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A-POSTERIORI ERROR ANALYSIS OF THE REDUCED BASIS METHOD FOR NON-AFFINE PARAMETRIZED NONLINEAR PDE'S

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ABSTRACT. In this paper, we present the a-posteriori error analysis for the Reduced Basis Method (RBM) applied to nonlinear variational problems that depend on a parameter in a non-affine manner. To this end, we generalize the analysis by Veroy and Patera ([16]) to non-affine parametrized partial differential equations. We use the Empirical Interpolation Method (EIM) in order to approximate the non-affine parameter dependencies by a linear combination of affine functions. We also investigate a standard dual problem formulation in particular for the computation of a general output functional, also in combination with the EIM.

First, we study the well-posedness of all involved problems in terms of the Brezzi-Rappaz-Raviart theory. Then, we develop a-posteriori error estimates for all problems and investigate offline/online decompositions. The a-posteriori error analysis allows us to introduce an adaptive sampling procedure for the choice of the snapshots. Numerical experiments for a convection-diffusion problem around a rotating propeller show the effectivity of the scheme.

1. INTRODUCTION

The Reduced Basis Method (RBM) is by now a well-established method to treat large scale problems that depend on a set of parameters. The basic idea is an offline/online-decomposition of the computation. In the offline stage, costly computations are performed, the results are stored and evaluated. Based on these computations, a small set of global problem adapted functions, called *modes*, *snapshots* or *reduced basis functions* are formed. In the online stage (where high efficiency is desired), this set of basis functions is used in order to form an algebraic system of small size which is solvable in real-time. This approach has been successfully used in several applications. A complete list of references goes far beyond the scope of the present paper, so let us just mention [5, 6, 7, 8, 10, 12, 13] and in particular for non-linear problems [3, 9, 11, 17, 15].

If the parameter enters into the variational problem in an affine way, the RBM is particularly efficient since stiffness matrices and other involved quantities can be computed offline. In the online stage, one just has to compute a cheap parameter-dependent linear combination of pre-computed terms. However, often the problem at hand does not allow for an affine dependency. One possible way-out is the so-called *Empirical Interpolation Method (EIM)*, introduced in [1]. The idea is to approximate the involved non-affine functions by a linear combination of affine functions in order to guarantee a certain approximation.

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Our work has been motivated by an application from hydromechanics, namely the flow around a rotating ship propeller. In this case, a rigid body is rotating with a prescribed movement. In this application, the orientation of the propeller during the rotation is the parameter, in the sense that the angle of attack at every position is variable. A typical way to treat such problems is to transform the flow domain onto a reference domain. This implies, however, that the parameter-dependent transformation enters into the variational form due to the change of variables in the integrals. This in turns means that the non-affine parameter dependency occurs within the variational form (and not e.g. on the right-hand side).

This is the reason why we cannot just use previous work in [16] or [3] for affine and certain specific non-affine parameter dependencies. Instead, we generalize the analysis in [16] to the case of nonlinear non-affine parameter dependencies within the variational form.

This paper is organized as follows. In Section 2, we collect all necessary facts on nonlinear parametrized variational problems, the Empirical Interpolation Method (EIM), the Reduced Basis Method (RBM) and the dual problem formulation. Section 3 is devoted to the investigation of the well-posedness of all involved nonlinear problems in terms of the Brezzi-Rappaz-Raviart (RBB) theory. We show in particular that the nonlinear problems are well-posed under the condition that certain indicators are below a tolerance. In Section 4, we present our a-posteriori error analysis and describe offline/online-decompositions of all relevant problems. We also present computable bounds for the involved continuity and inf-sup constants and introduce an adaptive sampling procedure for the choice of the parameters for the snapshots. In Section 5 we describe a more specific application, namely a convection-diffusion problem around a rotating propeller. We present several numerical results in Section 6.

2. Preliminaries

2.1. **The Nonlinear Parametrized Variational Problem.** In [16], a parametrized nonlinear problem has been studied that is induced by the form

$$g(u,v;\mu) := a_0(u,v) + \frac{1}{2}a_1(u,u,v) - \mu F(v), \tag{2.1}$$

where $a_0 : X^e \times X^e \to \mathbb{R}$ is a symmetric bilinear form, $a_1 : X^e \times X^e \times X^e \to \mathbb{R}$ is a trilinear form which is symmetric w.r.t. the first two arguments, $F : X^e \to \mathbb{R}$ is linear and bounded and $\mu \in \mathcal{D}$ is a parameter. Here, X^e is an appropriate function space, e.g., a Sobolev space. This means, that $g(\cdot,\cdot;\mu)$ is an affine function of the parameter and the analysis in [16] crucially relies on this assumption.

Here, we are interested in a non-affine parameter dependence. Such a situation occurs e.g. when considering a problem with moving domains when one uses a transformation to a reference situation (reference domain). The transformation of variables in the integral results in additional terms in the multilinear forms that can be non-affine. We will describe one application later in Section 5.

If we assume that the parameter $\mu \in \mathcal{D}$ describes the movement of the domain of interest (e.g. a propeller), then the transformation to a reference situation results in functions

$$h_0, h_1 : \Omega \times \mathcal{D} \to \mathbb{R},$$

where $\Omega \subset \mathbb{R}^d$ is the spatial domain (e.g. the flow domain). Typically, h_0 and h_1 are *not* affine w.r.t. the parameter $\mu \in \mathcal{D}$, but the resulting bi- and trilinear forms a_0 and a_1 are linear w.r.t. the transformation. Hence, we collect both transformations

$$h := (h_0, h_1) \in \big(L_\infty(\Omega) \times C^1_{\mathrm{loc}}(\mathcal{D})\big)^2$$

and consider the form

$$g(u,v;h(\cdot;\mu)) := a_0(u,v;h_0(\cdot;\mu)) + a_1(u,u,v;h_1(\cdot;\mu)) - F(v), \tag{2.2}$$

where $a_0 : X^e \times X^e \times L_\infty(\Omega) \to \mathbb{R}$ is bilinear w.r.t. the first two arguments and linear w.r.t. h_0, $a_1 : X^e \times X^e \times X^e \times L_\infty(\Omega) \to \mathbb{R}$ is trilinear w.r.t. the first three arguments and linear w.r.t. h_1. Note that we do not assume any kind of symmetry, which is also a slight generalization of [16]. Furthermore, note that the restriction to one bilinear form and one trilinear form, respectively, is for notational convenience, only. In fact, the application to be presented in Section 5 below consists of a linear combination of several forms. Finally, the function $F : X^e \to \mathbb{R}$ is as before.

Given this, we consider the following nonlinear variational problem (*Primal Problem*)

$$\begin{cases} \text{For } \mu \in \mathcal{D} \text{ find } u(\mu) \in X \text{ such that} \\ g(u(\mu),v;h(\cdot;\mu)) = 0, \quad v \in X, \end{cases} \tag{2.3}$$

where $h : \Omega \times \mathcal{D} \to \mathbb{R}^2$ is as above, in particular possibly non-affine in $\mu \in \mathcal{D}$. From now on, we will denote $h(\cdot;\mu)$ by $h(\mu)$ wherever unambiguous, i.e., dropping the explicit labeling of the dependency on the space variable. Furthermore, in (2.3) we have replaced the 'exact' function space X^e on which the original variational problem is posed (usually a Sobolev space) by the 'truth-approximation' function space X, a sufficiently rich finite-dimensional subspace of X^e, as it is common practice in the Reduced Basis context. In the sequel we will set $\mathcal{N} = \dim X$, keeping in mind that $\mathcal{N}$ is assumed to be very large.

In order to formulate conditions for existence and uniqueness of a solution to (2.3), we determine the Frèchet derivatives da_1 of a_1 and dg of g, respectively, at a point $z \in X$, which are readily seen to read

$$da_1(u,v;h_1)[z] = a_1(u,z,v;h_1) + a_1(z,u,v;h_1) \tag{2.4}$$

and

$$dg(u,v;h)[z] = a_0(u,v;h_0) + da_1(u,v;h_1)[z]. \tag{2.5}$$

If the trilinear form is symmetric w.r.t the first two arguments, we have

$$da_1(u,v;h_1)[z] = 2a_1(u,z,v;h_1)$$

which fits into the formulation in [16]. We define the *inf-sup-constant*

$$\begin{aligned} \beta(z;h) \; &:= \; \inf_{u\in X} \sup_{v\in X} \frac{dg(u,v;h)[z]}{\|u\|_X \, \|v\|_X}, \\ &= \; \inf_{v\in X} \sup_{u\in X} \frac{dg(u,v;h)[z]}{\|u\|_X \, \|v\|_X}, \quad z \in X, \; h \in (L_\infty(\Omega))^2, \end{aligned} \tag{2.6}$$

(note that this always exists since X is finite-dimensional) and the *continuity constant*

$$\gamma(z;h) := \sup_{u\in X} \sup_{v\in X} \frac{dg(u,v;h)[z]}{\|u\|_X \, \|v\|_X}, \quad z \in X, \; h \in (L_\infty(\Omega))^2. \tag{2.7}$$

Note that the equality in (2.6) holds thanks to the finite dimension of X (for a proof c.p. [10]). For convenience, we collect all requirements on the form $g(\cdot,\cdot;h)$ as follows.

Assumption 2.1. *We assume the following properties.*

(i) *Boundedness: The multilinear forms a_0 and a_1 are bounded, i.e. there exist constants $0 < \rho_i < \infty$, $i = 0,1$, such that (recall that $h_i \in L_\infty(\Omega)$)*

$$|a_0(u,v;h_0)| \le \rho_0 \, \|u\|_X \, \|v\|_X \, \|h_0\|_{L_\infty(\Omega)}, \quad u,v \in X, \tag{2.8}$$

as well as

$$|a_1(u,w,v;h_1)| \leq \rho_1 \, \|u\|_X \, \|w\|_X \, \|v\|_X \, \|h_1\|_{L_\infty(\Omega)}, \quad u,v,w \in X. \tag{2.9}$$

(ii) *Uniform inf-sup-constant: There exists a constant* $\beta_0 > 0$*, such that*

$$\beta(u(\mu); h(\mu)) \geq \beta_0, \quad \mu \in \mathcal{D}. \tag{2.10}$$

The assumptions (2.8) and (2.9) immediately imply the boundedness of g and dg. In fact, it is readily seen, that

$$\gamma(z;h) \leq \rho_0 \, \|h_0\|_{L_\infty(\Omega)} + 2\rho_1 \, \|z\|_X \, \|h_1\|_{L_\infty(\Omega)}. \tag{2.11}$$

Note that in the sequel for $\mu \in \mathcal{D}$ we will denote

$$\rho_i(\mu) := \rho_i \, \|h_i(\mu)\|_{L_\infty(\Omega)}, \quad i = 0,1. \tag{2.12}$$

Remark 2.1. *Assumption (2.10) seems somehow curious at a first glance, since it involves the solution* $u(\mu)$*, where existence is not yet known. One might expect a condition, such as*

$$\beta(v; h(\mu)) \geq \beta_0, \quad \mu \in \mathcal{D}, \; v \in X.$$

This, however might not be realistic, since the parameter-dependence enters into the variational form. Thus, we have to expect that there might be parameter values $\mu \in \mathcal{D}$*, such that no solution exists.*

Hence, we follow a slightly different philosophy. When simulating a given configuration (a given parameter $\mu \in \mathcal{D}$*), we develop and compute a-posteriori indicators. For these indicators, we develop and prove bounds that guarantee the well-posedness of the original problem. This allows to verify (2.10) a-posteriori.*

However, if the discretization (choice of X *and the basis for the RBM) does not allow to realize the bound for the indicator, this does not imply the original problem (2.3) is ill-posed. We only derive sufficient conditions.*

2.2. **Empirical Interpolation Method (EIM).**

Non-affine parameter dependencies usually prohibit offline computations to a large extent. This results in a strong negative influence on the efficiency of the online stage. One possible way-out is the so-called *Empirical Interpolation Method (EIM)* introduced in [1]. In order to apply it, we need that $h(\cdot;\mu) \in L_\infty(\Omega)$ is sufficiently smooth which is guaranteed by our general assumptions. The main idea is to approximate the non-affine components of $h(\mu)$ by a linear combination of separable functions, i.e.,

$$\widehat{h}_i(x;\mu) \;:=\; \sum_{m=1}^{M_i} \vartheta_m^i(\mu) \, \varphi_m^i(x), \quad i = 0,1, \tag{2.13}$$

where $W_i^{M_i} := \operatorname{span}\{\varphi_1^i, \ldots, \varphi_{M_i}^i\}$ is a finite dimensional approximation space and $\vartheta_m^i(\mu) \in \mathbb{R}$ are coefficients to be determined. Given a tolerance $\varepsilon_{\text{emp}} > 0$, one seeks a possibly small M_i, such that for all $\mu \in \mathcal{D}$

$$\left\| h_i(\mu) - \widehat{h}_i(\mu) \right\|_{L_\infty(\Omega)} \;\leq\; \varepsilon_{\text{emp}}. \tag{2.14}$$

We do not go into details of the EIM but refer the reader to the literature. We just mention two facts that will be needed in the sequel:

(a) The basis-functions φ_m^i, $1 \leq m \leq M_i$, are normalized in L_∞, i.e.

$$\left\|\varphi_m^i\right\|_{L_\infty(\Omega)} = 1. \tag{2.15}$$

(b) If $h_i(\mu) \in W_i^{M_i+1}$, then

$$h_i(\mu) - \widehat{h}_i(\mu) = \varepsilon_i^{M_i}(\mu)\, \varphi_i^{M_i+1}, \tag{2.16}$$

where the determination of $\varepsilon_i^{M_i}(\mu)$ is cheap by a single evaluation of $h_i(\mu)$ and $\widehat{h}_i(\mu)$.

