Recursion Formulae for the Characteristic Polynomial of Symmetric Banded Matrices

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Preprint Series: 2007-15

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Abstract. In this article we treat the algebraic eigenvalue problem for real, symmetric, and banded matrices of size $N \times N$, say. For symmetric, tridiagonal matrices, there is a well-known two-term recursion to evaluate the characteristic polynomials of its principal submatrices. This recursion is of complexity $O(N)$ and it requires additions and multiplications only. Moreover, it is used as the basis for a numerical algorithm to compute particular eigenvalues of the matrix via bisection. We derive similar recursion formulae with the same complexity $O(N)$ for symmetric matrices with arbitrary bandwidth, containing divisions. The main results are *divisionfree recursions for penta- and heptadiagonal* symmetric matrices. These recursions yield similarly as in the tridiagonal case effective (with complexity $O(N)$), fast, and stable algorithms to compute their eigenvalues.

Running head: Recursion formulae for banded matrices

Key words: Banded matrix, eigenvalue problem, Sturm-Liouville equation, pentadiagonal matrix, heptadiagonal matrix, bisection method

AMS subject classification: 15A18; 65F15, 15A15, 39A10, 15A24

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Recursion formulae for the characteristic polynomial of symmetric banded matrices

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1 Introduction

In this article we consider the algebraic eigenvalue problem for real, symmetric, and banded matrices of size $N \times N$, say. We are interested in algorithms for the numerical computation of some of their eigenvalues in an efficient and stable way. Concerning the efficiency we will always assume that their bandwidths are "very small" compared to N. Our main results here are recursion formulae for the evaluation of their characteristic polynomials rather than concrete numerical results.

For symmetric, tridiagonal matrices, there is a well-known two-term recursion to evaluate the characteristic polynomials of the principal submatrices of the given $N \times N$-matrix (see e.g. [10, Sec. 8.5]). This recursion requires $O(N)$ numerical operations, consisting of additions and multiplications only. By the Poincaré separation theorem the sequence of these polynomials constitutes a Sturmian chain [without *strict* separation] (it " ... forms the basis of one of the most effective methods of determining the eigenvalues" according to Givens 1954 [18, p. 300]). Therefore the recursion scheme yields an algorithm to compute all or special eigenvalues via bisection (see [10, Sec. 8.5], [14, Sec. 4.5.4], or [18, Ch. 5 Sec. 39]).

Every symmetric matrix can be transformed to (symmetric and) tridiagonal form by Givens' or Householder's method (see [14, Sec. 4.5] or [18, Ch. 5 Sec. 22-35]). These methods require $O(N^3)$ numerical operations (including square roots and divisions) for all symmetric matrices. This amount of work reduces to $O(N^2)$ for banded (but not tridiagonal) symmetric matrices with a modified Givens' method [13], where the O-constant naturally depends on the bandwidth. As this complexity of $O(N^2)$ still dominates the tridiagonal recursion, the complexity of the combined algorithm (i.e. Givens' method and tridiagonal recursion) becomes $O(N^2)$ altogether. Here we derive recursion schemes similar to the tridiagonal case for any bandwidth avoiding the above transformation to tridiagonal form. We obtain a *complexity of* $O(N)$ altogether with the O-constant depending on the bandwidth. For *penta- and heptadiagonal matrices* (i.e. the bandwidth equals 5 or 7 respectively) our algorithms require additions and multiplications only as in the tridiagonal case. These algorithms are given by formula (47) of Theorem 6 or formula (49) of Theorem 7 respectively, after performing the transformation via formula (4) from Section 2.

The basis of this work is the article [12] by Kratz. In [12] (see also [11]) a recursion scheme for the Sturmian chain [without *strict* separation] of the characteristic polynomials of the principal submatrices of any symmetric, banded matrix is derived [12, Th. 2 formula (17)]. This procedure in [12] requires a transformation to an equivalent *discrete Sturm-Liouville eigenvalue problem* via formula (4). In contrast to Givens' or Householder's method this transformation is of complexity $O(N)$, and it requires additions and multiplications only. Unfortunately the recursion scheme for the Sturm-Liouville difference equation or rather for the corresponding *Hamiltonian difference system* as described in Theorem A is **unstable** (in theory and in practice), because it contains a division and it needs the evaluation of a certain determinant at each step (compare the discussion in Remark 1 or see the diploma thesis by Tentler [17]).

To overcome this problem we consider a *Riccati difference system* in this article. It is derived from the Hamiltonian system above and leads to a new recursion as described in Theorem 1. The resulting algorithm (formulated in Theorem 2) contains divisions too, making it unstable in theory, although it turns out to be quite stable and very fast "in practice". Its complexity is also $O(N)$ (the O-constant depending on the bandwidth).

Actually, as follows from Lemma 1 (iii), the divisions occuring in the algorithm (based on the recursions via Riccati equations) can be carried out explicitly! It is the *main result* of this article to derive the corresponding recursion formulae in the case of *penta- and heptadiagonal symmetric matrices* (Theorems 4 and 5). The resulting algorithms (as formulated in Theorems 6 and 7) to compute the eigenvalues via bisection are effective (with complexity $O(N)$), fast, and stable. Algebraic eigenvalue problems for penta- and heptadiagonal symmetric matrices occur for example when computing numerically the eigenvalues of certain Schrödinger operators as done by V. Fack and G. Vanden Berghe ([7], [8]). In this article we give no numerical examples. Numerical results are contained in the dissertation by Tentler [16]. Additionally, *divisionfree recursions* are derived in [16] (as in our Theorems 4 and 5) for every bandwidth but in a different (and far more complicated) way.

The article is organized as follows: In the next section we introduce some necessary notation and we formulate the transformation of the given algebraic eigenvalue problem to a Sturm-Liouville eigenvalue problem with the corresponding recursion via Hamiltonian systems in Theorem A [12, Th. 2]. In Section 3 we derive the corresponding Riccati difference system (Theorem 1) and the resulting algorithm (Theorem 2). For our divisionfree recursions we need additional initial conditions, which are given in Section 4 (Theorem 3). In Section 5 we prove our main results (Theorems 4 and 5), namely divisionfree recursions in the penta- and the heptadiagonal case. Finally, in Section 6, we combine the results of Sections 3 to 5 to obtain the corresponding algorithms (Theorems 6 and 7).

