations are equivalent to

$$\begin{aligned}
\sqrt{-\mathcal{K}} &= \mathbf{K}^{(2)}\frac{g_{ij}z_t^i z_t^j}{e_{kl}z_\nu^k z_t^l} \\
&= -\frac{\mathbf{K}^{(2)}}{2}\frac{g_{ij}(z_u^i + z_v^i)(z_u^i - z_v^j)}{e_{kl}z_u^k z_v^l} \\
&= -\sigma\frac{\mathbf{K}^{(2)}}{2}\frac{\mathcal{G}_{11} - 2\mathcal{G}_{12} + \mathcal{G}_{22}}{\sqrt{\det(\mathcal{G}_{ij})}}
\end{aligned}$$

and similarly,

$$-\mathbf{K}^{(1)} = -\mathbf{K}^{(2)}\frac{g_{ij}z_\nu^i z_t^j}{e_{kl}z_\nu^k z_t^l} = -\sigma\frac{\mathbf{K}^{(2)}}{2}\frac{\mathcal{G}_{11} - \mathcal{G}_{22}}{\sqrt{\det(\mathcal{G}_{ij})}}$$

with $\sigma = \operatorname{sgn} J_z$ and $g_{ij}\dot{\zeta}^i\dot{\zeta}^j = 1$. Here $\mathcal{G}_{ij} = g_{kl}\partial_i z^k \partial_j z^l$ denotes the line element in conjugate-characteristic parametrisation. From both of these equations one equivalently arrives at

$$\begin{aligned}
-\sigma\sqrt{-\mathcal{K}\det(\mathcal{G}_{ij})} + \sigma\sqrt{\det(\mathcal{G}_{ij}))}\mathbf{K}^{(1)} &= \mathbf{K}^{(2)}(\mathcal{G}_{11} - \mathcal{G}_{12}), \\
\sigma\sqrt{-\mathcal{K}\det(\mathcal{G}_{ij})} + \sigma\sqrt{\det(\mathcal{G}_{ij}))}\mathbf{K}^{(1)} &= \mathbf{K}^{(2)}(\mathcal{G}_{12} - \mathcal{G}_{22}).
\end{aligned}$$

Now, consider the function $w(t) = \frac{1}{\sqrt{2}}(t,-t)$ with $\dot{w} = \frac{1}{\sqrt{2}}(1,-1)$, which leads via $z \circ w$ to a parametrisation of Γ. Moreover, one has in conjugate-characteristic parametrisation for the second fundamental form the equations

$$\mathcal{B}_{12} = \mathcal{B}_{21} = \sigma\sqrt{-\mathcal{K}\det(\mathcal{G}_{ij})} = \sqrt{-K(z)\det(g_{ij})}J_z, \quad \mathcal{B}_{11} = 0 = \mathcal{B}_{22}$$

and for the total antisymmetric surface element

$$\mathcal{E}_{12} = -\mathcal{E}_{21} = \sigma\sqrt{\det(\mathcal{G}_{ij})} = \sqrt{\det(g_{ij})}J_z, \quad \mathcal{E}_{11} = 0 = \mathcal{E}_{22}.$$

And therefore the equations above can be restated as

$$\mathcal{B}_{ij}\dot{w}^i = \left(\mathbf{K}^{(2)}\mathcal{G}_{ij} - \mathbf{K}^{(1)}\mathcal{E}_{ij}\right) w^i.$$

With the definition, especially that of $\mathcal{E}_{ij}$, this is a parameter invariant equation, which is valid in any parameter frame iff it is valid in one. After parameter transformation the original curve Γ is reached, because $z \circ w$ is a parametrisation of Γ by the initial condition. So one has reached equations (10) if a second fundamental form is defined as in lemma 4. The statement at the very beginning of this section leads to the first two equations of (9), the third one is always fulfilled. The surface $\mathbf{X}$ with normal $\mathbf{N}$ which exists by the fundamental theorem of surface theory, induces a normal cut $\{\mathbf{X}(\Gamma), \mathbf{N}(\Gamma)\}$ with the curvature vector $\mathbf{K}$ in virtue of lemma 9.