Using the approximation $\widehat{h}(\mu)$ instead of $h(\mu)$ yields the *Primal EIM-Problem*

$$\begin{cases} \text{For } \mu \in \mathcal{D} \text{ find } \widehat{u}(\mu) \in X \text{ such that} \\ g(\widehat{u}(\mu), v; \widehat{h}(\mu)) = 0, \quad v \in X. \end{cases} \tag{2.17}$$

Since $\widehat{h}(\mu)$ is component-wise affine separable w.r.t. the parameter μ and $g(\cdot,\cdot;\cdot)$ is affine w.r.t. the third argument, the form g in (2.17) is affine separable w.r.t. μ.

Remark 2.2. *Note that similar arguments hold as in Remark 2.1 for the original variational problem. We have to take the situation into account, that there might be parameter values $\mu \in \mathcal{D}$ that do not allow unique solutions $u(\mu)$ or $\widehat{u}(\mu)$. But we can verify the well-posedness a-posteriori.*

There is another point that is worth mentioning. Let us assume that the original problem admits a unique solution $u(\mu) \in X$. Then, the Primal EIM-Problem should also be well-posed. This is in fact the case, if we choose $\varepsilon_{\rm emp}$ in (2.14) small enough, which can be proven with techniques similar to the one presented in Section 3.

2.3. **Reduced Basis Approximation.** Now we introduce the RBM (Reduced Basis Method) both for the original and for the EIM-problem (2.17). For $1 \le N \le N^{\max}$ we use a standard Lagrange basis

$$W^N := \operatorname{span}\{\Xi^N\}, \quad \Xi^N := \{\xi_i := u(\mu_i),\, 1 \le i \le N\} \tag{2.18}$$

for a given set of samples

$$S^N := \{\mu_i, 1 \le i \le N\} \subset \mathcal{D}. \tag{2.19}$$

At this point, we assume that S^N is given to us. We investigate later in Section 4.6 how to determine these samples. The basis functions $u(\mu_i)$ are also called *snapshots*. This should not be mixed up with *spatial* snapshots known from the Proper Orthogonal Decomposition.

The *Primal RBM-Problem* then reads

$$\begin{cases} \text{For } \mu \in \mathcal{D} \text{ find } u^N(\mu) \in W^N \text{ such that} \\ g(u^N(\mu), v; h(\mu)) = 0, \quad v \in W^N, \end{cases} \tag{2.20}$$

and the *Primal RBM-EIM-Problem*:

$$\begin{cases} \text{For } \mu \in \mathcal{D} \text{ find } \widehat{u}^N(\mu) \in W^N \text{ such that} \\ g(\widehat{u}^N(\mu), v; \widehat{h}(\mu)) = 0, \quad v \in W^N. \end{cases} \tag{2.21}$$

As already mentioned above, problem (2.20) does not allow for an efficient online computation. Thus, we mainly consider (2.20) for completeness and theoretical investigations. The second problem (2.21) can be efficiently solved by a Newton iteration as follows.
Given an initial guess $\widehat{u}^{(0)}(\mu) \in W^N$, compute for $\nu = 0, 1, 2, \ldots$

(1) determine the Newton correction $\delta\widehat{u}^{(\nu)}(\mu) \in W^N$ as the solution of

$$dg\Big(\delta\widehat{u}^{(\nu)}(\mu), v; \widehat{h}(\mu)\Big)\,[\widehat{u}^{(\nu)}(\mu)] = -g\Big(\widehat{u}^{(\nu)}(\mu), v; \widehat{h}(\mu)\Big), \quad v \in W^N,$$

(2) Newton update:

$$\widehat{u}^{(\nu+1)}(\mu) = \widehat{u}^{(\nu)}(\mu) + \delta\widehat{u}^{(\nu)}(\mu).$$

Offline/Online Decomposition. Now we investigate the required number of operations for an offline/online-decomposition. Due to the affine dependence, we can compute the following quantities in the offline stage:

$$A^0_m := \big(a_0(\xi_j, \xi_i; \varphi^0_m)\big)_{1\le i,j\le N} \in \mathbb{R}^{N\times N}, \quad 1 \le m \le M_0, \tag{2.22}$$

$$A^1_{m,k} := \big(a_1(\xi_j, \xi_k, \xi_i; \varphi^1_m)\big)_{1\le i,j\le N} \in \mathbb{R}^{N\times N}, \quad 1 \le m \le M_1, \tag{2.23}$$

$$C_{m,k} := \big(da_1(\xi_j, \xi_i; \varphi^1_m)[\xi_k]\big)_{1\le i,j\le N} \in \mathbb{R}^{N\times N}, \quad 1 \le m \le M_1 \tag{2.24}$$

the latter two for $1 \le k \le N$, as well as

$$F := (f(\xi_i))_{1\le i\le N} \in \mathbb{R}^N. \tag{2.25}$$

Thus, the offline complexity for computing these quantities is

$$\mathcal{O}\big(M_0 N^2 \mathcal{N}\big) + \mathcal{O}\big(M_1 N^3 \mathcal{N}\big) + \mathcal{O}(N\mathcal{N}) .$$

We now express the first step of the Newton scheme in terms of the offline quantities in (2.22)–(2.25). In order to do so, let us collect the coefficients

$$\underline{\delta\widehat{u}}^{(\nu)}(\mu) := \Big(\delta\widehat{u}^{(\nu)}_i(\mu)\Big)_{1\le i\le N} \in \mathbb{R}^N, \quad \underline{\widehat{u}}^{(\nu)}(\mu) := \Big(\widehat{u}^{(\nu)}_i(\mu)\Big)_{1\le i\le N} \in \mathbb{R}^N$$

of the Newton update $\delta\widehat{u}^{(\nu)}(\mu)$ and the known previous iteration $\widehat{u}^{(\nu)}(\mu)$, respectively, i. e.,

$$\delta\widehat{u}^{(\nu)}(\mu) := \sum_{i=1}^N \delta\widehat{u}^{(\nu)}_i(\mu)\xi_i \in W^N, \quad \widehat{u}^{(\nu)}(\mu) := \sum_{i=1}^N \widehat{u}^{(\nu)}_i(\mu)\xi_i \in W^N.$$

Then we can write the first step as

$$B^{(\nu)}(\mu)\, \underline{\delta\widehat{u}}^{(\nu)}(\mu) = b^{(\nu)}(\mu), \tag{2.26}$$

where

$$B^{(\nu)}(\mu) := \sum_{m=1}^{M_0} \vartheta^0_m(\mu) A^0_m + \sum_{m=1}^{M_1} \vartheta^1_m(\mu) \sum_{k=1}^N C_{m,k}(\mu)\widehat{u}^{(\nu)}_k(\mu), \tag{2.27}$$

$$b^{(\nu)}(\mu) := F - \left(\sum_{m=1}^{M_0} \vartheta^0_m(\mu) A^0_m + \sum_{m=1}^{M_1} \vartheta^1_m(\mu) \sum_{k=1}^N A^1_{m,k}\widehat{u}^{(\nu)}_k(\mu)\right) \underline{\widehat{u}}^{(\nu)}(\mu). \tag{2.28}$$

Thus, the assembly of $B^{(\nu)}(\mu) \in \mathbb{R}^{N\times N}$ requires $\mathcal{O}\big(M_0N^2\big) + \mathcal{O}\big(M_1N^3\big)$operations and the right-hand side $b^{(\nu)}(\mu) \in \mathbb{R}^N$ $\mathcal{O}(N) + \mathcal{O}\big(M_0N^2\big) + \mathcal{O}\big(M_1N^3\big)$. Combined with the solution of the linear system of equations in (2.26), which is typically densely populated, hence requires $\mathcal{O}\big(N^3\big)$ operations, we obtain

$$\mathcal{O}\big(M_1N^3\big)$$

operations in the online stage for one Newton step.

2.4. **The Dual Problem.** The consideration of a dual problem is a well-known technique e. g. for adaptive methods, in particular for the so-called *goal oriented* error estimates, where one is not (only) interested in an approximation to the state $u(\mu) \in X$, but on some functional of it, say $s(\mu) := \ell(u(\mu))$, where

$$\ell : X^e \to \mathbb{R}$$

is a linear and bounded functional.

Given the solution $\widehat{u}^N(\mu) \in W^N \subset X$ of the *Primal RBM-EIM-Problem* (2.21), the solution $u(\mu)$ of the *Primal Problem* (2.3) and denoting the error by

$$e^N(\mu) := u(\mu) - \widehat{u}^N(\mu),$$

the *Dual Problem* for (2.3) reads:

$$\begin{cases} \text{For } \mu \in \mathcal{D} \text{ find } \psi^N(\mu) \in X \text{ such that} \\ dg(v, \psi^N(\mu); h(\mu))[\widehat{u}^N(\mu) + \frac{1}{2}e^N(\mu)] = -\ell(v), \quad v \in X. \end{cases} \tag{2.29}$$

It should be noted that the *Dual Problem* (2.29) is a *linear* problem, whereas the *Primal Problem* is *nonlinear*. Thus, the complexity to numerically solve (2.29) corresponds to only one Newton-iteration for solving (2.3). Before we consider an EIM-variant of this problem, let us investigate the error in the output of interest.

Lemma 2.1. *Provided solutions $u(\mu) \in X$, $\widehat{u}^N(\mu) \in W^N$ and $\psi^N(\mu) \in X$ of (2.3), (2.21) and (2.29), respectively, exist, we have*

$$s(\mu) - \widehat{s}^N(\mu) = g\big(\widehat{u}^N(\mu), v; h(\mu)\big) + g\big(\widehat{u}^N(\mu), \psi^N(\mu) - v; h(\mu)\big) \tag{2.30}$$

for all $v \in X$, where we use the abbreviations

$$s(\mu) := \ell(u(\mu)), \quad \widehat{s}^N(\mu) := \ell(\widehat{u}^N(\mu)).$$

Proof. We use the particular choice $v = e^N(\mu)$ in dg and obtain for the left-hand side of (2.29)

$$\begin{aligned} & dg\big(e^N(\mu), \psi^N(\mu); h(\mu)\big)\,[\widehat{u}^N(\mu) + \frac{1}{2}e^N(\mu)] \\ =\; & a_0\big(e^N(\mu), \psi^N(\mu); h_0(\mu)\big) + a_1\left(e^N(\mu), \widehat{u}^N(\mu) + \frac{1}{2}e^N(\mu), \psi^N(\mu); h_1(\mu)\right) \\ & \quad + a_1\left(\widehat{u}^N(\mu) + \frac{1}{2}e^N(\mu), e^N(\mu), \psi^N(\mu); h_1(\mu)\right). \end{aligned}$$

Now, due to trilinearity of $a_1(\cdot,\cdot,\cdot; h_1(\mu))$ and the simple fact

$$\widehat{u}^N(\mu) + \frac{1}{2}e^N(\mu) = \frac{1}{2}\big(u(\mu) + \widehat{u}^N(\mu)\big)$$

we get

$$\begin{aligned} & dg\big(e^N(\mu), \psi^N(\mu); h(\mu)\big)\,[\widehat{u}^N(\mu) + \frac{1}{2}e^N(\mu)] \\ =\; & a_0\big(e^N(\mu), \psi^N(\mu); h_0(\mu)\big) \\ & + \frac{1}{2}a_1\big(u(\mu) - \widehat{u}^N(\mu), u(\mu) + \widehat{u}^N(\mu), \psi^N(\mu); h_1(\mu)\big) \\ & + \frac{1}{2}a_1\big(u(\mu) + \widehat{u}^N(\mu), u(\mu) - \widehat{u}^N(\mu), \psi^N(\mu); h_1(\mu)\big) \\ =\; & a_0\big(e^N(\mu), \psi^N(\mu); h_0(\mu)\big) + a_1\big(u(\mu), u(\mu), \psi^N(\mu); h_1(\mu)\big) \\ & - a_1\big(\widehat{u}^N(\mu), \widehat{u}^N(\mu), \psi^N(\mu); h_1(\mu)\big). \end{aligned}$$