2 Recursion via Hamiltonian difference systems

We consider the algebraic eigenvalue problem for symmetric banded matrices as in [12] (see also [11]). More precisely,

$$\mathcal{A}y = \lambda y\,, \quad y \in \mathbb{R}^{N+1-n} \setminus \{0\}\,, \tag{1}$$

where $1 \le n \le N$, and $\mathcal{A} = \mathcal{A}_{N+1} = (a_{\mu\nu})_{\mu,\nu=1}^{N+1-n} \in \mathbb{R}^{(N+1-n)\times(N+1-n)}$ satisfies

$$a_{\mu\nu} = a_{\nu\mu} \ \text{ for all } \ \mu, \nu, \ \text{ and } \ a_{\mu\nu} = 0 \ \text{ if } \ \mu \ge n + \nu + 1\,, \tag{2}$$

i.e. $\mathcal{A}$ is symmetric and banded with bandwidth $2n+1$. For convenience, we assume that the $a_{\mu\nu}$ are given reals satisfying (2) for all integers μ, ν . We put

$$\mathcal{A}_m = (a_{\mu\nu})_{\mu,\nu=1}^{m-n} \in \mathbb{R}^{(m-n)\times(m-n)}\,, \quad d(\lambda) := \det(\mathcal{A} - \lambda I) \tag{3}$$

for $n+1 \le m \le N+1$. Here, and in the following, I denotes the identity matrix of suitable size, and $\det$ abbreviates determinant. It is the aim of this note to calculate the characteristic polyomial $d(\lambda)$ of $\mathcal{A}$, and moreover, of all the $\mathcal{A}_m$'s for $n + 1 \le m \le N + 1$ via recursions. Such a recursion is well-known and used in numerical analysis for tridiagonal symmetric matrices (cf. [18, Ch. 5 Sec. 36] or [10, Sec. 8.5]), which corresponds in our setting to the *scalar case* $n = 1$.

For $0 \le \mu \le n, \quad k \in \mathbb{Z}$, we define [12, formula (3)]

$$r_\mu(k+\mu) = \tag{4}$$
$$(-1)^\mu \sum_{s=\mu}^{n} \left\{ \binom{s}{\mu} a_{k+1,k+1+s} + \sum_{l=1}^{s-\mu} \frac{s}{l} \binom{\mu+l-1}{l-1} \binom{s-l-1}{s-\mu-l} a_{k+1-l,k+1-l+s} \right\}$$

so that, in particular,

$$r_n(k+n) = (-1)^n a_{k+1,k+1+n} \quad \text{for all } k\,. \tag{5}$$

With this setting our eigenvalue problem (1) is equivalent to a Sturm-Liouville difference eigenvalue problem of the even order $2n$ by [12, Theorem 1]. The corresponding Sturm-Liouville difference equation (where $\Delta y_k = y_{k+1} - y_k$)

$$L(y)_k := \sum_{\mu=0}^{n} (-\Delta)^\mu \left\{ r_\mu(k) \Delta^\mu y_{k+1-\mu} \right\} = \lambda y_{k+1}$$

is the basis of our main results, which are recursion formulae to compute the characteristic polynomials of the matrices $\mathcal{A}_m$, in particular, of the given matrix $\mathcal{A}$. As is well-known (cf. [6] or [2, Remark 2]), these Sturm-Liouville equations are in turn equivalent to linear Hamiltonian difference systems, which will play a central role here. Therefore we need the following notation, which will be used throughout this paper:

$$A := \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix} \quad \text{denotes the } n \times n\text{-companion matrix,}$$

$$\tilde{A} := (I - A)^{-1} = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}, \quad B_k := \frac{1}{r_n(k)} B \quad \text{if } r_n(k) \neq 0\,, \tag{6}$$

where $B = \operatorname{diag}(0, \ldots, 0, 1)$, $C_k = \operatorname{diag}(r_0(k), \ldots, r_{n-1}(k))$,

and $\tilde{C} = \operatorname{diag}(1, 0, \ldots, 0)$,

where diag denotes diagonal matrices, and where all these matrices are of size $n \times n$.

Using this notation, we have the following result [12, Theorem 2]:

Theorem A *(Recursion via Hamiltonian system).*
Assume (2), (4), and suppose that

$$r_n(k) \neq 0 \quad \textit{for all } \; 0 \leq k \leq N\,. \tag{7}$$

Then, for all $n \leq m \leq N$, $\lambda \in \mathbb{R}$ *, we have that (with notation (3))*

$$\det(\mathcal{A}_{m+1} - \lambda I) = r_n(0) \cdots r_n(m) \det X_{m+1}(\lambda), \tag{8}$$

where $Z = (X, U) = (X_k, U_k) = (X_k(\lambda), U_k(\lambda))_{k=0}^{N+1}$ *is the* principal solution *of the following corresponding* Hamiltonian difference system

$$X_{k+1} = \widetilde{A}(X_k + B_k U_k), \quad U_{k+1} = (C_k - \lambda \widetilde{C}) X_{k+1} + (I - A^T)\, U_k \tag{9}$$

for $0 \leq k \leq N$ *with the initial condition*

$$X_0 = 0\,, \quad U_0 = I\,. \tag{10}$$

Note that $X_k = X_k(\lambda)$, $U_k = U_k(\lambda)$ are real $n \times n$ -matrices such that $Z = (X, U)$ is a *conjoined basis* of (9) (cf. e.g. [1, Definition 1]), which means that they satisfy (9) and that

$$\operatorname{rank}\left(X_k^T, U_k^T\right) = n\,, \quad X_k^T\, U_k = U_k^T\, X_k \;\text{ for all }\; 0 \leq k \leq N+1\,. \tag{11}$$

As in the formulae (9)-(11), as a rule we shall omit the argument λ , when it is any fixed real "playing no role". Finally, under the assumptions and with the notation of Theorem A we have that, for all $\lambda \in \mathbb{R}$, (cf. [12, Lemma 4]):

$$\operatorname{rank} X_k(\lambda) \equiv k \;\text{ for }\; 0 \leq k \leq n-1\,, \quad r_n(0) \cdots r_n(n-1) \det X_n(\lambda) \equiv 1\,, \tag{12}$$

so that, in particular, $X_n(\lambda)$ is always invertible.

Remark 1 The recursion (9) with initial values (10) constitutes an algorithm to compute recursively the characteristic polynomials of the given matrices $\mathcal{A}_m$ and in particular of $\mathcal{A}$, if condition (7) holds. In the scalar case $n = 1$ this algorithm is the well-known procedure to evaluate the characteristic polynomial of tridiagonal matrices [18, pp. 299], except that we have the division by $r_n(k)$ in each step, which makes the algorithm unstable in general and which can be avoided for all n as we will see later on. Besides these divisions the algorithm of Theorem A requires at each step the evaluation of the $n \times n$ -determinant of $X_{m+1}(\lambda)$ by (8), and this makes the procedure also unstable in general for $n \geq 2$. This instability is caused by the fact that the matrix elements of X_m become large while its determinant remains small. For $n = 2$ the "standard problem" $\Delta^4 y_{k-1} = \lambda y_{k+1}$ shows already this effect, which can be seen by an explicit calculation. Besides eliminating the divisions by $r_n(k)$ it is the main goal of the next sections to avoid the calculation of the determinants of X_m , and this is achieved by a transition from the Hamiltonian system (9) to a corresponding Riccati matrix difference equation, and this transition means essentially a multiplication of (9) by the inverse matrices X_{k+1}^{-1} at each step. This procedure is carried out in the next section.

3 Recursion via Riccati matrix difference equations

Besides the notation of the previous section we need some further notation concerning the $n \times n$ -matrices X_k and U_k of the Hamiltonian system (9).