Let the normal cut $\{\mathbf{X}(\Gamma), \mathbf{N}(\Gamma)\}$ induce the curvature vector $\mathbf{K}$. Then the equations (9) are fulfilled. Due to parameter invariance of this equations, they are also valid in conjugate-characteristic parametrisation. Inspeckting the proof of the first part and taking the linear independence of $g_{ij}\dot{\zeta}^i$ and $e_{ij}\dot{\zeta}^j$ into account, one sees the validity of the assuption. □

5. The existence and uniqueness theorems

In this section, the existence problem will be formulated in a strict way. Moreover, an existence theorem will be proven. The question of uniqueness will be discussed in a quite satisfactory way.

Problem 1.

(1) *In an open set $V \subset \mathbb{R}^2$, a line element with symmetric, positive-definite coefficients*

$$g_{ij} : V \to \mathbb{R} \in C^4(V)$$

is given. The Gaussian curvature $K = K(g_{ij})$ from formula (1) *is assumed to be negative in V.*

(2) *Let $\Gamma \subset V$ with $\operatorname{dist}(\Gamma, \partial V) > 0$ be a regular curve with an injective parametrisation*

$$\zeta = \zeta(t) = (\zeta^1(t), \zeta^2(t)) : (0, T) \to V \in C^2(0, T),$$

which fulfilles the condition $g_{ij}(\zeta)\dot{\zeta}^i\dot{\zeta}^j \equiv 1$. Moreover, the geodesic curvature of Γ with respect to g_{ij} is denoted by $\mathbf{K}^{(3)}$ from formula (9)*. Furthermore assume $\zeta \in C^2[0, T]$.*

(3) *Let $\mathbf{Y} : [0, T] \to \mathbb{R}^3 \in C^4(0, T)$ be a regular curve with $\mathbf{P} = \mathbf{Y}'$ and $\mathbf{P}^2 \equiv 1$. The curvature of $\mathbf{Y}$ is denoted by $\kappa = |\mathbf{P}'|$. It is assumed that the compatibility inequality*

$$\kappa^2 - \left(\mathbf{K}^{(3)}\right)^2 > 0$$

holds true in $[0, T]$.

(4) *Now, search for an open set* U *with* $\Gamma \subset U \subset V$ *and for a vector-valued mapping*