Next, we use (2.29) (again for $v = e^N(\mu)$)

$$\begin{aligned} & -(s(\mu) - \widehat{s}^N(\mu)) = -\ell(u(\mu) - \widehat{u}^N(\mu)) \\ =\; & -\ell(e^N(\mu)) \\ =\; & dg\big(e^N(\mu), \psi^N(\mu); h(\mu)\big)\,[\widehat{u}^N(\mu) + \frac{1}{2}e^N(\mu)] \\ =\; & a_0\big(u(\mu), \psi^N(\mu); h_0(\mu)\big) + a_1\big(u(\mu), u(\mu), \psi^N(\mu); h_1(\mu)\big) \\ & - a_0\big(\widehat{u}^N(\mu), \psi^N(\mu); h_0(\mu)\big) - a_1\big(\widehat{u}^N(\mu), \widehat{u}^N(\mu), \psi^N(\mu); h_1(\mu)\big) \\ =\; & F(\psi^N(\mu)) - a_0\big(\widehat{u}^N(\mu), \psi^N(\mu); h_0(\mu)\big) \\ & - a_1\big(\widehat{u}^N(\mu), \widehat{u}^N(\mu), \psi^N(\mu); h_1(\mu)\big) \\ =\; & -g\big(\widehat{u}^N(\mu), \psi^N(\mu); h(\mu)\big). \end{aligned}$$

Adding and subtracting $v \in X$ and using linearity of g w.r.t the second argument yields the assertion. $\square$

This result will serve as a starting point for developing a-posteriori error estimates for $s(\mu)$. This involves also the numerical solution of the dual problem (2.29). For computational efficiency we thus need a *Dual RBM-Problem*, which we introduce now. We use the superscript '$\widetilde{\ }$' to indicate all quantities related to the dual problem. For $1 \le \widetilde{N} \le \widetilde{N}^{\max}$, let

$$\widetilde{S}^{\widetilde{N}} := \left\{ \widetilde{\mu}_i, 1 \le i \le \widetilde{N} \right\} \subset \mathcal{D}. \tag{2.31}$$

be a given set of samples. We define

$$\widetilde{W}^{\widetilde{N}} := \operatorname{span}\left\{ \widetilde{\Xi}^{\widetilde{N}} \right\}, \quad \widetilde{\Xi}^{\widetilde{N}} := \left\{ \widetilde{\xi}_i := \psi^{N^{\max}}(\widetilde{\mu}_i), 1 \le i \le \widetilde{N} \right\}, \tag{2.32}$$

where $\psi^{N^{\max}}(\mu)$ is the solution of (2.29) (corresponding to $\widehat{u}^{N^{\max}}(\mu) \in W^{N^{\max}}$, the solution of (2.21)). With this notation at hand, we obtain the *Dual RBM-Problem* as follows

$$\begin{cases} \text{For } \mu \in \mathcal{D} \text{ find } \psi^{N,\widetilde{N}}(\mu) \in \widetilde{W}^{\widetilde{N}} \text{ such that} \\ dg(v, \psi^{N,\widetilde{N}}(\mu); h(\mu))[\widehat{u}^N(\mu)] = -\ell(v), \quad v \in \widetilde{W}^{\widetilde{N}}. \end{cases} \tag{2.33}$$

Note that, as opposed to (2.29), the term

$$\frac{1}{2} e^N(\mu)$$

is missing in the argument of dg, since this would involve the unknown true solution $u(\mu)$ of (2.3). Replacing $h(\mu)$ in (2.33) by its EIM-approximation $\widehat{h}(\mu)$ leads to the *Dual RBM-EIM-Problem*:

$$\begin{cases} \text{For } \mu \in \mathcal{D} \text{ find } \widehat{\psi}^{N,\widetilde{N}}(\mu) \in \widetilde{W}^{\widetilde{N}} \text{ such that} \\ dg(v, \widehat{\psi}^{N,\widetilde{N}}(\mu); \widehat{h}(\mu))[\widehat{u}^N(\mu)] = -\ell(v), \quad v \in \widetilde{W}^{\widetilde{N}}. \end{cases} \tag{2.34}$$

Offline/Online Decomposition. As for the primal problem, we obtain a corresponding offline/online decomposition as follows. In the offline stage, we compute the following the following quantities

$$\widetilde{A}^0_m := \left(a_0(\widetilde{\xi}_i, \widetilde{\xi}_j; \varphi^0_m) \right)_{1 \le i,j \le \widetilde{N}} \in \mathbb{R}^{\widetilde{N} \times \widetilde{N}}, \quad 1 \le m \le M_0, \tag{2.35}$$

$$\widetilde{C}_{m,k} := \left(da_1(\widetilde{\xi}_i, \widetilde{\xi}_j; \varphi^1_m)[\xi_k] \right)_{1 \le i,j \le \widetilde{N}} \in \mathbb{R}^{\widetilde{N} \times \widetilde{N}}, \ 1 \le m \le M_1, 1 \le k \le N, \tag{2.36}$$

as well as

$$\widetilde{F} := \left(-\ell(\widetilde{\xi}_i) \right)_{1 \le i \le \widetilde{N}} \in \mathbb{R}^{\widetilde{N}}. \tag{2.37}$$

Then, (2.34) becomes

$$\widetilde{B}(\mu) \underline{\widehat{\psi}}^{N,\widetilde{N}}(\mu) = \widetilde{F}, \tag{2.38}$$

where $\underline{\widehat{\psi}}^{N,\widetilde{N}}(\mu) = \left(\widehat{\psi}^{N,\widetilde{N}}_i(\mu) \right)_{1 \le i \le \widetilde{N}} \in \mathbb{R}^{\widetilde{N}}$ is the vector of the unknown expansion coefficients of the desired solution

$$\widehat{\psi}^{N,\widetilde{N}}(\mu) = \sum_{i=1}^{\widetilde{N}} \widehat{\psi}^{N,\widetilde{N}}_i(\mu) \widetilde{\xi}_i \in \widetilde{W}^{\widetilde{N}}$$

and

$$\widetilde{B}(\mu) := \sum_{m=1}^{M_0} \vartheta^0_m(\mu) \widetilde{A}^0_m + \sum_{m=1}^{M_1} \vartheta^1_m(\mu) \sum_{k=1}^{N} \widetilde{C}_{m,k} \widehat{u}^N_k(\mu) \in \mathbb{R}^{\widetilde{N} \times \widetilde{N}}.$$

Thus, the offline complexity for the assembly of (2.35)–(2.37) is

$$\mathcal{O}\Big(\widetilde{N}\mathcal{N}\Big) + \mathcal{O}\Big(M_0\widetilde{N}^2\mathcal{N}\Big) + \mathcal{O}\Big(M_1 N\widetilde{N}^2\mathcal{N}\Big).$$

The setup in the online phase requires $\mathcal{O}\Big(M_0\widetilde{N}^2\Big) + \mathcal{O}\Big(M_1 N\widetilde{N}^2\Big)$ operations plus $\mathcal{O}\Big(\widetilde{N}^3\Big)$ to solve the linear system (2.38). Recall that the dual problem is *linear*, so that the complexity corresponds to one Newton iteration (for solving the *Primal RBM-EIM-Problem*), only.

3. Well-Posedness

Before constructing any kind of numerical scheme for the primal problem (2.3), its RBM-formulation (2.20) or the RBM-EIM-version (2.21), we should investigate the question of existence and uniqueness of a solution. Recall that (2.3), (2.20), (2.21) are nonlinear, thus there might be several branches of solutions. One framework to tackle such question is the *Brezzi-Rappaz-Raviart (BRR) theory* (see, e.g., [2]) which has also been used in [16]. We have to modify the proofs in [16] here, since we have a more complicated parameter dependency and we also have to take the EIM into account.

We start by collecting all involved quantities. First,

$$R^N(v;\mu) := g\Big(\widehat{u}^N(\mu), v; \widehat{h}(\mu)\Big), \quad R^N(\mu) := \big\|R^N(\cdot;\mu)\big\|_{X'} \tag{3.1}$$

is the the primal RBM-EIM residual and its dual norm, respectively, where as usual $\|\ell\|_{X'} = \sup_{x\in X} \frac{\ell(x)}{\|x\|_X}$ denotes the dual norm. In addition to [16] we have to keep track of the additional error introduced by the approximation of $h(\mu)$ by $\widehat{h}(\mu)$ via the EIM. Thus, we consider

$$E^N(v;\mu) := g\Big(\widehat{u}^N(\mu), v; h(\mu) - \widehat{h}(\mu)\Big), \quad E^N(\mu) := \big\|E^N(\cdot;\mu)\big\|_{X'}. \tag{3.2}$$

Next, we need (as in [16]) inf-sup- and continuity constants according to the derivative dg at $\widehat{u}^N(\mu)$, namely

$$\beta^N(\mu) := \beta\big(\widehat{u}^N(\mu); h(\mu)\big), \tag{3.3}$$

$$\gamma^N(\mu) := \gamma\big(\widehat{u}^N(\mu); h(\mu)\big), \tag{3.4}$$

where β and γ are defined by (2.6) and (2.7), respectively. Next, we introduce a *lower bound* $\overline{\beta}^N(\mu)$ for $\beta^N(\mu)$ (to be developed in Section 4.5), i. e.,

$$0 \le \overline{\beta}^N(\mu) \le \beta^N(\mu), \quad \mu \in \mathcal{D}. \tag{3.5}$$

Having this at hand, we introduce the key parameter of the BRR theory, namely a *proximity indicator*

$$\tau^N(\mu) := 4\rho_1(\mu)\Big(\overline{\beta}^N(\mu)\Big)^{-2}\big(R^N(\mu) + E^N(\mu)\big), \tag{3.6}$$

where $\rho_1(\mu)$ is defined in (2.12) and, furthermore, an error bound

$$\Delta^N(\mu) := \overline{\beta}^N(\mu)(2\rho_1(\mu))^{-1}\Big(1 - \sqrt{1-\tau^N(\mu)}\Big). \tag{3.7}$$

Note that the error bound is defined exactly as in [16], whereas the definition of $\tau^N(\mu)$ involves the additional term $E^N(\mu)$ due to the EIM-approximation. This also implies that we have to modify the proof of the corresponding result [16, Proposition 2.1], which in turns is a modification of [2, Theorem 2.1]. However, the main idea of the proof is the same, we only need some more technical work. We use the standard notation

$$B(x_0, r) := \{x \in X : \|x - x_0\|_X < r\}, \quad x_0 \in X,\ r \in \mathbb{R}^+,$$

for a ball (in X) of radius r around x_0.

Proposition 3.1. *If $\tau^N(\mu) < 1$, then there exists a unique solution*

$$u(\mu) \in B\Big(\widehat{u}^N(\mu), \overline{\beta}^N(\mu)(2\rho_1(\mu))^{-1}\Big)$$

of the nonlinear problem (2.3) and the following error estimate holds

$$\big\|u(\mu) - \widehat{u}^N(\mu)\big\|_X \le \Delta^N(\mu). \tag{3.8}$$

Proof. As in [16], the fundamental theorem of calculus yields

$$g(w^2, v; h(\mu)) - g(w^1, v; h(\mu)) = \int_0^1 dg(w^2 - w^1, v; h(\mu))[w^1 + t(w^2 - w^1)]\ dt$$

for any $w^1, w^2 \in X$. For the Frèchet derivative dg we obtain

$$\begin{aligned} &\big|dg(w, v; h(\mu))[z^2] - dg(w, v; h(\mu))[z^1]\big| \\ &\quad = \big|da_1(w, v; h_1(\mu))[z^2 - z^1]\big| \\ &\quad \le 2\rho_1(\mu)\,\|v\|_X\,\|w\|_X\,\big\|z^2 - z^1\big\|_X \end{aligned} \tag{3.9}$$

by (2.4) and Assumption 2.1(i), (2.9). Next, we consider the operator $H^\mu : X \to X$ defined by

$$dg(H^\mu(w), v; h(\mu))\,[\widehat{u}^N(\mu)] = dg(w, v; h(\mu))\,[\widehat{u}^N(\mu)] - g(w, v; h(\mu)), \quad v \in X,$$

for a given $w \in X$. Since X is finite-dimensional, it is readily seen that H^μ is well posed due to the assumption $\tau^N(\mu) < 1$ which in turns implies $\overline{\beta}^N(\mu) > 0$. If we can show that H^μ has a fixed point w^* then we get $g(w^*, v; h(\mu)) = 0$, $v \in X$, which would prove existence. As in [16], we use the Banach fixed point theorem.