Let be given real $n \times n$ -matrices X_k and U_k , $k \in \mathbb{Z}$, which may depend on the eigenvalue parameter λ . We denote (compare assertion (8) of Theorem A):

$$d_k := r_n(0) \cdots r_n(k-1) \det X_k \quad \text{for } 1 \leq k \leq N+1 , \quad d_0 := \det X_0 , \tag{13}$$

and, for $0 \leq k \leq N+1$ with $d_k \neq 0$ (then X_k is invertible), we introduce further $n \times n$ -matrices by

$$\begin{aligned} Q_k &= (q_{\mu\nu}(k))_{\mu,\nu=0}^{n-1} := U_k X_k^{-1} , \\ W_k &= (w_{\mu\nu}(k))_{\mu,\nu=0}^{n-1} := d_k Q_k , \\ P_k &= (p_{\mu\nu}(k))_{\mu,\nu=0}^{n-1} := \frac{1}{d_k} W_k \{ w_{n-1,n-1}(k) I - B W_k \} . \end{aligned} \tag{14}$$

Now, we can formulate and prove the main result of this section.

Theorem 1 *(Recursion via Riccati equation). Let* $k \in \{0, \ldots, N\}$ *and* $\lambda \in \mathbb{R}$ *be fixed, and suppose that* $X_k, U_k, X_{k+1}, U_{k+1}$ *are real* $n \times n$ *-matrices, such that the recursion formula (9) holds and such that* $d_k \neq 0$ *and* $d_{k+1} \neq 0$ *. Then,*

$$d_{k+1} = w_{n-1,n-1}(k) + r_n(k) d_k \,, \tag{15}$$

$$W_{k+1} = d_{k+1}(C_k - \lambda \widetilde{C}) + (I - A^T)\{r_n(k) W_k + P_k\}(I - A) \,, \tag{16}$$

where $d_k, d_{k+1}, W_k, W_{k+1}, P_k$ *are defined by (13) and (14).*

Proof. By our assumptions all occuring quantities are well-defined, and $r_n(k) \neq 0$ by (13) because $d_{k+1} \neq 0$. Since $\widetilde{A}^{-1} = (I - A)$ and $\det \widetilde{A} = 1$, we obtain from (9) and (14) that

$$0 \neq \det X_{k+1} = \det(X_k + B_k U_k) = \det(I + B_k Q_k) \det X_k \,. \tag{17}$$

Hence, $I + B_k Q_k$ is invertible, and by (9),

$$X_{k+1}^{-1} = X_k^{-1}(I + B_k Q_k)^{-1}(I - A). \tag{18}$$

Moreover, by the definition of B_k by (6), we have that

$$I + B_k Q_k = \begin{pmatrix} 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ \star & \cdots & \star & 1 + \frac{1}{r_n(k)}\, q_{n-1,n-1}(k) \end{pmatrix}.$$

It follows from (13) and (17) that

$$d_{k+1} = r_n(k) d_k \det(I + B_k Q_k) = r_n(k) d_k + d_k\, q_{n-1,n-1}(k),$$

and this yields assertion (15) using (14).
Next, we obtain from (9) and (14) that

$$Q_{k+1} = \left\{ (C_k - \lambda \widetilde{C}) X_{k+1} + (I - A^T) U_k \right\} X_{k+1}^{-1} \,.$$

Hence, by (18), we get the well-known Riccati matrix difference equation (cf. e.g. [1, Remark 7] or [3]):

$$Q_{k+1} = (C_k - \lambda \widetilde{C}) + (I - A^T) Q_k (I + B_k Q_k)^{-1}(I - A). \tag{19}$$

The definitions of B, B_k, and W_k imply that

$$B_k Q_k B = \frac{q_{n-1,n-1}(k)}{r_n(k)}\, B = \frac{w_{n-1,n-1}(k)}{d_k}\, B_k \,.$$

It follows from assertion (15) that

$$(I+B_kQ_k)\left(I-\frac{d_k}{d_{k+1}}BQ_k\right)=I+\left(1-\frac{r_n(k)d_k}{d_{k+1}}-\frac{w_{n-1,n-1}(k)}{d_{k+1}}\right)B_kQ_k=I\,,$$

and therefore

$$(I+B_kQ_k)^{-1}=I-\frac{d_k}{d_{k+1}}\,BQ_k\,. \tag{20}$$

It follows from (15), (19), and (20) that

$$W_{k+1}=$$

$$d_{k+1}(C_k-\lambda\widetilde{C})+(I-A^T)\left\{r_n(k)W_k+\frac{1}{d_k}\,W_k(w_{n-1,n-1}(k)I-BW_k)\right\}(I-A),$$

which is assertion (16) by the definition of P_k via (14). $\square$

We need the following auxiliary lemma.

Lemma 1 *Assume (7) and (9) (for $0\le k\le N$), and suppose that X_0,U_0 are real $n\times n$ -matrices, which do not depend on λ (as e.g. in (10)). Then,*

(i) $X_k=X_k(\lambda),U_k=U_k(\lambda)$ *are $n\times n$ -matrix polynomials in λ , and $d_k=d_k(\lambda)$ is a polynomial in λ for all $0\le k\le N+1$.*

(ii) $W_k=W_k(\lambda)$ *is an $n\times n$ -matrix polynomial in λ for all $0\le k\le N+1$, for which it is defined, i.e. if $d_k(\lambda)\not\equiv 0$.*

(iii) $P_k=P_k(\lambda)$ *is an $n\times n$ -matrix polynomial in λ for all $0\le k\le N$, if $d_k(\lambda)\not\equiv 0$ and $d_{k+1}(\lambda)\not\equiv 0$.*

(iv) $p_{\mu\nu}(k)=\dfrac{1}{d_k}\det\begin{pmatrix} w_{\mu\nu} & w_{\mu,n-1}\\ w_{n-1,\nu} & w_{n-1,n-1}\end{pmatrix}(k)$ *, in particular $p_{\mu,n-1}(k)=p_{n-1,\nu}(k)=0$ for $0\le\mu,\nu\le n-1$, and for all $0\le k\le N+1$, if $d_k\ne 0$.*

(v) the matrices $Q_k\,,W_k\,,P_k$ are symmetric for $0\le k\le N+1$, if $d_k\ne 0$, and if $Z=(X_k,U_k)_{k=0}^{N+1}$ is a conjoined basis of (9), i.e. if (11) holds additionally.

Proof. Assertion (i) follows immediately from the recursion (9) and from (13), because X_0 and U_0 are independent of λ . The assertion (ii) follows from (i) and from the fact that the matrix elements of $d_kX_k^{-1}$ are, up to constant factors, subdeterminants (cofactors) of $X_k(\lambda)$. The assertion (iii) follows from (i), (ii), and from formula (16) of Theorem 1, which gives

$$P_k = \widetilde{A}^T \left\{ W_{k+1} - d_{k+1}(C_k - \lambda \widetilde{C}) \right\} \widetilde{A} - r_n(k) W_k \,, \tag{21}$$

where $\widetilde{A} = (I - A)^{-1}$ from (6). The assertion (iv) is a direct consequence of the definition of P_k by (14) and of B by (6). Finally, the symmetry of the matrices $Q_k\,, W_k\,, P_k$ follows from (11) and from their definition by (14). $\square$

Similarly as Theorem A in the previous section, our Theorem 1 leads to an algorithm to compute the characteristic polynomial of the given matrix $\mathcal{A}$ recursively. We formulate the procedure in the following theorem.