$$\mathbf{X} = \mathbf{X}(z^1, z^2) : U \to \mathbb{R}^3 \in C^4(U)$$

subject to the conditions

$$(\partial_i \mathbf{X}, \partial_j \mathbf{X}) = g_{ij} \quad in \quad U$$

and

$$\mathbf{X}(\zeta(t)) = \mathbf{Y}(t) \quad for \quad t \in (0, T).$$

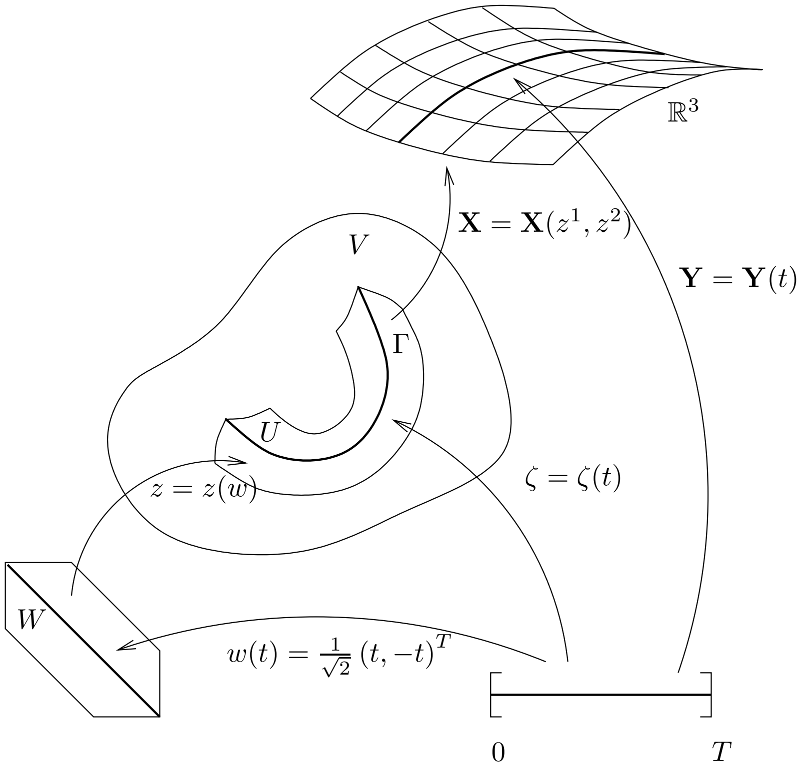

Figure 1. This figure shows the situation of the problem 1 and the theorem 4 below. There is sketched only one of the in general two different solutions

Theorem 4. *There is a sufficiently small open set* U *with* $\Gamma \subset U \subset V$ *such that the problem 1 has exactly two solutions, namely* $\mathbf{X}_\pm$.
Proof.

(1) In virtue of lemma 4 and theorem 1 one considers a solution z of the differential equation (3). This is a system of the form

$$z^k_{uv} = h^k_{ij}(z) \left(z^i_u z^j_v + z^i_v z^j_u \right)$$

with certain coefficients $h^k_{ij} \in C^1(V)$. The initial data $\mathbf{Y}$ is translated by theorem 2 into two curvature vectors $\mathbf{K}_\pm$. From theorem 3 there are imposed two sets of initial values, namely:

$$z^i \left(\frac{1}{\sqrt{2}}(t, -t) \right) = \zeta^i(t), \quad \partial_\nu z^i \left(\frac{1}{\sqrt{2}}(t, -t) \right) = \overline{\zeta}^i_\pm(t)$$

with a function

$$\overline{\zeta}^i_\pm(t) = \frac{1}{\sqrt{-K(\zeta(t)}}\left(-\mathbf{K}^{(1)}_\pm(t)\dot{\zeta}^i(t) + \mathbf{K}^{(2)}_\pm(t)g^{ij}e_{jk}\dot{\zeta}^k(t)\right).$$

By the regularity assumption from problem 1 one has $\zeta \in C^2[0,T]$ and $\overline{\zeta}_\pm \in C^1[0,T]$. With the theory in [13, Chapter XI, §4] and the theorem 4 as stated there, one has a unique solution $z_\pm \in C^1(W')$ of the Cauchy initial value problem with existing mixed derivatives defined in a set W'. Due to $h^k_{ij} \in C^1(V)$, one can do some regularity theory similiar to parametrised ordinary differential equations and obtain $z_\pm \in C^2(W')$. This is explicitely done in [4]. There is left to show that $z_\pm$ is a diffeomorphism. The proof of theorem 3 gives the formula

$$\frac{\mathbf{K}^{(2)}_\pm}{\sqrt{-K(z_\pm)}} = e_{ij}\partial_\nu z^i_\pm \partial_t z^j_\pm = \sqrt{\det(g_{ij})}J_{z_\pm}$$

on $u+v=0$. Therefore $z_\pm$ is in an environment of the diagonal $u+v=0$ a local diffeomorphism. Because ζ is injective, an environment $W \subset W'$ of $u+v=0$ can be found such that $z_\pm : W \to U := z_\pm(W)$ is a global diffeomorphism. Therefore theorems 4 and 1 state the existence of a surface $\mathbf{X}_\pm$ with the prescribed line element. The theorems 3, 2 and lemma 7 state that the curves $\mathbf{X}_\pm(\Gamma)$ and $\mathbf{Y}$ coincide.

(2) Let $\mathbf{X}$ be another solution with normal $\mathbf{N}$, then the normal cut induced by the surface $\{\mathbf{X}(\Gamma), \mathbf{N}(\Gamma)\}$ has by theorem 2 exactly two possible curvature vectors, namely $\mathbf{K}_\pm$. Then consider a conjugate-characteristic reparametrisation $\overline{z}$ whose preimage $\overline{z}^{-1}(\Gamma)$ is a curve parametrised by

$$\eta(t) = \left(\eta^1(t), \eta^2(t)\right) \in C^2,$$

with $g_{kl}\partial_i\overline{z}^k\partial_j\overline{z}^l\dot{\eta}^i\dot{\eta}^j \equiv 1$, furthermore it defines a curvature vector $\mathbf{K} = \mathbf{K}_\pm$ on Γ. The existence of a conjugate-characteristic transformation can be concluded from [13, chapter XI, §3]. Consider now the transformation $\hat{z} \in C^2$ given by

$$\hat{z}(u,v) = \left(\eta^1\left(\sqrt{2}u\right), \eta^2\left(-\sqrt{2}v\right)\right).$$