In order to show that H^μ is contractive on $\overline{B}\big(\widehat{u}^N(\mu), \alpha\big)$ for suitable values of $\alpha > 0$ to be determined, let $w^1, w^2 \in \overline{B}\big(\widehat{u}^N(\mu), \alpha\big)$. Then we get

$$\begin{aligned} &dg\big(H^\mu(w^2) - H^\mu(w^1), v; h(\mu)\big)\,[\widehat{u}^N(\mu)] \\ &\quad = dg\big(w^2 - w^1, v; h(\mu)\big)\,[\widehat{u}^N(\mu)] - g\big(w^2 - w^1, v; h(\mu)\big) \\ &\quad = dg\big(w^2 - w^1, v; h(\mu)\big)\,[\widehat{u}^N(\mu)] \\ &\qquad - \int_0^1 dg\big(w^2 - w^1, v; h(\mu)\big)\,[w^1 + t(w^2 - w^1)]\ dt \\ &\quad = \int_0^1 \Big\{dg\big(w^2 - w^1, v; h(\mu)\big)\,[\widehat{u}^N(\mu)] \\ &\qquad - dg\big(w^2 - w^1, v; h(\mu)\big)\,[w^1 + t(w^2 - w^1)]\Big\}\ dt \end{aligned}$$

by definition of H^μ and the above arguments. Applying (3.9) to the integrand above and using $w^1 + t(w^2 - w^1) \in \overline{B}\big(\widehat{u}^N(\mu), \alpha\big)$ for $t \in [0, 1]$ yields

$$\begin{aligned} &\big|dg\big(H^\mu(w^2) - H^\mu(w^1), v; h(\mu)\big)\,[\widehat{u}^N(\mu)]\big| \\ &\quad \le \int_0^1 2\rho_1(\mu)\,\big\|w^2 - w^1\big\|_X\,\|v\|_X\,\big\|\widehat{u}^N(\mu) - \big(w^1 + t(w^2 - w^1)\big)\big\|_X\ dt \\ &\quad \le 2\alpha\rho_1(\mu)\,\big\|w^2 - w^1\big\|_X\,\|v\|_X\,. \end{aligned} \tag{3.10}$$

Next, we use our assumption (3.5) for the inf-sup constant $\overline{\beta}^N(\mu)$, namely

$$\|w\|_X \le \Big(\overline{\beta}^N(\mu)\Big)^{-1} \sup_{v\in X} \frac{dg(w,v;h(\mu))\,[\widehat{u}^N(\mu)]}{\|v\|_X}$$

and apply this for $w := H^\mu(w^2) - H^\mu(w^1) \in X$, i.e.,

$$\begin{aligned}
\big\|H^\mu(w^2) - H^\mu(w^1)\big\|_X &\le \Big(\overline{\beta}^N(\mu)\Big)^{-1} \sup_{v\in X} \frac{dg\big(H^\mu(w^2) - H^\mu(w^1), v; h(\mu)\big)\,[\widehat{u}^N(\mu)]}{\|v\|_X} \\
&\le 2\alpha\rho_1(\mu)\Big(\overline{\beta}^N(\mu)\Big)^{-1} \big\|w^2 - w^1\big\|_X ,
\end{aligned}$$

so that H^μ is a contraction for

$$\alpha \in I_1 := \Big[0, \overline{\beta}^N(\mu)(2\rho_1(\mu))^{-1}\Big) .$$

Next, we have to investigate for which values of α, H^μ maps $\overline{B}\big(\widehat{u}^N(\mu),\alpha\big)$ into itself. This part of the proof differs from [16] due to the EIM. For any $w \in \overline{B}\big(\widehat{u}^N(\mu),\alpha\big)$, we have by definition of H^μ

$$\begin{aligned}
&dg\big(H^\mu(w) - \widehat{u}^N(\mu), v; h(\mu)\big)\,[\widehat{u}^N(\mu)] \\
&\quad = dg(w,v;h(\mu))\,[\widehat{u}^N(\mu)] - g(w,v;h(\mu)) - dg\big(\widehat{u}^N(\mu),v;h(\mu)\big)\,[\widehat{u}^N(\mu)] \\
&\quad = dg\big(w - \widehat{u}^N(\mu), v; h(\mu)\big)\,[\widehat{u}^N(\mu)] - g\big(w - \widehat{u}^N(\mu), v; h(\mu)\big) \\
&\qquad - g\Big(\widehat{u}^N(\mu), v; h(\mu) - \widehat{h}(\mu)\Big) - g\Big(\widehat{u}^N(\mu), v; \widehat{h}(\mu)\Big) .
\end{aligned}$$

We can rewrite the first two terms as

$$\begin{aligned}
&dg\big(w - \widehat{u}^N(\mu), v; h(\mu)\big)\,[\widehat{u}^N(\mu)] - g\big(w - \widehat{u}^N(\mu), v; h(\mu)\big) \\
&\quad = \int_0^1 \Big\{ dg\big(w - \widehat{u}^N(\mu), v; h(\mu)\big)\,[\widehat{u}^N(\mu)] \\
&\qquad - dg\big(w - \widehat{u}^N(\mu), v; h(\mu)\big)\,[\widehat{u}^N(\mu) + t(w - \widehat{u}^N(\mu))]\Big\}\, dt \\
&\quad \le 2\rho_1(\mu)\,\big\|w - \widehat{u}^N(\mu)\big\|_X^2\, \|v\|_X \int_0^1 t\, dt
\end{aligned}$$

using the above estimates for dg (c.p. 3.9), so that by (3.1) and (3.2)

$$\begin{aligned}
&\big\|dg\big(H^\mu(w) - \widehat{u}^N(\mu), h(\mu)\big)\,[\widehat{u}^N(\mu)]\big\|_{X'} \\
&\quad \le E^N(\mu) + R^N(\mu) + 2\rho_1(\mu)\,\big\|w - \widehat{u}^N(\mu)\big\|_X^2 \int_0^1 t\, dt \\
&\quad \le E^N(\mu) + R^N(\mu) + \alpha^2\rho_1(\mu)
\end{aligned}$$

under the assumption $w \in \overline{B}\big(\widehat{u}^N(\mu),\alpha\big)$, i.e. $\big\|w - \widehat{u}^N(\mu)\big\|_X \le \alpha$. Using exactly the same reasoning as above yields

$$\big\|H^\mu(w) - \widehat{u}^N(\mu))\big\|_X \le \Big(\overline{\beta}^N(\mu)\Big)^{-1}\big(E^N(\mu) + R^N(\mu) + \alpha^2\rho_1(\mu)\big) ,$$

and this is bounded by α if

$$\alpha \in I_2 := \Big[\Delta^N(\mu), \overline{\beta}^N(\mu)(2\rho_1(\mu))^{-1}\Big(1 + \sqrt{1 - \tau^N(\mu)}\Big)\Big] .$$

This means that H^μ satisfies the assumption of the Banach fixed point theorem on $B\big(\widehat{u}^N(\mu),\alpha\big)$ for

$$\alpha \in I_1 \cap I_2 = \Big[\Delta^N(\mu), \overline{\beta}^N(\mu)(2\rho_1(\mu))^{-1}\Big)$$

which proofs the existence and uniqueness statement. Choosing $\alpha = \Delta^N(\mu)$ yields (3.8). $\square$

Exactly as in [16], one can show the following implication. We omit the proof.

Corollary 3.1. *For $\tau^N(\mu) \leq \frac{1}{2}$, we have*

$$\beta(u(\mu); h(\mu)) \geq \frac{1}{\sqrt{2}} \overline{\beta}^N(\mu) \tag{3.11}$$

for β defined by (2.6). □

This latter result shows that our Assumption 2.1(ii) (2.10) is in fact realistic, we may choose the minimum over $\mathcal{D}$ of right-hand side as β_0.

4. A-posteriori Error Estimates

The ultimate goal is to control the number N of samples or snapshots, possibly also the numbers M_0, M_1 of EIM-terms and also to construct an adaptive scheme for the selection of the samples. It is well-known that a-posteriori error estimators are the key for these goals. It should be noted that Proposition 3.1 already gives rise to an a-posteriori error estimate for the error

$$e^N(\mu) := u(\mu) - \widehat{u}^N(\mu) \tag{4.1}$$

since the bound $\Delta^N(\mu)$ in (3.7) is in fact computable a-posteriori.

4.1. **The Primal Problem.**

As in [16], we start by investigating the *effectivity* of this estimator, i.e., we consider

$$\eta^N(\mu) := \frac{\Delta^N(\mu)}{\|e^N(\mu)\|_X}. \tag{4.2}$$

Again, due to the EIM, the analysis is more involved. Setting

$$\kappa^N(\mu) := \frac{\gamma^N(\mu)}{\overline{\beta}^N(\mu)}, \tag{4.3}$$

where $\gamma^N(\mu)$ defined in (3.4), we have the following result.

Proposition 4.1. *Assume that*

$$E^N(\mu) \leq c(\mu) R^N(\mu) \tag{4.4}$$

for some $c(\mu) \in [0, 1)$ and set

$$C(\mu) := \frac{1 - c(\mu)}{1 + c(\mu)}. \tag{4.5}$$

If $\tau^N(\mu) \leq \frac{1}{2} C(\mu)$, we obtain

$$\eta^N(\mu) \leq 4 (C(\mu))^{-1} \kappa^N(\mu). \tag{4.6}$$

Proof. It is immediate to show that (4.4) implies

$$\begin{aligned} R^N(\mu) + E^N(\mu) &\leq \frac{1 + c(\mu)}{1 - c(\mu)} \big(R^N(\mu) - E^N(\mu)\big) \\ &= (C(\mu))^{-1} \big(R^N(\mu) - E^N(\mu)\big). \end{aligned} \tag{4.7}$$

Next, we use the fact that

$$\begin{aligned} g(z + w, v; h(\mu)) &= a_0 (z + w, v; h_0(\mu)) + a_1 (z + w, z + w, v; h_1(\mu)) - F(v) \\ &= g(z, v; h(\mu)) + a_0 (w, v; h_0(\mu)) \\ &\quad + a_1 (z, w, v; h_1(\mu)) + a_1 (w, z, v; h_1(\mu)) \\ &\quad + a_1 (w, w, v; h_1(\mu)) \end{aligned}$$

(which follows directly from the definitions (2.2) and (2.4)) for $z = \widehat{u}^N(\mu)$ and $w = e^N(\mu)$ to obtain by (2.3)

$$\begin{aligned} 0 &= g(u(\mu), v; h(\mu)) \\ &= g\big(\widehat{u}^N(\mu), v; h(\mu)\big) + dg\big(e^N(\mu), v; h(\mu)\big)\,[\widehat{u}^N(\mu)] \\ &\qquad + a_1\left(e^N(\mu), e^N(\mu), v; h_1(\mu)\right). \end{aligned}$$

Then, by continuity of dg and a_1

$$g\big(\widehat{u}^N(\mu), v; h(\mu)\big) \le \gamma^N(\mu)\left\|e^N(\mu)\right\|_X \|v\|_X + \rho_1(\mu)\left\|e^N(\mu)\right\|_X^2 \|v\|_X. \tag{4.8}$$

On the other hand, for any $v \in X$

$$\begin{aligned} g\big(\widehat{u}^N(\mu), v; h(\mu)\big) &= g\Big(\widehat{u}^N(\mu), v; \widehat{h}(\mu)\Big) + g\Big(\widehat{u}^N(\mu), v; h(\mu) - \widehat{h}(\mu)\Big) \\ &= \big(\mathcal{R}^N(\mu), v\big)_X + \big(\mathcal{E}^N(\mu), v\big)_X, \end{aligned} \tag{4.9}$$

where by duality $\mathcal{R}^N(\mu) \in X$ and $\mathcal{E}^N(\mu) \in X$ exist such that $\left\|\mathcal{R}^N(\mu)\right\|_X = R^N(\mu)$, $\left\|\mathcal{E}^N(\mu)\right\|_X = E^N(\mu)$ and

$$R^N(v;\mu) = \big(\mathcal{R}^N(\mu), v\big)_X, \quad E^N(v;\mu) = \big(\mathcal{E}^N(\mu), v\big)_X$$

for all $v \in X$. First, we use (4.9) and (4.8) for $v = \mathcal{R}^N(\mu)$ and obtain

$$\begin{aligned} &\left\|\mathcal{R}^N(\mu)\right\|_X^2 + \big(\mathcal{E}^N(\mu), \mathcal{R}^N(\mu)\big)_X \\ &\quad\le \gamma^N(\mu)\left\|e^N(\mu)\right\|_X \left\|\mathcal{R}^N(\mu)\right\|_X + \rho_1(\mu)\left\|e^N(\mu)\right\|_X^2 \left\|\mathcal{R}^N(\mu)\right\|_X \end{aligned}$$

and again for $v = -\mathcal{E}^N(\mu)$

$$\begin{aligned} &-\big(\mathcal{R}^N(\mu), \mathcal{E}^N(\mu)\big)_X - \left\|\mathcal{E}^N(\mu)\right\|_X^2 \\ &\quad\le \gamma^N(\mu)\left\|e^N(\mu)\right\|_X \left\|\mathcal{E}^N(\mu)\right\|_X + \rho_1(\mu)\left\|e^N(\mu)\right\|_X^2 \left\|\mathcal{E}^N(\mu)\right\|_X. \end{aligned}$$