Theorem 2 *(Algorithm via Riccati equation). Assume (2), (3), (4), and (7), and suppose that* $Z = (X_k, U_k)_{k=0}^{N+1}$ *is the principal solution of (9). Let* $\lambda \in \mathbb{R}$ *be such that*

$$\det(\mathcal{A}_k - \lambda I) \neq 0 \quad \textit{for all } n+1 \leq k \leq N \,. \tag{22}$$

Then,

$$\det(\mathcal{A}_k - \lambda I) = d_k \quad \textit{for } n+1 \leq k \leq N+1 \,, \tag{23}$$

in particular

$$\det(\mathcal{A} - \lambda I) = d_{N+1} \,,$$

where the $d_k = d_k(\lambda)$ *are obtained by the recursion of Theorem 1, i.e. (15) and (16):*

$$\begin{aligned} d_{k+1} &= w_{n-1,n-1}(k) + r_n(k) d_k \,, \\ W_{k+1} &= d_{k+1}(C_k - \lambda\, \widetilde{C}) + (I - A^T)\{r_n(k) W_k + P_k\}(I - A) \,, \end{aligned}$$

where $P_k := \dfrac{1}{d_k} W_k \{w_{n-1,n-1}(k) I - B W_k\}$ *for* $n \leq k \leq N$ *as in (14) with the initial conditions (for* $k = n$ *):*

$$d_n = 1 \,, \quad W_n = U_n X_n^{-1} \,, \tag{24}$$

where $X_n = X_n(\lambda), U_n = U_n(\lambda)$ *are obtained by the recursion (9) for* $k = 0, \ldots, n-1$ *starting with the initial values (10).*

Proof. First, the initial values (24) follow from (12) and from the notation (13) and (14). Then our assertion (23) is obtained from Theorem A, formulae (8), (13), and from Theorem 1, formulae (15), (16). □

Remark 2

(i) Note that the *assumption (7)* is not needed explicitly for $k \geq n$. It is required in the proof above when applying Theorem A. Moreover, it is needed for $0 \leq k < n$ to calculate the initial value for W_n via recursion (9), which needs the matrices B_k .

(ii) If (7) holds, then $d_k(\lambda) \neq 0$ for all $n + 1 \leq k \leq N + 1$ up to finitely many λ , because it is the characteristic polynomial of $\mathcal{A}_k$ by assertion (23) of Theorem 2 or by (8) of Theorem A and notation (13).

(iii) The initial values (10) for X_0 and U_0 imply that $d_k \equiv 0$ for $0 \leq k \leq n - 1$ by (12) and (13). Hence, by (14), W_k is not defined for these k , and the recursion of Theorem 1 or Theorem 2 cannot start earlier than for $k = n$. On the other hand, we shall see that the recursion does not require any divisions, and the data, i.e. d_k and W_k , are polynomials in λ (under (7)) by Lemma 1. In the next section we will derive the appropriate initial values for $k = 0$, so that the algorithm of Theorem 2 can start with $k = 0$, and the Hamiltonian system (9) is not needed anymore.

(iv) Altogether, the situation is still unsatisfactory essentially because of two reasons already mentioned above, namely: the assumption (7) is required “formally” for the proofs of our results. Moreover, the algorithm of Theorem 2 requires divisions by d_k to compute P_k . But the P_k ’s are matrix polynomials in λ (see Lemma 1), which means that they are divisible by the polynomials $d_k(\lambda)$. This division will be carried out recursively for *pentadiagonal* $(n = 2)$ and *heptadiagonal* $(n = 3)$ matrices in Section 5. The resulting new recursion formulae can be considered as the main results of this paper. The cases $n \geq 4$ require a new method, which is the main content of the dissertation by Tentler [16].

4 Initial values

First, we motivate the initial values of W_k , P_k etc. for $k = 0$ by the following continuity argument, where we perturb the initial values (10) for X_0

and U_0 and the numbers $r_n(k)$ in such a way that the crucial assumptions (7) and the nonvanishing of the d_k 's of our previous results hold. Therefore, let $\mathcal{A} = (a_{\mu\nu})$ be given such that (2) holds as in Section 2, and suppose that the $r_\mu(k)$ are defined by (4). We fix any real ε such that

$$r_n(k, \varepsilon) := r_n(k) + \varepsilon \neq 0 \text{ for all } 0 \leq k \leq N\,. \tag{25}$$

Then we consider the $n \times n$ -matrix-valued solution

$$Z(\varepsilon) := (X_k(\varepsilon, \lambda), U_k(\varepsilon, \lambda))_{k=0}^{N+1}$$

of the Hamiltonian system (9) with $r_n(k, \varepsilon)$ instead of $r_n(k)$ and with the initial condition

$$X_0(\varepsilon, \lambda) = \varepsilon I\,, \quad U_0(\varepsilon, \lambda) = I \text{ for } \lambda \in \mathbb{R}\,. \tag{26}$$

It follows from (9), (25), (26) that the solution matrices $X_k(\varepsilon, \lambda)$ and $U_k(\varepsilon, \lambda)$ are rational functions in ε and polynomials in λ . Moreover, it is easily seen that the corresponding quadratic functional is positive definite for sufficiently small $\varepsilon > 0$ and sufficiently large $-\lambda > 0$. Therefore, by the Reid Roundabout Theorem [4, Theorem 3.2] or [5], the matrices $X_k(\varepsilon, \lambda)$ are invertible for all $0 \leq k \leq N+1$ and $\lambda \in \mathbb{R} \setminus \mathcal{N}(\varepsilon)$, where $\mathcal{N}(\varepsilon)$ is a finite set for $0 < \varepsilon < \varepsilon_0$. Hence, by (13) and (26),

$$\begin{aligned} &d_k(\varepsilon, \lambda) := r_n(0, \varepsilon) \cdots r_n(k-1, \varepsilon) \det X_k(\varepsilon, \lambda) \neq 0 \\ &\text{for all } 1 \leq k \leq N+1\,,\ \lambda \in \mathbb{R} \setminus \mathcal{N}(\varepsilon), \\ &d_0(\varepsilon, \lambda) := \det X_0(\varepsilon\,,\ \lambda) = \varepsilon^n \neq 0\,. \end{aligned} \tag{27}$$

Therefore, $W_k(\varepsilon, \lambda) = (w_{\mu\nu}(k, \varepsilon, \lambda))_{\mu,\nu=0}^{n-1}$ and $P_k(\varepsilon, \lambda) = (p_{\mu\nu}(k, \varepsilon, \lambda))_{\mu,\nu=0}^{n-1}$ are well defined by (14), and they are also rational in ε and polynomials in λ (see Lemma 1). Letting $\varepsilon \to 0$ we obtain the desired formulae, if the limits exist. Of course, these limits do exist, if all quantities are polynomials in ε , which will be the case in the forthcoming sections, when the divisions are carried out.