Taking the second equation of (9) and $\mathbf{K}^{(2)} \neq 0$ into account, one has

$$\frac{d}{dt}\eta^1 \neq 0 \quad \text{and} \quad \frac{d}{dt}\eta^2 \neq 0$$

and therefore, $\hat{z}$ is a diffeomorphism. It should be remarked that $\hat{z}$ transformes a characteristic form into a characteristic form. Then $z = \overline{z} \circ \hat{z}$ is a solution of the system (3). Because z has prescribed ζ as initial data, it follows from theorem 3 that $\partial_\nu z$ coincides with either $\overline{\zeta}_+$ or $\overline{\zeta}_-$. Therefore z coincides with one of the $z_\pm$ und therefore the surface $\mathbf{X}$ coincides by theorem 1 with one of the $\mathbf{X}_\pm$.

□

Lemma 13. *Let $\mathbf{Y}$ be a flat curve, w.l.g. in the $x_1 - x_2$-plane. Then one of the two solutions from theorem 4 emenates from the other by a reflection with respect to the $x_1 - x_2$-plane.*

Proof. Denote by $\mathbf{X}_+ = (\mathbf{X}_+^{(1)}, \mathbf{X}_+^{(2)}, \mathbf{X}_+^{(3)})$ the component functions of $\mathbf{X}$. Then one has with

$$\mathbf{X} = (\mathbf{X}_+^{(1)}, \mathbf{X}_+^{(2)}, -\mathbf{X}_+^{(3)})$$

another solution of the problem. Assume $\mathbf{X} \neq \mathbf{X}_+$, then $\mathbf{X}$ has to agree with the second solution.

If $\mathbf{X}$ and $\mathbf{X}_+$ agree, then $\mathbf{X}_+$ is symmetric with respect to the $x_1 - x_2$-plane. In that case consider the normals $\mathbf{N}_+$ and $\mathbf{N}$ of $\mathbf{X}_+$ and $\mathbf{X}$. One has reached at $\mathbf{N} = \mathbf{N}_-$ for the normal $\mathbf{N}_-$ of $\mathbf{X}_-$ which follows from $\mathbf{N}_+ \equiv -\mathbf{N}$. □

Remark 14. *Under the condition that* $\mathbf{Y}$ *is flat, the proof of the lemma reveals, that if*

$$\{\mathbf{X}_+(z) : z \in U\} \cap K = \{\mathbf{X}_-(z) : z \in U\} \cap K$$

holds true with an adequate open set $K \supset \mathbf{Y}$*, then the equations* $\mathbf{K}^{(3)} \equiv 0$, $\mathbf{K}_+^{(2)} = -\mathbf{K}_-^{(2)}$ *and* $\mathbf{K}_\pm^{(1)} \equiv 0$ *are valid neccesarily. But this is not sufficient, as the following example shows:*

$$\mathbf{X}(z^1, z^2) = \left(z^1, z^2, \frac{(z^1)^3}{3} + \frac{(z^1)^2}{2} - \frac{(z^2)^2}{2}\right)$$

is a solution of the problem, where the line element of $\mathbf{X}$ *is prescribed with*

$$K(z^1, z^2) = -2z^1 - 1 < 0 \quad \textit{for} \quad z^1 > -\frac{1}{2}$$

together with the initial data

$$\mathbf{Y}(t) = \left(0, t, -\frac{t^2}{2}\right) = \mathbf{X}(0, t).$$

For the sake of simplicity the condition $\mathbf{P}^2 \equiv 1$ *is not imposed here. One has to rescale the initial condition with the inverse of*

$$s(t) = \int_0^t \sqrt{1 + t^2}\, dt$$

to obtain formaly a proper example.

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