Next, we sum up latter two inequalities

$$\begin{aligned} &\left\|\mathcal{R}^N(\mu)\right\|_X^2 - \left\|\mathcal{E}^N(\mu)\right\|_X^2 \\ &\quad\le \Big(\gamma^N(\mu)\left\|e^N(\mu)\right\|_X + \rho_1(\mu)\left\|e^N(\mu)\right\|_X^2\Big)\big(\left\|\mathcal{R}^N(\mu)\right\|_X + \left\|\mathcal{E}^N(\mu)\right\|_X\big), \end{aligned}$$

thus

$$\left\|\mathcal{R}^N(\mu)\right\|_X - \left\|\mathcal{E}^N(\mu)\right\|_X \le \left\|e^N(\mu)\right\|_X \big(\gamma^N(\mu) + \rho_1(\mu)\left\|e^N(\mu)\right\|_X\big),$$

and by (4.7)

$$C(\mu)\big(\left\|\mathcal{R}^N(\mu)\right\|_X + \left\|\mathcal{E}^N(\mu)\right\|_X\big) \le \left\|e^N(\mu)\right\|_X \big(\gamma^N(\mu) + \rho_1(\mu)\left\|e^N(\mu)\right\|_X\big). \tag{4.10}$$

Now, we estimate $\Delta^N(\mu)$. First, note that $1 - \sqrt{1 - \tau^N(\mu)} \le \tau^N(\mu)$ which follows from

$$\tau^N(\mu) \le \frac{1}{2}C(\mu) = \frac{1}{2}\frac{1 - c(\mu)}{1 + c(\mu)} \le 1$$

since $c(\mu) \in [0, 1)$. Hence,

$$\begin{aligned} \Delta^N(\mu) &= \overline{\beta}^N(\mu)(2\rho_1(\mu))^{-1}\left(1 - \sqrt{1 - \tau^N(\mu)}\right) \\ &\le \overline{\beta}^N(\mu)(2\rho_1(\mu))^{-1}\,\tau^N(\mu) \\ &= 2\Big(\overline{\beta}^N(\mu)\Big)^{-1}\big(R^N(\mu) + E^N(\mu)\big), \end{aligned} \tag{4.11}$$

where for the last equality we used the definition of $\tau^N(\mu)$ (c.p. (3.6)).

The next step is to use (4.10), (4.11) and Proposition 3.1 to estimate

$$\begin{aligned}\frac{1}{2}C(\mu)\overline{\beta}^N(\mu)\Delta^N(\mu) &\leq \gamma^N(\mu)\left\|e^N(\mu)\right\|_X + \rho_1(\mu)\left\|e^N(\mu)\right\|_X^2 \\ &\leq \gamma^N(\mu)\left\|e^N(\mu)\right\|_X + \rho_1(\mu)\big(\Delta^N(\mu)\big)^2 \\ &= \gamma^N(\mu)\left\|e^N(\mu)\right\|_X + \frac{1}{2}\Delta^N(\mu)\big(2\rho_1(\mu)\Delta^N(\mu)\big).\end{aligned}$$

Finally, we employ (4.11) again (i.e., the second line) to obtain

$$\frac{1}{2}C(\mu)\overline{\beta}^N(\mu)\Delta^N(\mu) \leq \gamma^N(\mu)\left\|e^N(\mu)\right\|_X + \frac{1}{2}\tau^N(\mu)\overline{\beta}^N(\mu)\Delta^N(\mu),$$

which is equivalent to

$$\begin{aligned}\eta^N(\mu) = \frac{\Delta^N(\mu)}{\|e^N(\mu)\|_X} &\leq 2\frac{\gamma^N(\mu)}{\overline{\beta}^N(\mu)}\big(C(\mu) - \tau^N(\mu)\big)^{-1} \\ &\leq 4(C(\mu))^{-1}\,\kappa^N(\mu),\end{aligned}$$

since $\tau^N(\mu) \leq \frac{1}{2}C(\mu)$, which proofs the claim (4.6). □

Offline/Online Decomposition. As we are not only interested in the rapid evaluation of $\widehat{u}^N(\mu)$ and $\underline{\widehat{u}}^N(\mu)$, respectively, but also of its a-posteriori error estimator $\Delta^N(\mu)$, we have to perform the offline/online decomposition for $\Delta^N(\mu)$, too. We will separately treat $R^N(\mu)$ and $E^N(\mu)$, starting with the first one.

From duality we know, that $\mathcal{R}^N(\mu) \in X$ exist, such that $\left\|\mathcal{R}^N(\mu)\right\|_X = R^N(\mu)$. Furthermore, by linear superposition, we find

$$\mathcal{R}^N(\mu) = \mathcal{F} + \sum_{n_1=1}^{N}\widehat{u}^N_{n_1}(\mu)\sum_{m=1}^{M_0}\left[\vartheta^0_m(\mu)\mathcal{A}^m_{n_1} + \sum_{n_2=1}^{N}\widehat{u}^N_{n_2}(\mu)\sum_{m=1}^{M_1}\vartheta^1_m(\mu)\mathcal{A}^m_{n_1,n_2}\right],$$

where we have defined $\mathcal{F}$, $\mathcal{A}^m_{n_1}$ and $\mathcal{A}^m_{n_1,n_2}$, such that for all $v \in X$

$$\begin{aligned}(\mathcal{F}, v)_X &= -f(v), \\ \big(\mathcal{A}^m_{n_1}, v\big)_X &= a_0(\xi_{n_1}, v; \varphi^0_m), \qquad 1 \leq m \leq M_0 < M_0^{\max}, \\ \big(\mathcal{A}^m_{n_1,n_2}, v\big)_X &= a_1(\xi_{n_1}, \xi_{n_2}, v; \varphi^1_m), \quad 1 \leq m \leq M_1 < M_1^{\max},\end{aligned}$$

each with $1 \leq n_1, n_2 \leq N$. Consequently, we can compute $\left\|\mathcal{R}^N(\mu)\right\|_X$ by the following nested quadruple sum:

$$\begin{aligned}\left\|\mathcal{R}^N(\mu)\right\|_X^2 = {} & (\mathcal{F},\mathcal{F})_X + \sum_{n_1=1}^{N}\widehat{u}^N_{n_1}(\mu)\Bigg\{2\sum_{m=1}^{M_0}\vartheta^0_m(\mu)\big(\mathcal{F},\mathcal{A}^m_{n_1}\big)_X \\ & + \sum_{n_2=1}^{N}\widehat{u}^N_{n_2}(\mu)\Bigg\{2\sum_{m=1}^{M_1}\vartheta^1_m(\mu)\big(\mathcal{F},\mathcal{A}^m_{n_1,n_2}\big)_X \\ & + \sum_{m=1}^{M_0}\sum_{m'=1}^{M_0}\vartheta^0_m(\mu)\vartheta^0_{m'}(\mu)\Big(\mathcal{A}^m_{n_1},\mathcal{A}^{m'}_{n_2}\Big)_X \\ & + \sum_{n_3=1}^{N}\widehat{u}^N_{n_3}(\mu)\Bigg\{2\sum_{m=1}^{M_0}\sum_{m'=1}^{M_1}\vartheta^0_m(\mu)\vartheta^1_{m'}(\mu)\Big(\mathcal{A}^m_{n_1},\mathcal{A}^{m'}_{n_2,n_3}\Big)_X \\ & + \sum_{n_4=1}^{N}\widehat{u}^N_{n_4}(\mu)\sum_{m=1}^{M_1}\sum_{m'=1}^{M_1}\vartheta^1_m(\mu)\vartheta^1_{m'}(\mu)\Big(\mathcal{A}^m_{n_1,n_2},\mathcal{A}^{m'}_{n_3,n_4}\Big)_X\Bigg\}\Bigg\}\Bigg\}.\end{aligned}$$

Hence, the online complexity for evaluating $R^N(\mu)$ is

$$\mathcal{O}\big(M_1^2N^4\big) + \mathcal{O}\big(M_0M_1N^3\big) + \mathcal{O}\big(\big(M_0^2 + M_1\big)\,N^2\big) + \mathcal{O}(M_0N).$$

Next, we derive an offline/online decomposition for the evaluation of $E^N(\mu)$. First, note that in the case of non-affine coefficient functions $h(\mu)$ the error within the empirical interpolation, $h(\mu) - \widehat{h}(\mu)$, is non-affine, too. As pointed out earlier, non-affine dependencies prohibit an efficient offline/online decomposition, i.e., a decomposition such that the online complexity is independent of $\mathcal{N}$. The usual way to overcome this (c.p. [1, 9]), is to pose the assumption $h_i(\mu) \in W_i^{M_i+1}$, $i = 0, 1$, such that we can take advantage of (2.16). However, the price to pay is to loose complete rigorousity of $\Delta^N(\mu)$. This loss is usually compensated by posing an additional 'Safety-Condition' (c.p. [9]), i.e., M_i, $i = 0, 1$ should be chosen sufficiently large, such that

$$\frac{E^N(\mu)}{R^N(\mu)} \leq \frac{1}{2}, \tag{4.12}$$

which fits in the context of Proposition 4.1, too.

With (2.16) at hand, we can proceed similar to the investigation of $R^N(\mu)$. Due to duality there exists $\mathcal{E}^N(\mu) \in X$, such that $\left\|\mathcal{E}^N(\mu)\right\|_X = E^N(\mu)$, where by linear superposition

$$\mathcal{E}^N(\mu) = \sum_{n_1=1}^{N} \widehat{u}_{n_1}^N(\mu)\left[\varepsilon_0^{M_0}(\mu)\mathcal{A}_{n_1}^{M_0+1} + \sum_{n_2=1}^{N} \widehat{u}_{n_2}^N(\mu)\varepsilon_1^{M_1}(\mu)\mathcal{A}_{n_1,n_2}^{M_1+1}\right].$$

Therefore, $\left\|\mathcal{E}^N(\mu)\right\|_X$ can be computed by the following nested quadruple sum:

$$\begin{aligned}\left\|\mathcal{E}^N(\mu)\right\|_X^2 \;=\; & \sum_{n_1=1}^{N} \widehat{u}_{n_1}^N(\mu) \sum_{n_2=1}^{N} \widehat{u}_{n_2}^N(\mu)\bigg\{ \Big(\varepsilon_0^{M_0}(\mu)\Big)^2 \big(\mathcal{A}_{n_1}^{M_0+1}, \mathcal{A}_{n_2}^{M_0+1}\big)_X \\ & + \sum_{n_3=1}^{N} \widehat{u}_{n_3}^N(\mu)\bigg\{ 2\varepsilon_0^{M_0}(\mu)\varepsilon_1^{M_1}(\mu)\big(\mathcal{A}_{n_1}^{M_0+1}, \mathcal{A}_{n_2,n_3}^{M_1+1}\big)_X \\ & + \sum_{n_4=1}^{N} \widehat{u}_{n_4}^N(\mu)\Big(\varepsilon_1^{M_1}(\mu)\Big)^2 \big(\mathcal{A}_{n_1,n_2}^{M_1+1}, \mathcal{A}_{n_3,n_4}^{M_1+1}\big)_X \bigg\}\bigg\},\end{aligned}$$

with complexity $\mathcal{O}(N^4)$.

4.2. **The Dual Problem.** Also for the dual problem, the a-posteriori error analysis differs from [16]. We start by defining analogous quantities for $R^N(\mu)$ in (3.1) and $E^N(\mu)$ in (3.2) namely

$$\widetilde{R}^{N,\widetilde{N}}(\mu) := \left\| dg\Big(\cdot, \widehat{\psi}^{N,\widetilde{N}}(\mu); \widehat{h}(\mu)\Big)\,[\widehat{u}^N(\mu)] + \ell(\cdot)\right\|_{X'}, \tag{4.13}$$

the dual norm of the dual RBM-EIM residual and

$$\widetilde{E}^{N,\widetilde{N}}(\mu) := \left\| dg\Big(\cdot, \widehat{\psi}^{N,\widetilde{N}}(\mu); h(\mu) - \widehat{h}(\mu)\Big)\,[\widehat{u}^N(\mu)]\right\|_{X'}, \tag{4.14}$$

the EIM-approximation error. We also introduce the quantities

$$\widetilde{e}^{N,\widetilde{N}}(\mu) := \psi^N(\mu) - \widehat{\psi}^{N,\widetilde{N}}(\mu), \tag{4.15}$$

where $\psi^N(\mu)$ denotes the solution of the *Dual Problem* (2.29), whereas $\widehat{\psi}^{N,\widetilde{N}}(\mu)$ is the solution of the *Dual RBM-EIM-Problem* (2.34). The error bound is

$$\widetilde{\Delta}^{N,\widetilde{N}}(\mu) \;:=\; \frac{2\Big(\widetilde{R}^{N,\widetilde{N}}(\mu) + \widetilde{E}^{N,\widetilde{N}}(\mu)\Big)}{\overline{\beta}^N(\mu)\Big(1 + \sqrt{1 - \tau^N(\mu)}\Big)} + \frac{1 - \sqrt{1 - \tau^N(\mu)}}{1 + \sqrt{1 - \tau^N(\mu)}} \left\|\widehat{\psi}^{N,\widetilde{N}}(\mu)\right\|_X, \tag{4.16}$$

and we obtain the following error estimate.