Here, we compute the initial values of $W_k\,, P_k$ etc. for $k = 0$, where we let ε tend to zero. We consider the following limits, which do naturally not depend on λ :

$$\begin{aligned} &d_0 := \lim_{\varepsilon \to 0} d_0(\varepsilon, \lambda), \quad W_0 = (w_{\mu\nu}(0)) := \lim_{\varepsilon \to 0} W_0(\varepsilon, \lambda), \\ &P_0 = (p_{\mu\nu}(0)) := \lim_{\varepsilon \to 0} P_0(\varepsilon, \lambda), \quad \text{and } p_0 := \lim_{\varepsilon \to 0} p_0(\varepsilon, \lambda)\,, \end{aligned}$$

where

$$p_k(\varepsilon,\lambda) := \frac{\det W_k(\varepsilon,\lambda)}{d_k(\varepsilon,\lambda)^{n-1}}\,, \tag{28}$$

and, more generally,

$$v_{i_1\cdots i_r j_1\cdots j_r}(0) := \lim_{\varepsilon\to 0} v_{i_1\cdots i_r j_1\cdots j_r}(0,\varepsilon,\lambda),$$

where

$$v_{i_1\cdots i_r j_1\cdots j_r}(k,\varepsilon,\lambda) := \frac{\det(w_{i_\mu j_\nu}(k,\varepsilon,\lambda))_{\mu,\nu=1}^{r}}{d_0(k,\varepsilon,\lambda)^{r-1}} \tag{29}$$

for all $1 \le r \le n, \quad 0 \le i_1 < \cdots < i_r \le n-1, \quad 0 \le j_1 < \cdots < j_r \le n-1.$

Thus,

$$v_{\mu\nu}(k,\varepsilon,\lambda) = w_{\mu\nu}(k,\varepsilon,\lambda)$$

(where $r = 1$),

$$v_{\mu\, n-1\, \nu\, n-1}(k,\varepsilon,\lambda) = p_{\mu\nu}(k,\varepsilon,\lambda)$$

by Lemma 1 (iv) (where $r = 2 \le n$), and

$$v_{0\cdots n-1\, 0\cdots n-1}(k,\varepsilon,\lambda) = p_k(\varepsilon,\lambda)$$

by (28) (where $r = n$).

Since, $d_0(\varepsilon,\lambda) = \varepsilon^n$ by (27) and $W_0(\varepsilon,\lambda) = \varepsilon I$ by (14) and (26), we obtain immediately

Theorem 3 *(Initial values). With the notation above we have that*

$$d_0 = 0,\ v_{i_1\cdots i_r j_1\cdots j_r}(0) = 0$$

for all $1 \le r \le n-1, \quad 0 \le i_1 < \cdots < i_r \le n-1, \quad 0 \le j_1 < \cdots < j_r \le n-1,$ *and*

$$p_0 = v_{0\cdots n-1\, 0\cdots n-1}(0) = 1\,.$$

Remark 3 For tri-, penta-, and heptadiagonal matrices we get from Theorem 3 the following initial values:

$n = 1:\ d_0 = 0,\ W_0 = p_0 = 1 \quad (P_k = 0$ for all $k)$.

$n = 2:\ d_0 = 0,\ W_0 = 0_{2\times 2},\ P_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ p_0 = 1\,.$

$n = 3:\ d_0 = 0,\ W_0 = P_0 = 0_{3\times 3},\ p_0 = 1\,.$

5 Divisionfree recursions for penta- and heptadiagonal matrices

We shall use the following the following elementary facts from matrix theory (cf. e.g. [9]).

Lemma 2 *Let* $X \in \mathbb{R}^{n\times n}$ *be a matrix with eigenvalues* $\lambda_0, \ldots, \lambda_{n-1} \in \mathbb{C}$ *and characteristic polynomial*

$$\det(X - \lambda I) = \sum_{\nu=o}^{n} \alpha_\nu (-\lambda)^{n-\nu} = (\lambda_0 - \lambda)\cdots(\lambda_{n-1} - \lambda).$$

Then,

(i) $\alpha_n = \det X = \lambda_0 \cdots \lambda_{n-1},\ \alpha_0 = 1\,.$

(ii) $\alpha_1 = \operatorname{trace} X = \sum_{\nu=0}^{n-1} \lambda_\nu\,.$

(iii) $\det X \cdot \det(X^{-1} - \lambda I) = \sum_{\nu=0}^{n-1} \alpha_{n-\nu}(-\lambda)^{n-\nu},$ *if* $\det X \neq 0\,.$

(iv) $\sum_{\nu=0}^{n} (-1)^{n-\nu} \alpha_\nu X^{n-\nu} = 0$ *(Caley-Hamilton theorem).*

(v) $\det(I + X) = \sum_{\nu=0}^{n} \alpha_\nu\,.$

We use the same notation as before, which was already introduced in Section 2, formula (6), namely:

A denotes the $n \times n$ -companion matrix, $\widetilde{A} = (I - A)^{-1}$, and

$B = \operatorname{diag}(0, \ldots, 0, 1)\,.$

Throughout this section there are *given*:

$$W = (w_{\mu\nu})_{\mu,\nu=0}^{n-1} \in \mathbb{R}^{n\times n}\ ,\ \text{reals } r_0, \ldots, r_{n-1}\,, r\,, d \text{ with } d \neq 0\,. \tag{30}$$

As in the Sections 3 and 4, formulae (14) and (28) we put

$$\begin{aligned} P &:= \frac{1}{d}\, W\{w_{n-1,n-1} I - BW\}\,,\ p := \frac{\det W}{d^{\,n-1}}\ ,\ \text{and} \\ \widetilde{P} &:= \frac{1}{d}\, \widetilde{W}\{\widetilde{w}_{n-1,n-1} I - B\widetilde{W}\}\,,\ \widetilde{p} := \frac{\det \widetilde{W}}{\widetilde{d}^{\,n-1}}\ ,\ \text{if } \widetilde{d} \neq 0\,, \end{aligned} \tag{31}$$

where $\widetilde{W} = (\widetilde{w}_{\mu\nu})_{\mu,\nu=0}^{n-1}$ and $\widetilde{d}$ are given by the recursion formulae (15) and (16) of Theorem 1, namely:

$$\widetilde{d} = w_{n-1,n-1} + rd\,,$$
$$\widetilde{W} = \widetilde{d}C + (I - A^T)\{rW + P\}(I - A), \tag{32}$$
$$\text{where } C := \operatorname{diag}(r_0, \ldots, r_{n-1}).$$

Our goal in this section is to derive from (32) *divisionfree* recursions for $\widetilde{P}$ and $\widetilde{p}$ (i.e. formulae without the above divisions by $\widetilde{d}$) for the cases $n = 1$ (trivial/tridiagonal), $n = 2$ (pentadiagonal), and $n = 3$ (heptadiagonal). Therefore, we need the following lemma.