Proposition 4.2. *If $\tau^N(\mu) < 1$, then*

$$\left\|\widetilde{e}^{N,\widetilde{N}}(\mu)\right\|_X \leq \widetilde{\Delta}^{N,\widetilde{N}}(\mu). \tag{4.17}$$

Proof. Recall that $\psi^N(\mu) \in X$ is the solution of (2.29), i.e.,

$$dg\Big(v, \psi^N(\mu); h(\mu))[\widehat{u}^N(\mu) + \frac{1}{2}e^N(\mu)]\Big) = -\ell(v), \quad v \in X,$$

for $e^N(\mu) = u(\mu) - \widehat{u}^N(\mu)$ and the solutions $u(\mu) \in X$ and $\widehat{u}^N(\mu) \in W^N$ of (2.3) and (2.21), respectively.

Then, straightforward calculations show that

$$\begin{aligned}
&dg\Big(v, \widetilde{e}^{N,\widetilde{N}}(\mu); h(\mu)\Big)\,[\widehat{u}^N(\mu)] = \\
&= \; dg\big(v, \psi^N(\mu); h(\mu)\big)\,[\widehat{u}^N(\mu) + \frac{1}{2}e^N(\mu)] - dg\big(v, \psi^N(\mu); h(\mu)\big)\,[\frac{1}{2}e^N(\mu)] \\
&\quad - dg\Big(v, \widehat{\psi}^{N,\widetilde{N}}(\mu); h(\mu) - \widehat{h}(\mu)\Big)\,[\widehat{u}^N(\mu)] \\
&\quad - dg\Big(v, \widehat{\psi}^{N,\widetilde{N}}(\mu); \widehat{h}(\mu)\Big)\,[\widehat{u}^N(\mu)] \\
&= \; -\ell(v) - dg\Big(v, \widehat{\psi}^{N,\widetilde{N}}(\mu); \widehat{h}(\mu)\Big)\,[\widehat{u}^N(\mu)] \\
&\quad - dg\Big(v, \widehat{\psi}^{N,\widetilde{N}}(\mu); h(\mu) - \widehat{h}(\mu)\Big)\,[\widehat{u}^N(\mu)] \\
&\quad - dg\big(v, \psi^N(\mu); h(\mu)\big)\,[\frac{1}{2}e^N(\mu)].
\end{aligned}$$

Using the inf-sup condition (2.6) for dg yields

$$\begin{aligned}
\overline{\beta}^N(\mu)\left\|\widetilde{e}^{N,\widetilde{N}}(\mu)\right\|_X &\leq \sup_{v\in X} \frac{1}{\|v\|_X}\left\|dg\Big(v, \widetilde{e}^{N,\widetilde{N}}(\mu); h(\mu)\Big)\,[\widehat{u}^N(\mu)]\right\|_X \\
&\leq \widetilde{R}^{N,\widetilde{N}}(\mu) + \widetilde{E}^{N,\widetilde{N}}(\mu) \\
&\qquad + \left\|dg\big(\cdot, \psi^N(\mu); h(\mu)\big)\,[\frac{1}{2}e^N(\mu)]\right\|_{X'}.
\end{aligned} \tag{4.18}$$

Hence, we need to estimate the last term. In order to do so, note that

$$\begin{aligned}
&dg\big(v, \psi^N(\mu); h(\mu)\big)\,[\frac{1}{2}e^N(\mu)] \\
&= \; \frac{1}{2}dg\Big(v, \psi^N(\mu) - \widehat{\psi}^{N,\widetilde{N}}(\mu); h(\mu)\Big)\,[e^N(\mu)] \\
&\quad + \frac{1}{2}dg\Big(v, \widehat{\psi}^{N,\widetilde{N}}(\mu); h(\mu)\Big)\,[e^N(\mu)] \\
&\leq \; \rho_1(\mu)\,\|e^N(\mu)\|_X\Big(\left\|\widetilde{e}^{N,\widetilde{N}}(\mu)\right\|_X + \left\|\widehat{\psi}^{N,\widetilde{N}}(\mu)\right\|_X\Big)\,\|v\|_X \\
&\leq \; \rho_1(\mu)\Delta^N(\mu)\Big(\left\|\widetilde{e}^{N,\widetilde{N}}(\mu)\right\|_X + \left\|\widehat{\psi}^{N,\widetilde{N}}(\mu)\right\|_X\Big)\,\|v\|_X\,,
\end{aligned}$$

in view of Proposition 3.1 and (3.8), respectively. This leads us with (4.18) to

$$\begin{aligned}
\overline{\beta}^N(\mu)\left\|\widetilde{e}^{N,\widetilde{N}}(\mu)\right\|_X &\leq \widetilde{R}^{N,\widetilde{N}}(\mu) + \widetilde{E}^{N,\widetilde{N}}(\mu) \\
&\qquad + \rho_1(\mu)\Delta^N(\mu)\Big(\left\|\widetilde{e}^{N,\widetilde{N}}(\mu)\right\|_X + \left\|\widehat{\psi}^{N,\widetilde{N}}(\mu)\right\|_X\Big),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}\left\|\widetilde{e}^{N,\widetilde{N}}(\mu)\right\|_X &\le \frac{1}{\overline{\beta}^N(\mu)-\rho_1(\mu)\Delta^N(\mu)}\Big(\widetilde{R}^{N,\widetilde{N}}(\mu)+\widetilde{E}^{N,\widetilde{N}}(\mu)\Big)\\ &\quad+\frac{\rho_1(\mu)\Delta^N(\mu)}{\overline{\beta}^N(\mu)-\rho_1(\mu)\Delta^N(\mu)}\left\|\widehat{\psi}^{N,\widetilde{N}}(\mu)\right\|_X.\end{aligned} \tag{4.19}$$

Finally, by (3.7) we have

$$\rho_1(\mu)\Delta^N(\mu)=\frac{1}{2}\overline{\beta}^N(\mu)\Big(1-\sqrt{1-\tau^N(\mu)}\Big),$$

and, furthermore,

$$\overline{\beta}^N(\mu)-\rho_1(\mu)\Delta^N(\mu)=\frac{1}{2}\overline{\beta}^N(\mu)\Big(1+\sqrt{1-\tau^N(\mu)}\Big).$$

Together with (4.19) this proofs the claim (4.17) for $\widetilde{\Delta}^{N,\widetilde{N}}(\mu)$ in (4.16). □

Offline/Online Decomposition. For the offline/online decomposition of $\widetilde{\Delta}^{N,\widetilde{N}}(\mu)$ we can proceed along the lines of Section 4.1. First, from duality we know the existence of $\widetilde{\mathcal{R}}^{N,\widetilde{N}}(\mu)\in X$, such that $\left\|\widetilde{\mathcal{R}}^{N,\widetilde{N}}(\mu)\right\|_X=\widetilde{R}^{N,\widetilde{N}}(\mu)$. Furthermore, by linear superposition, we find

$$\widetilde{\mathcal{R}}^{N,\widetilde{N}}(\mu)=\mathcal{L}+\sum_{n_1=1}^{\widetilde{N}}\widehat{\psi}_{n_1}^{N,\widetilde{N}}(\mu)\sum_{m=1}^{M_0}\left[\vartheta_m^0(\mu)\mathcal{B}_{n_1}^m+\sum_{n_2=1}^{N}\widehat{u}_{n_2}^N(\mu)\sum_{m=1}^{M_1}\vartheta_m^1(\mu)\mathcal{B}_{n_1,n_2}^m\right],$$

where we have defined $\mathcal{L}$, $\mathcal{B}_{n_1}^m$ and $\mathcal{B}_{n_1,n_2}^m$, such that for all $v\in X$

$$\begin{aligned}(\mathcal{L},v)_X &= \ell(v),\\ \big(\mathcal{B}_{n_1}^m,v\big)_X &= a_0(v,\widetilde{\xi}_{n_1};\varphi_m^0), && 1\le m\le M_0<M_0^{\max},\\ \big(\mathcal{B}_{n_1,n_2}^m,v\big)_X &= da_1(v,\widetilde{\xi}_{n_1};\varphi_m^1)[\xi_{n_2}], && 1\le m\le M_1<M_1^{\max},\end{aligned}$$

each for $1\le n_1\le\widetilde{N}$, $1\le n_2\le N$. Consequently, we can compute $\left\|\widetilde{\mathcal{R}}^{N,\widetilde{N}}(\mu)\right\|_X$ by the following quadruple sum:

$$\begin{aligned}\left\|\widetilde{\mathcal{R}}^{N,\widetilde{N}}(\mu)\right\|_X^2 &= (\mathcal{L},\mathcal{L})_X+\sum_{n_1=1}^{\widetilde{N}}\widehat{\psi}_{n_1}^{N,\widetilde{N}}(\mu)\Bigg\{2\sum_{m=1}^{M_0}\vartheta_m^0(\mu)\big(\mathcal{L},\mathcal{B}_{n_1}^m\big)_X\\ &\quad+2\sum_{n_2=1}^{N}\widehat{u}_{n_2}^N(\mu)\sum_{m=1}^{M_1}\vartheta_m^1(\mu)\big(\mathcal{L},\mathcal{B}_{n_1,n_2}^m\big)_X\\ &\quad+\sum_{n_2=1}^{\widetilde{N}}\widehat{\psi}_{n_2}^{N,\widetilde{N}}(\mu)\Bigg\{\sum_{m=1}^{M_0}\sum_{m'=1}^{M_0}\vartheta_m^0(\mu)\vartheta_{m'}^0(\mu)\Big(\mathcal{B}_{n_1}^m,\mathcal{B}_{n_2}^{m'}\Big)_X\\ &\quad+\sum_{n_3=1}^{N}\widehat{u}_{n_3}^N(\mu)\Bigg\{2\sum_{m=1}^{M_0}\sum_{m'=1}^{M_1}\vartheta_m^0(\mu)\vartheta_{m'}^1(\mu)\Big(\mathcal{B}_{n_1}^m,\mathcal{B}_{n_2,n_3}^{m'}\Big)_X\\ &\quad+\sum_{n_4=1}^{N}\widehat{u}_{n_4}^N(\mu)\sum_{m=1}^{M_1}\sum_{m'=1}^{M_1}\vartheta_m^1(\mu)\vartheta_{m'}^1(\mu)\Big(\mathcal{B}_{n_1,n_3}^m,\mathcal{B}_{n_2,n_4}^{m'}\Big)_X\Bigg\}\Bigg\}\Bigg\}.\end{aligned}$$

Thus, the online complexity for evaluating $\widetilde{R}^{N,\widetilde{N}}(\mu)$ is

$$\mathcal{O}\Big(M_1^2N^2\widetilde{N}^2\Big)+\mathcal{O}\Big(M_0M_1N\widetilde{N}^2\Big)+\mathcal{O}\Big(M_0^2\widetilde{N}^2\Big)+\mathcal{O}\Big(M_1N\widetilde{N}\Big)+\mathcal{O}\Big(M_0\widetilde{N}\Big).$$

Next, to derive an offline/online decomposition for $\widetilde{E}^{N,\widetilde{N}}(\mu)$ we again have to assume that $h_i(\mu) \in W_i^{M_i+1}$, $i = 0, 1$, hence (2.16) holds. Furthermore, in analogy to (4.12) we assume the following 'Safety-Condition' to hold:

$$\frac{\widetilde{E}^{N,\widetilde{N}}(\mu)}{\widetilde{R}^{N,\widetilde{N}}(\mu)} \leq \frac{1}{2}.$$

Again, due to duality there exists $\widetilde{\mathcal{E}}^{N,\widetilde{N}}(\mu) \in X$, such that $\left\|\widetilde{\mathcal{E}}^{N,\widetilde{N}}(\mu)\right\|_X = \widetilde{E}^{N,\widetilde{N}}(\mu)$, where by linear superposition (thanks to (2.16))

$$\widetilde{\mathcal{E}}^{N,\widetilde{N}}(\mu) = \sum_{n_1=1}^{\widetilde{N}} \widehat{\psi}_{n_1}^{N,\widetilde{N}}(\mu) \left[\varepsilon_0^{M_0}(\mu)\mathcal{B}_{n_1}^{M_0+1} + \sum_{n_2=1}^{N} \widehat{u}_{n_2}^N(\mu)\varepsilon_1^{M_1}(\mu)\mathcal{B}_{n_1,n_2}^{M_1+1}\right],$$

and, furthermore, due to

$$\begin{aligned} \left\|\widetilde{\mathcal{E}}^{N,\widetilde{N}}(\mu)\right\|_X^2 &= \sum_{n_1=1}^{\widetilde{N}} \widehat{\psi}_{n_1}^{N,\widetilde{N}}(\mu) \sum_{n_2=1}^{\widetilde{N}} \widehat{\psi}_{n_2}^{N,\widetilde{N}}(\mu)\Bigg\{\left(\varepsilon_0^{M_0}(\mu)\right)^2 \left(\mathcal{B}_{n_1}^{M_0+1}, \mathcal{B}_{n_2}^{M_0+1}\right)_X \\ &\quad + \sum_{n_3=1}^{N} \widehat{u}_{n_3}^N(\mu)\Bigg\{2\varepsilon_0^{M_0}(\mu)\varepsilon_1^{M_1}(\mu)\left(\mathcal{B}_{n_1}^{M_0+1}, \mathcal{B}_{n_2,n_3}^{M_1+1}\right)_X \\ &\quad + \sum_{n_4=1}^{N} \widehat{u}_{n_4}^N(\mu)\left(\varepsilon_1^{M_1}(\mu)\right)^2 \left(\mathcal{B}_{n_1,n_3}^{M_1+1}, \mathcal{B}_{n_2,n_4}^{M_1+1}\right)_X \Bigg\}\Bigg\}, \end{aligned}$$

the complexity for evaluating $\widetilde{E}^{N,\widetilde{N}}(\mu)$ is $\mathcal{O}\left(N^2\widetilde{N}^2\right)$.

4.3. **Output of Interest.** As already mentioned in Section 2.4, the development of a-posteriori error estimators for $s(\mu)$ is mainly based on Lemma 2.1. Again, the analysis slightly differs from [16]. Furthermore, we define an additional output approximation and a corresponding a-posteriori error estimator.

Note that, if we do not (want to) take advantage of the dual problem formulation, i.e., using the primal problem for the output approximation only, we define

$$\widehat{s}_1^N(\mu) := \ell\big(\widehat{u}^N(\mu)\big)\,, \tag{4.20}$$

and the error bound

$$\Delta_{s_1}^N(\mu) := \|\ell\|_{X'}\,\Delta^N(\mu). \tag{4.21}$$

Then, we obtain the following error estimate.

Proposition 4.3. *If* $\tau^N(\mu) < 1$*, then*

$$\left|s(\mu) - \widehat{s}_1^N(\mu)\right| \leq \Delta_{s_1}^N(\mu). \tag{4.22}$$

Proof. The result directly follows from the continuity of ℓ and Proposition 3.1, as

$$\begin{aligned} \left|s(\mu) - \widehat{s}_1^N(\mu)\right| &= \left|\ell\big(u(\mu) - \widehat{u}^N(\mu)\big)\right| \\ &= \frac{\left|\ell\big(u(\mu) - \widehat{u}^N(\mu)\big)\right|}{\|u(\mu) - \widehat{u}^N(\mu)\|_X} \left\|u(\mu) - \widehat{u}^N(\mu)\right\|_X \leq \|\ell\|_{X'}\,\Delta^N(\mu), \end{aligned}$$

which proves the claim. □

Next, if we aim at improving the error bound for $\widehat{s}_1^N(\mu)$ using the dual problem (c.p. Section 2.4) we can proceed similar to [16] and define the error bound by

$$\begin{aligned} \widetilde{\Delta}_{s_1}^{N,\widetilde{N}}(\mu) &:= \left|R^N(\widehat{\psi}^{N,\widetilde{N}}(\mu);\mu)\right| + \left|E^N(\widehat{\psi}^{N,\widetilde{N}}(\mu);\mu)\right| \\ &\quad + \big(R^N(\mu) + E^N(\mu)\big)\,\widetilde{\Delta}^{N,\widetilde{N}}(\mu). \end{aligned} \tag{4.23}$$

Then, the improved error estimate reads as follows.

Proposition 4.4. *If* $\tau^N(\mu) < 1$, *then*

$$\left|s(\mu) - \widehat{s}_1^N(\mu)\right| \leq \widetilde{\Delta}_{s_1}^{N,\widetilde{N}}(\mu). \tag{4.24}$$

Proof. From Lemma 2.1 we find for $v = \widehat{\psi}^{N,\widetilde{N}}(\mu)$

$$\begin{aligned} s(\mu) - \widehat{s}_1^N(\mu) &= g\Big(\widehat{u}^N(\mu), \widehat{\psi}^{N,\widetilde{N}}(\mu); h(\mu)\Big) \\ &\quad + g\Big(\widehat{u}^N(\mu), \psi^N(\mu) - \widehat{\psi}^{N,\widetilde{N}}(\mu); h(\mu)\Big). \end{aligned} \tag{4.25}$$

We can estimate the right-hand side by

$$\begin{aligned} \left|g\Big(\widehat{u}^N(\mu), \widehat{\psi}^{N,\widetilde{N}}(\mu); h(\mu)\Big)\right| &\leq \left|g\Big(\widehat{u}^N(\mu), \widehat{\psi}^{N,\widetilde{N}}(\mu); \widehat{h}(\mu)\Big)\right| \\ &\quad + \left|g\Big(\widehat{u}^N(\mu), \widehat{\psi}^{N,\widetilde{N}}(\mu); h(\mu) - \widehat{h}(\mu)\Big)\right| \\ &= \left|R^N(\widehat{\psi}^{N,\widetilde{N}}(\mu); \mu)\right| + \left|E^N(\widehat{\psi}^{N,\widetilde{N}}(\mu); \mu)\right|, \end{aligned}$$

and

$$\begin{aligned} &\left|g\Big(\widehat{u}^N(\mu), \psi^N(\mu) - \widehat{\psi}^{N,\widetilde{N}}(\mu); h(\mu)\Big)\right| \\ &\quad \leq \left\|g\big(\widehat{u}^N(\mu), \cdot; h(\mu)\big)\right\|_{X'} \left\|\psi^N(\mu) - \widehat{\psi}^{N,\widetilde{N}}(\mu)\right\|_X \\ &\quad \leq \big(R^N(\mu) + E^N(\mu)\big)\, \widetilde{\Delta}^{N,\widetilde{N}}(\mu), \end{aligned}$$

by Proposition 4.2 and (4.17), respectively. Using the triangle inequality for (4.25) and applying the above estimates completes the proof. □

However, if we are not only interested in improving the error bound $\Delta_{s_1}^N(\mu)$, but $\widehat{s}_1^N(\mu)$ itself, we can proceed as follows. Let

$$\widehat{s}_2^{N,\widetilde{N}}(\mu) := \widehat{s}_1^N(\mu) + R^N(\widehat{\psi}^{N,\widetilde{N}}(\mu); \mu) \tag{4.26}$$

and the corresponding error bound

$$\widetilde{\Delta}_{s_2}^{N,\widetilde{N}}(\mu) := \left|E^N(\widehat{\psi}^{N,\widetilde{N}}(\mu); \mu)\right| + \big(R^N(\mu) + E^N(\mu)\big)\, \widetilde{\Delta}^{N,\widetilde{N}}(\mu). \tag{4.27}$$

Then, we obtain the following result.

Proposition 4.5. *If* $\tau^N(\mu) < 1$, *then*

$$\left|s(\mu) - \widehat{s}_2^{N,\widetilde{N}}(\mu)\right| \leq \widetilde{\Delta}_{s_2}^{N,\widetilde{N}}(\mu). \tag{4.28}$$

Proof. Note, that from (4.25) and (4.26) we have

$$\begin{aligned} s(\mu) - \widehat{s}_2^{N,\widetilde{N}}(\mu) &= g\Big(\widehat{u}^N(\mu), \widehat{\psi}^{N,\widetilde{N}}(\mu); h(\mu) - \widehat{h}(\mu)\Big) \\ &\quad + g\Big(\widehat{u}^N(\mu), \psi^N(\mu) - \widehat{\psi}^{N,\widetilde{N}}(\mu); h(\mu)\Big). \end{aligned}$$

The result can be obtained straightforward by proceeding along the lines of the proof of Proposition 4.4. □

Note that $\widehat{s}_2^{N,\widetilde{N}}(\mu)$ and $\widetilde{\Delta}_{s_2}^{N,\widetilde{N}}(\mu)$, respectively, will give a better result than $\widehat{s}_1^N(\mu)$ in conjunction with $\widetilde{\Delta}_{s_1}^{N,\widetilde{N}}(\mu)$. On the other hand, the effectivity of $\widetilde{\Delta}_{s_1}^{N,\widetilde{N}}(\mu)$ will in general be better (i.e., closer to one), than the one of $\widetilde{\Delta}_{s_2}^{N,\widetilde{N}}(\mu)$. We will point this out later in our numerical results.

Offline/Online Decomposition. The offline/online decomposition of the additional terms needed for $\widetilde{\Delta}_{s_1}^{N,\widetilde{N}}(\mu)$ and $\widetilde{\Delta}_{s_2}^{N,\widetilde{N}}(\mu)$, respectively, namely $R^N(\widehat{\psi}^{N,\widetilde{N}}(\mu);\mu)$ and $E^N(\widehat{\psi}^{N,\widetilde{N}}(\mu);\mu)$ is readily admitted. The online complexities are $\mathcal{O}\Big(M_1\widetilde{N}N^2\Big)+\mathcal{O}\Big(M_0\widetilde{N}N\Big)$ and $\mathcal{O}\Big(\widetilde{N}N^2\Big)+\mathcal{O}\Big(\widetilde{N}N\Big)$, respectively, where for the latter one we have to take advantage of the assumption $h_i(\mu)\in W_i^{M_i+1}$, $i=0,1$, again (c.p. Section 4.1).

4.4. **The Continuity Constants.** To finish the derivation of rapidly evaluable a-posteriori error estimators, we will now address the computation/estimation of $\rho_i(\mu)$, $i=0,1$, and $\overline{\beta}^N(\mu)$. We start by $\rho_i(\mu)$.

The determination of the continuity constants from Assumption 2.1, i.e., ρ_i, $i=0,1$, themselves, is of course highly dependent on the particular problem to solve. At this point we just want to mention, that the determination of ρ_1 will involve a Sobolev embedding constant (due to the quadratic non-linearity), which can be determined by a homotopy procedure, that is detailed in [16].

Compared to [16], due to the more sophisticated dependence on the parameter, for the determination of $\rho_i(\mu)$, $i=0,1$, we have to estimate the $L_\infty(\Omega)$-norm of the the involved coefficient functions $h_i(\mu)$. To preserve the independence of $\mathcal{N}$, we can use the following estimate

$$
\begin{aligned}
\|h_i(\mu)\|_{L_\infty(\Omega)} &\leq \left\|\widehat{h}_i(\mu)\right\|_{L_\infty(\Omega)}+\left\|h_i(\mu)-\widehat{h}_i(\mu)\right\|_{L_\infty(\Omega)} \\
&\leq \left\|\underline{\vartheta}^i(\mu)\right\|_1+\varepsilon_{\mathrm{emp}},
\end{aligned}
\tag{4.29}
$$

where $\underline{\vartheta}^i(\mu)$ is the vector of the expansion coefficients $\vartheta^i_m(\mu)$, $1\leq m\leq M_i$, due to (2.13)–(2.15) and the triangle-inequality.

4.5. **The Inf-Sup Constant.** Now, we consider the computation of $\overline{\beta}^N(\mu)$, i.e., a rapidly evaluable (w.r.t. the online stage) lower bound for the inf-sup constant $\beta^N(\mu)$. Due to the presence of a non-affine parameter dependence, we will split the derivation into two parts (this idea has been pursued in e.g. [9], too). First, we will construct a lower bound for $\beta(\widehat{u}^N(\mu);\widehat{h}(\mu))$. Afterwards, we will introduce a correction term to derive a lower bound for $\beta^N(\mu)$ itself.

Furthermore, note that we will restrict ourselves to the case of an one-dimensional parameter space, i.e., $\mathcal{D}\in\mathbb{R}$, only. Even though expanding to higher dimensional parameter spaces is readily admitted, more recent approaches (c.p. [4]) are advantageous in that case.

4.5.1. *Affine Parameter Dependence.* We will follow the main idea of [16], namely expanding $\beta^N(\mu)$ in the parameter $\mu\in\mathcal{D}$, although we have to generalize it significantly, due to the more sophisticated parameter dependence (c.p. (2.1)). As already said before, we will first treat affine-parameter dependence.