Lemma 3 *Assume (30)-(32). Then,*

(i) $rW + P = \dfrac{1}{d} W(\widetilde{d} I - BW)\,.$

(ii) $\det\{rW + P\} = rp\,\widetilde{d}^{\,n-1}\,.$

(iii) $BWB = (\widetilde{d} - rd)B = w_{n-1,n-1}B\,.$

(iv) $\{rW + P\}^{-1} = \dfrac{d}{\widetilde{d}}\, W^{-1} + \dfrac{1}{r\widetilde{d}}\, B\,,$ *if* $rp\widetilde{d} \neq 0\,.$

Proof. The assertions (i)-(iii) follow immediately from our notation. Assertion (iv) follows from (i) and (iii) via the following equations:

$$\{rW + P\}\left(\frac{d}{\widetilde{d}}\, W^{-1} + \frac{1}{r\widetilde{d}}\, B\right) W = \frac{1}{d} W(\widetilde{d} I - BW)\left(\frac{d}{\widetilde{d}}\, I + \frac{1}{r\widetilde{d}}\, BW\right) =$$
$$W\left(I - \frac{1}{\widetilde{d}}\, BW + \frac{1}{rd} BW - \frac{1}{rd\,\widetilde{d}}\,(\widetilde{d} - rd)\, BW\right) = W,$$

which yields (iv). $\square$

First, we treat the pentadiaginal case $n = 2$. In this case we have that $P = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$ by Lemma 1 (iv) or just by (31).

Theorem 4 *(Pentadiagonal recursion). Assume (30)-(32), and $n = 2$. Then,*

$$\widetilde{p} = rp + r_0\widetilde{w}_{11} + r_1\widetilde{w}_{00} - r_0r_1\widetilde{d}\,,\ \widetilde{P} = \begin{pmatrix} \widetilde{p} & 0 \\ 0 & 0 \end{pmatrix}\,. \tag{33}$$

Proof. We put $X := \dfrac{1}{\widetilde{d}}\,(I - A^T)\{rW + P\}(I - A)C^{-1}$, provided that $\tilde{d} \cdot \det C \neq 0$. Then, by (32),

$$\widetilde{W} = \widetilde{d}\,(I + X)\,C\,.$$

These formulae, (31), Lemma 2 (i), (ii), (v), Lemma 3 (ii), and $\det(I - A) = 1$ imply that

$$\widetilde{p} = \frac{\det \widetilde{W}}{\widetilde{d}} = \widetilde{d}\,\det C \det(I + X) = \widetilde{d} \det C(1 + \operatorname{trace} X + \det X) =$$

$$\widetilde{d}\,\det C + \det C \cdot \operatorname{trace}\,[(\widetilde{W} - \widetilde{d}C)\,C^{-1}] + \frac{1}{\widetilde{d}}\,\det\{rW + P\} =$$

$$\widetilde{d} \det C + \det C \cdot \operatorname{trace}\,(\widetilde{W} C^{-1}) - 2\widetilde{d} \det C + rp\,.$$

This yields the first part of assertion (33) using that $C = \operatorname{diag}(r_0, r_1)$ by (32). The second part of (33) follows e.g. from Lemma 1 (iv) as mentioned above for P instead of $\widetilde{P}$. $\square$

Next, we treat the heptadiagonal case $n = 3$. Therefore, we introduce the following additional notation:

$$Z = (z_{\mu\nu})_{\mu,\nu=0}^{n-1} := \frac{\det W}{d}\,W^{-1}\,,\ \widetilde{Z} := \frac{\det \widetilde{W}}{\widetilde{d}}\,\widetilde{W}^{-1}\,. \tag{34}$$

In this case we have that

$$P = \begin{pmatrix} z_{11} & -z_{01} & 0 \\ -z_{10} & z_{00} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{35}$$

by Lemma 1 (iv) and by the formula for the inverse of a matrix via Cramer's rule (see e.g. [15, Ch. 4]). We need another lemma.

Lemma 4 *Let* $X \in \mathbb{R}^{3\times 3}$ *with* $\det X \cdot \det(I + X) \neq 0$. *Then,*

$$\det(I + X) \cdot (I + X)^{-1} = \det X \cdot X^{-1} + (1 + \operatorname{trace} X)I - X\,. \tag{36}$$

Proof. It follows from Lemma 2 (i)-(v) (Caley-Hamilton) that

$$(I + X)[\det X \cdot X^{-1} + (1 + \operatorname{trace} X)I - X]X =$$

$$-X^3 + (1 + \operatorname{trace} X - 1)X^2 + (1 + \operatorname{trace} X + \det X)X + \det X \cdot I =$$

$$(1 + \operatorname{trace} X + \det X + \det X \cdot \operatorname{trace} X^{-1})X = \det(I + X) \cdot X.$$

This yields the assertion (36). $\square$

Theorem 5 *(Heptadiagonal recursion). Assume (30)-(32), (34), and* $n = 3$. *Then,*

$$\widetilde{Z} = \widetilde{A}\{rZ + pB\}\widetilde{A}^T + \det C \cdot C^{-1}[\operatorname{trace}(\widetilde{W}C^{-1})I - \widetilde{W}C^{-1} - \widetilde{d}\, I]\,,$$

$$\widetilde{P} = \begin{pmatrix} \widetilde{z}_{11} & -\widetilde{z}_{01} & 0 \\ -\widetilde{z}_{10} & \widetilde{z}_{00} & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \tag{37}$$

$$\widetilde{p} = rp + \operatorname{trace}(C\widetilde{Z}) - \det C \cdot \operatorname{trace}(\widetilde{W}C^{-1}) + \det C \cdot \widetilde{d}\,, \tag{38}$$

provided that all occuring quantities exist.

Proof. We put, similarly as in the proof of Theorem 4,

$$Y := (I - A^T)\{rW + P\}(I - A)\,,\ X := \frac{1}{\widetilde{d}}\, YC^{-1}\,,$$

so that

$$\widetilde{W} = \widetilde{d}C + Y = \widetilde{d}\,(I + X)\,C\,.$$

First, we obtain from Lemma 4 (formula (36)), and our notation that

$$\widetilde{Z} = \frac{\det \widetilde{W}}{\widetilde{d}}\,\widetilde{W}^{-1} = \frac{\widetilde{d}^3 \det(I + X)\det C}{\widetilde{d}}\,\frac{C^{-1}}{\widetilde{d}}\,(I + X)^{-1}\,.$$

Hence,

$$\widetilde{Z} = \widetilde{d}\,\det C \cdot C^{-1}[\det X \cdot X^{-1} + (1 + \operatorname{trace} X)I - X]\,. \tag{39}$$

By the notation above (observe that $n = 3$), we have that

$$X = \frac{1}{\widetilde{d}}\,\widetilde{W}C^{-1} - I\,,\ \operatorname{trace} X = \frac{1}{\widetilde{d}}\,\operatorname{trace}(\widetilde{W}C^{-1}) - 3\,, \tag{40}$$

and Lemma 3 (ii) yields

$$\det X = \frac{rp}{\widetilde{d}\,\det C}\,. \tag{41}$$

Now, applying Lemma 3 (iv), (31), (34), and (41) we get that

$$\det X \cdot X^{-1} = \frac{rp}{\widetilde{d}\,\det C}\,\widetilde{d}\,C\widetilde{A}\,\{rW + P\}^{-1}\widetilde{A}^{T}\,.$$

Thus,

$$\det X \cdot X^{-1} = \frac{1}{\widetilde{d}\,\det C}\,C\widetilde{A}\{rZ + pB\}\widetilde{A}^{T}\,. \tag{42}$$

Putting the formulae (40) and (42) into (39) we obtain the first part of the assertion (37). The second part of (37) follows from (35) with $\widetilde{P}$ instead of P.

Using the same notation as above, (31), Lemma 2 (ii), (iii), (v), $n = 3$, and the formulae (40), (42), (41), and (37) we obtain that:

$$\widetilde{p} = \frac{\det \widetilde{W}}{\widetilde{d}^{2}} = \widetilde{d}\,\det C \cdot \det(I + X) =$$

$$\widetilde{d}\,\det C\,[1 + \operatorname{trace} X + \det X \cdot \operatorname{trace} X^{-1} + \det X] =$$

$$\widetilde{d}\det C\left[1+\frac{1}{\widetilde{d}}\operatorname{trace}(\widetilde{W}C^{-1})-3+\frac{1}{\widetilde{d}\det C}\operatorname{trace}(C\widetilde{A}\{rZ+pB\}\widetilde{A}^{T})+\frac{rp}{\widetilde{d}\det C}\right]$$

$$= -2\widetilde{d}\,\det C + \det C \cdot \operatorname{trace}(\widetilde{W}C^{-1}) + rp + \operatorname{trace}(C\widetilde{Z})$$

$$-\operatorname{trace}[C\det C \cdot C^{-1}\{\operatorname{trace}(\widetilde{W}C^{-1})I - \widetilde{W}C^{-1} - \widetilde{d}\,I\}]$$

$$= rp + \operatorname{trace}(C\widetilde{Z}) - \det C \cdot \operatorname{trace}(\widetilde{W}C^{-1}) + \det C \cdot \widetilde{d}\,.$$

This proves assertion (38). $\square$

In contrast to our notation (31) and (34) the new recursions (37) and (38) of Theorem 5 do not require any divisions or inverse matrices. This follows from our next lemma, which is a direct consequence of the fact that $C = \operatorname{diag}(r_0, r_1, r_2)$ by (32).

Lemma 5 *Under the assumptions and with the notation of Theorem 5, we have that* *(i)* $\det C = r_0 r_1 r_2$.

(ii) $\det C \cdot C^{-1} = \mathrm{diag}\,(r_1 r_2, r_0 r_2, r_0 r_1)$.

(iii) $\det C \cdot \mathrm{trace}\,(\widetilde{W} C^{-1}) = r_1 r_2 \widetilde{w}_{00} + r_0 r_2 \widetilde{w}_{11} + r_0 r_1 \widetilde{w}_{22}$.

(iv) $\det C \cdot C^{-1} \{\mathrm{trace}\,(\widetilde{W} C^{-1}) I - \widetilde{W} C^{-1}\} =$

$$\mathrm{diag}\,(r_2\widetilde{w}_{11}+r_1\widetilde{w}_{22}, r_2\widetilde{w}_{00}+r_0\widetilde{w}_{22}, r_1\widetilde{w}_{00}+r_0\widetilde{w}_{11}) + \begin{pmatrix} 0 & -r_2\widetilde{w}_{01} & -r_1\widetilde{w}_{02} \\ -r_2\widetilde{w}_{10} & 0 & -r_0\widetilde{w}_{12} \\ -r_1\widetilde{w}_{20} & -r_0\widetilde{w}_{21} & 0 \end{pmatrix}.$$

6 Divisionfree algorithms for penta- and heptadiagonal matrices

In this section we combine the results of the Sections 3-5. This yields a recursive algorithm for the computation of the characteristic polynomials of symmetric banded matrices with bandwidths 3, 5, and 7. As in Section 2 let be given a symmetric banded matrix

$$\mathcal{A} = (a_{\mu\nu})$$

with bandwidth $2n+1$, i.e. (2) holds, and let the associated $r_\mu(k)$ for $\mu = 0, \ldots, n$, be defined by (4). Moreover, the matrices

$$A, B, C_k, \widetilde{C}$$

are given by (6). We consider the submatrices (see (3) in Section 2)

$$\mathcal{A}_m = (a_{\mu\nu})_{\mu,\nu=1}^{m-n} \ \text{ for } \ m \geq n+1\,.$$

Then, by the Poincaré separation theorem (cf. [10, Th. 8.5-1] or [14, Sätze 4.9 and 4.11] or [18, Ch. 2 Section 47]) their characteristic polynomials

$$\det(\mathcal{A}_k - \lambda I) \ \text{ for } \ k \geq n+1$$

constitute a Sturmian chain [without *strict* separation] as already mentioned in the introduction. Our recursive algorithms require in addition sequences $\{W_k(\lambda)\}$, $\{Z_k(\lambda)\}$, and $\{P_k(\lambda)\}$ of symmetric $n \times n$ -matrix polynomials (cf. formulae (14), (31), and (34) above), and $\{p_k(\lambda)\}$ of scalar polynomials (cf. formulae (28), and (31)). These formulae contain divisions (by d_k), which are carried out according to the previous section. The corresponding recursion formulae were derived under assumption (7) (i.e. $r_n(k) \neq 0$ or $a_{k,k+n} \neq 0$) and $d_k \neq 0$. By the continuity argument in Section 4 these formulae remain valid without these assumptions. Our divisionfree recursion schemes, as derived in the foregoing sections, read as follows:

First, we note the well-known tridiagonal case $n = 1$ in the following remark.

Remark 4 *(Tridiagonal algorithm).* If $n = 1$, then, by (6), $B = I = 1$, $A = 0$, and therefore, by (14),

$$W_k = w_{00}(k),\ P_k = 0.$$

Hence, we obtain from Theorem A, formula (8), Theorem 1, formulae (15), (16) (with notation (13)), and from Theorem 3, Remark 3, the following algorithm (with $\lambda \in \mathbb{R}$):

$$d_{k+1} = w_{00}(k) + r_1(k)d_k\,,\ w_{00}(k+1) = \{r_0(k) - \lambda\}d_{k+1} + r_1(k)w_{00}(k) \quad (43)$$

for $k = 0, 1, 2, \ldots$ with the initial conditions

$$d_0 = 0\,,\ w_{00}(0) = 1\,, \quad (44)$$

such that

$$d_m = \det(\mathcal{A}_m - \lambda I) \text{ for } m \geq 2\,.$$

Actually, in this case the $w_{00}(k)$ can be eliminated. This leads to the well-known *two-term recursion* for the characteristic polynomials of tridiagonal matrices [18, Ch. 5 formula (37.2)]:

$$d_{k+1} = \{r_1(k) + r_1(k-1) + r_0(k-1) - \lambda\}d_k - r_1(k-1)^2 d_{k-1} \quad (45)$$

for $k = 1, 2, \ldots$ with the initial conditions

$$d_0 = 0\,,\ d_1 = 1\,. \quad (46)$$

Next, we consider the pentadiagonal case $n = 2$. Similarly as in the previous remark, we apply Theorem A, formulae (8) and (12), notation (13), Theorem 1, formulae (15), (16), Theorem 3, Remark 3, and additionally Theorem 4, formula (33) with [$W = W_k$, $r_0 = r_0(k) - \lambda$, $r_1 = r_1(k)$, $d = d_k$, $P = P_k$, $p = p_k$, and $\widetilde{W} = W_{k+1}$, $\widetilde{d} = d_{k+1}$, $\widetilde{P} = P_{k+1}$, $\widetilde{p} = p_{k+1}$]. Thereby we obtain our next theorem.

Theorem 6 *(Pentadiagonal algorithm). Assume (2), (3), (4), (6),* $n = 2$*, and let* $\lambda \in \mathbb{R}$*. Define sequences* $\{d_k\}$*,* $\{p_k\}$ *of reals, and* $\{W_k = (w_{\mu\nu}(k))\}$*,* $\{P_k\}$ *of symmetric* 2×2 *-matrices by the recursive algorithm:*

$$d_{k+1} = w_{11}(k) + r_2(k)d_k\,,$$

$$W_{k+1} = d_{k+1}(C_k - \lambda\,\widetilde{C}) + (I - A^T)\{r_2(k)W_k + P_k\}(I - A)\,,$$

$$\text{with } P_k = \begin{pmatrix} p_k & 0 \\ 0 & 0 \end{pmatrix}\,, \tag{47}$$

$$p_{k+1} = r_2(k)p_k + \{r_0(k) - \lambda\}w_{11}(k+1)$$
$$+r_1(k)w_{00}(k+1) - \{r_0(k) - \lambda\}r_1(k)d_{k+1}$$

for $k = 0, 1, 2, \ldots$ *with the initial conditions*

$$d_0 = 0,\, W_0 = 0_{2\times 2},\, p_0 = 1\,. \tag{48}$$

Then, $d_1 = 0,\, d_2 = 1,$ *and*

$$d_m = \det(\mathcal{A}_m - \lambda I) \ \textit{for}\ m \geq 3\,.$$

Remark 5 Observe that the recursion (47) consists of a system of $5 = 1 + 3 + 1$ scalar recursions (use that W_k is symmetric by (47) and (48), see also Lemma 1 (v)). By eliminating W_k and p_k one might expect that (47) leads to a *5-term recursion* for the "relevant" sequence $\{d_k\}$ of the characteristic polynomials similarly as in Remark 4 in the scalar case $n = 1$. But in the present case $n = 2$ an elimination (particularly without divisions!) is not possible in general. Hence, there is no 5-term recursion for the d_k 's only. This impossibility can be shown by explicit examples, which we do not present here. Such an example is contained in the dissertation by Tentler [16].

Finally, we treat the heptadiagonal case $n = 3$. We apply the same results and formulae as before, but with Theorem 5 (combined with Lemma 5) instead of Theorem 4 (where $r_2 = r_2(k)$,$r = r_3(k)$,$Z = Z_k$, $\widetilde{Z} = Z_{k+1}$ in addition). This leads to our last theorem.

Theorem 7 *(Heptadiagonal algorithm). Assume (2), (3), (4), (6),* $n = 3$*, and let* $\lambda \in \mathbb{R}$*. Define sequences* $\{d_k\}$*,* $\{p_k\}$ *of reals, and* $\{W_k = (w_{\mu\nu}(k))\}$*,* $\{Z_k = (z_{\mu\nu}(k))\}$*,* $\{P_k\}$ *of symmetric* 3×3 *-matrices by the recursive algorithm:*

$$
\begin{aligned}
d_{k+1} &= w_{22}(k) + r_3(k) d_k\,,\\
W_{k+1} &= d_{k+1}(C_k - \lambda \widetilde{C}) + (I - A^T)\{r_3(k) W_k + P_k\}(I - A)\,,\\
&\quad \textit{with}\;\; P_k = \begin{pmatrix} z_{11}(k) & -z_{01}(k) & 0\\ -z_{01}(k) & z_{00}(k) & 0\\ 0 & 0 & 0 \end{pmatrix},\\
z_{00}(k+1) &= p_k + r_3(k)[z_{00}(k) + z_{11}(k) + z_{22}(k) + 2z_{01}(k) + 2z_{02}(k)\\
&\quad +2z_{12}(k)] + r_2(k) w_{11}(k+1)\\
&\quad +r_1(k) w_{22}(k+1) - r_1(k) r_2(k) d_{k+1}\,,\\
z_{01}(k+1) &= p_k + r_3(k)[z_{11}(k) + z_{22}(k) + z_{01}(k) + z_{02}(k) + 2z_{12}(k)]\\
&\quad -r_2(k) w_{01}(k+1)\,,\\
z_{11}(k+1) &= p_k + r_3(k)[z_{11}(k) + z_{22}(k) + 2z_{12}(k)]\\
&\quad +r_2(k) w_{00}(k+1) + \{r_0(k) - \lambda\} w_{22}(k+1)\\
&\quad -\{r_0(k) - \lambda\} r_2(k) d_{k+1}\,,\\
z_{02}(k+1) &= p_k + r_3(k)[z_{22}(k) + z_{02}(k) + z_{12}(k)] - r_1(k) w_{02}(k+1)\,,\\
z_{12}(k+1) &= p_k + r_3(k)[z_{22}(k) + z_{12}(k)] - \{r_0(k) - \lambda\} w_{12}(k+1)\,,\\
z_{22}(k+1) &= p_k + r_3(k) z_{22}(k) + r_1(k) w_{00}(k+1)\\
&\quad +\{r_0(k) - \lambda\} w_{11}(k+1) - \{r_0(k) - \lambda\} r_1(k) d_{k+1}\,,\\
p_{k+1} &= r_3(k) p_k + \{r_0(k) - \lambda\} z_{00}(k+1) + r_1(k) z_{11}(k+1)\\
&\quad +r_2(k) z_{22}(k+1) - r_1(k) r_2(k) w_{00}(k+1)\\
&\quad -\{r_0(k) - \lambda\} r_2(k) w_{11}(k+1)\\
&\quad -\{r_0(k) - \lambda\} r_1(k) w_{22}(k+1)\\
&\quad +\{r_0(k) - \lambda\} r_1(k) r_2(k) d_{k+1}
\end{aligned}
\tag{49}
$$

for $k = 0, 1, 2, \ldots$ *with the initial conditions*

$$
d_0 = 0,\; W_0 = P_0 = Z_0 = 0_{3\times 3},\; p_0 = 1\,. \tag{50}
$$

Then, $d_1 = d_2 = 0,\; d_3 = 1,$ *and*

$$
d_m = \det(\mathcal{A}_m - \lambda I)\;\; \textit{for}\;\; m \geq 4\,.
$$

Remark 6 The recursion (49) consists of a system of $14 = 1 + 6 + 6 + 1$ scalar recursions (use again that W_k is symmetric). Similarly as in Remark 5 the recursion scheme does not lead to a *14-term recursion* for the "relevant" sequence $\{d_k\}$ of the characteristic polynomials.

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