First, for $\mu\in\mathcal{D}$ and $w\in X$ we introduce a linear operator $T_\mu^N: X\to X$ by

$$
\left(T_\mu^N w,v\right)_X = dg\Big(w,v;\widehat{h}(\mu)\Big)\,[\widehat{u}^N(\mu)],\quad v\in X.
\tag{4.30}
$$

Using this, for $\bar{\mu} \in \mathcal{D}$ and $t \in \mathbb{R}$ we further introduce

$$\begin{aligned} \mathcal{T}(w, v; t; \bar{\mu}) \;:=\; & \big(T_{\bar{\mu}}^{N^{\max}} w, T_{\bar{\mu}}^{N^{\max}} v\big)_X \\ & + t\Big\{ dg\Big(w, T_{\bar{\mu}}^{N^{\max}} v; \partial_\mu \widehat{h}(\bar{\mu})\Big)\,[\widehat{u}^{N^{\max}}(\bar{\mu})] \\ & \quad + dg\Big(v, T_{\bar{\mu}}^{N^{\max}} w; \partial_\mu \widehat{h}(\bar{\mu})\Big)\,[\widehat{u}^{N^{\max}}(\bar{\mu})] \\ & \quad + da_1\Big(w, T_{\bar{\mu}}^{N^{\max}} v; \widehat{h}(\bar{\mu})\Big)\,[\partial_\mu \widehat{u}^{N^{\max}}(\bar{\mu})] \\ & \quad + da_1\Big(v, T_{\bar{\mu}}^{N^{\max}} w; \widehat{h}(\bar{\mu})\Big)\,[\partial_\mu \widehat{u}^{N^{\max}}(\bar{\mu})] \Big\}, \end{aligned} \tag{4.31}$$

where $\partial_\mu \widehat{h}(\bar{\mu})$ denotes $\partial \widehat{h}(\bar{\mu})/\partial\mu$ and $\partial_\mu \widehat{u}^{N^{\max}}(\bar{\mu}) \in X$ satisfies for all $v \in X$:

$$dg\Big(\partial_\mu \widehat{u}^{N^{\max}}(\bar{\mu}), v; \widehat{h}(\bar{\mu})\Big)\,[\widehat{u}^{N^{\max}}(\bar{\mu})] = -g\Big(\widehat{u}^{N^{\max}}(\bar{\mu}), v; \partial_\mu \widehat{h}(\bar{\mu})\Big) - f(v). \tag{4.32}$$

Finally, we use (4.31) to define the desired expansion of $\beta^N(\mu)$ around μ (or more precisely of $\big(\beta^N(\mu)\big)^2$), namely

$$\mathcal{F}(t; \bar{\mu}) := \inf_{v \in X} \frac{\mathcal{T}(v, v; t; \bar{\mu})}{\|v\|_X^2}. \tag{4.33}$$

Additionally, we will take advantage of a second order correction to the expansion defined above, namely for $\bar{\mu} \in \mathcal{D}$ and $t \in \mathbb{R}$:

$$\begin{aligned} \delta^N(t; \bar{\mu}) \;:=\; & \rho_0 \Big\| \widehat{h}_0(\bar{\mu} + t) - \widehat{h}_0(\bar{\mu}) - t\partial_\mu \widehat{h}_0(\bar{\mu}) \Big\|_{L_\infty(\Omega)} \\ & + 2\rho_1 \textstyle\sum_{m=1}^{M_1} \Big\| \vartheta_m^1(\bar{\mu} + t)\widehat{u}^N(\bar{\mu} + t) - \vartheta_m^1(\bar{\mu})\widehat{u}^{N^{\max}}(\bar{\mu}) \\ & \quad - t\big(\vartheta_m^1(\bar{\mu})\partial_\mu \widehat{u}^{N^{\max}}(\bar{\mu}) + \partial_\mu \vartheta_m^1(\bar{\mu})\widehat{u}^{N^{\max}}(\bar{\mu})\big) \Big\|_X. \end{aligned} \tag{4.34}$$

Now, with (4.30)–(4.34) at hand, we can show the following crucial properties.

Lemma 4.1. *The function $\mathcal{F}(t; \bar{\mu})$ is concave in t, i.e., for all $t \in [t_1, t_2]$ we have*

$$\mathcal{F}(t; \bar{\mu}) \geq \min\{\mathcal{F}(t_1; \bar{\mu}), \mathcal{F}(t_2; \bar{\mu})\}. \tag{4.35}$$

Proof. The result is an immediate consequence of (4.33) and the affine-linearity of $\mathcal{T}(w, v; t; \bar{\mu})$ in t (c.p. (4.31)). ☐

Furthermore, we have the following bound for the inf-sup constant.

Lemma 4.2. *For given $\bar{\mu}, \mu \in \mathcal{D}$ and $t = \mu - \bar{\mu}$, it holds*

$$\beta^N(\mu) \geq \Big(\sqrt{\mathcal{F}(t; \bar{\mu})} - \delta^N(t; \bar{\mu})\Big)^+. \tag{4.36}$$

Proof. We start by observing that for

$$\sigma(w) := \frac{\|T_\mu^N w\|_X}{\|w\|_X} \tag{4.37}$$

we have (c.p. (2.6))

$$\beta^N(\mu) = \inf_{w \in X} \sigma(w). \tag{4.38}$$

Using

$$T_\mu^N w = T_{\bar{\mu}}^{N^{\max}} w + \Big(T_\mu^N w - T_{\bar{\mu}}^{N^{\max}} w\Big), \tag{4.39}$$

we find for (4.37)

$$\begin{aligned} \sigma^2(w)\,\|w\|_X^2 \;=\; & \Big\|T_{\bar{\mu}}^{N^{\max}} w\Big\|_X^2 + \Big\|T_\mu^N w - T_{\bar{\mu}}^{N^{\max}} w\Big\|_X^2 \\ & + 2\Big(T_\mu^N w - T_{\bar{\mu}}^{N^{\max}} w, T_{\bar{\mu}}^{N^{\max}} w\Big)_X, \end{aligned} \tag{4.40}$$

or, equivalently,

$$\begin{aligned}(4.41)\qquad &\sigma^2(w)\,\|w\|_X^2 - \left\|T_\mu^N w - T_{\bar\mu}^{N^{\max}} w\right\|_X^2 - \mathcal{T}(w,w;t;\bar\mu)\\ &= \left\|T_{\bar\mu}^{N^{\max}} w\right\|_X^2 - \mathcal{T}(w,w;t;\bar\mu) + 2\Big(T_\mu^N w - T_{\bar\mu}^{N^{\max}} w, T_{\bar\mu}^{N^{\max}} w\Big)_X .\end{aligned}$$

First, we look at the scalar product in (4.40) to find by the definition of T_μ^N (c.p. (4.30)) and the definition of dg itself (c.p. (2.4)):

$$\begin{aligned}&\Big(T_\mu^N w - T_{\bar\mu}^{N^{\max}} w, T_{\bar\mu}^{N^{\max}} w\Big)_X\\ =\;& dg\Big(w, T_{\bar\mu}^{N^{\max}} w; \widehat{h}(\mu)\Big)\,[\widehat{u}^N(\mu)] - dg\Big(w, T_{\bar\mu}^{N^{\max}} w; \widehat{h}(\bar\mu)\Big)\,[\widehat{u}^{N^{\max}}(\bar\mu)]\\ =\;& a_0\Big(w, T_{\bar\mu}^{N^{\max}} w; \widehat{h}(\mu) - \widehat{h}(\bar\mu)\Big)\\ &+ \sum_{m=1}^{M_1} da_1\Big(w, T_{\bar\mu}^{N^{\max}} w; \varphi_m^1\Big)\,[\vartheta_m^1(\mu)\widehat{u}^N(\mu) - \vartheta_m^1(\bar\mu)\widehat{u}^{N^{\max}}(\bar\mu)].\end{aligned}$$

On the other hand, we have by the definition of $\mathcal{T}(w,v;t;\bar\mu)$ (c.p. (4.31)):

$$\begin{aligned}&\left\|T_{\bar\mu}^{N^{\max}} w\right\|_X^2 - \mathcal{T}(w,w;t;\bar\mu)\\ =\;& -2dg\Big(w, T_{\bar\mu}^{N^{\max}} w; t\partial_\mu\widehat{h}(\bar\mu)\Big)\,[\widehat{u}^{N^{\max}}(\bar\mu)]\\ &-2da_1\Big(w, T_{\bar\mu}^{N^{\max}} w; \widehat{h}(\bar\mu)\Big)\,[t\partial_\mu\widehat{u}^{N^{\max}}(\bar\mu)].\end{aligned}$$

Hence, for the right-hand side of (4.41) we find

$$\begin{aligned}&\left\|T_{\bar\mu}^{N^{\max}} w\right\|_X^2 - \mathcal{T}(w,w;t;\bar\mu) + 2\Big(T_\mu^N w - T_{\bar\mu}^{N^{\max}} w, T_{\bar\mu}^{N^{\max}} w\Big)_X\\ =\;& 2a_0\Big(w, T_{\bar\mu}^{N^{\max}} w; \widehat{h}(\mu) - \widehat{h}(\bar\mu) - t\partial_\mu\widehat{h}(\bar\mu)\Big)\\ &+ 2\sum_{m=1}^{M_1} da_1\Big(w, T_{\bar\mu}^{N^{\max}} w; \varphi_m^1\Big)\Big[\vartheta_m^1(\mu)\widehat{u}^N(\mu) - \vartheta_m^1(\bar\mu)\widehat{u}^{N^{\max}}(\bar\mu)\\ &\quad - t\Big(\vartheta_m^1(\bar\mu)\partial_\mu\widehat{u}^{N^{\max}}(\bar\mu) + \partial_\mu\vartheta_m^1(\bar\mu)\widehat{u}^{N^{\max}}(\bar\mu)\Big)\Big].\end{aligned}$$

Next, we estimate the absolute value of both summands above, where by Assumption 2.1 the first one is bounded by

$$2\rho_0\,\|w\|_X \left\|T_{\bar\mu}^{N^{\max}} w\right\|_X \left\|\widehat{h}(\mu) - \widehat{h}(\bar\mu) - t\partial_\mu\widehat{h}(\bar\mu)\right\|_{L_\infty(\Omega)},$$

and the second one (again by Assumption 2.1 and (2.15)) by

$$\begin{aligned}&4\rho_1\,\|w\|_X \left\|T_{\bar\mu}^{N^{\max}} w\right\|_X \sum_{m=1}^{M_1}\Big\|\vartheta_m^1(\mu)\widehat{u}^N(\mu) - \vartheta_m^1(\bar\mu)\widehat{u}^{N^{\max}}(\bar\mu)\\ &\quad - t\Big(\vartheta_m^1(\bar\mu)\partial_\mu\widehat{u}^{N^{\max}}(\bar\mu) + \partial_\mu\vartheta_m^1(\bar\mu)\widehat{u}^{N^{\max}}(\bar\mu)\Big)\Big\|_X .\end{aligned}$$

Using these estimates and the definition of $\delta^N(t;\bar\mu)$ (c.p. (4.34)), (4.41) yields

$$\begin{aligned}(4.42)\qquad &\sigma^2(w)\,\|w\|_X^2 - \left\|T_\mu^N w - T_{\bar\mu}^{N^{\max}} w\right\|_X^2 - \mathcal{T}(w,w;t;\bar\mu)\\ &\geq -2\delta^N(t;\bar\mu)\,\|w\|_X \left\|T_{\bar\mu}^{N^{\max}} w\right\|_X .\end{aligned}$$

The last step is to estimate the right-hand side of (4.42). To do so, we find by (4.39), the triangle-inequality, the definition of $\sigma(w)$ (c.p. (4.37)) and the simple fact $2ab \leq a^2 + b^2$ that

$$\begin{aligned} & 2\delta^N(t;\bar\mu)\,\|w\|_X \left\| T_{\bar\mu}^{N^{\max}} w \right\|_X \\ \leq\ & 2\delta^N(t;\bar\mu)\,\|w\|_X \Big(\left\| T_\mu^N w \right\|_X + \left\| T_{\bar\mu}^{N^{\max}} w - T_\mu^N w \right\|_X \Big) \\ \leq\ & 2\delta^N(t;\bar\mu)\sigma(w)\,\|w\|_X^2 + \big(\delta^N(t;\bar\mu)\big)^2\,\|w\|_X^2 + \left\| T_{\bar\mu}^{N^{\max}} w - T_\mu^N w \right\|_X^2 . \end{aligned}$$

Hence, (4.42) yields a quadratic inequality for $\sigma(w)$, namely

$$\sigma^2(w) \quad \geq \quad \frac{\mathcal{T}(w,w;t;\bar\mu)}{\|w\|_X^2} - 2\delta^N(t;\bar\mu)\sigma(w) - \big(\delta^N(t;\bar\mu)\big)^2,$$

which holds, if (note that $\sigma(w) \geq 0$)

$$\sigma(w) \quad \geq \quad \left(\sqrt{\frac{\mathcal{T}(w,w;t;\bar\mu)}{\|w\|_X^2}} - \delta^N(t;\bar\mu) \right)^+ .$$

Recalling (4.38) and the definition of $\mathcal{F}(t;\bar\mu)$ (c.p. (4.33)) completes the proof. □

The remaining construction of the (fast evaluable) lower bound for $\beta^N(\mu)$ is done along the lines of [16]. We will recapitulate it at this point for the sake of completeness without going into details.

Let $\mathcal{P}^J := \{\mathcal{R}^j := (\bar\mu_-^j, \bar\mu_+^j), 1 \leq j \leq J\}$ be a partition of $\mathcal{D}$, i.e.

(i) $\mathcal{R}^i \cap \mathcal{R}^j = \emptyset$ for $i \neq j$ and

(ii) $\overline{\cup_{j\in J}\mathcal{R}^j} = \mathcal{D}$.

Furthermore, for each $\mathcal{R}^j$, we define a center $\bar\mu^j$, e.g. by the geometric mean of its lower and upper bound, i.e., $\log(\bar\mu^j) = \frac{1}{2}(\log(\bar\mu_-^j) + \log(\bar\mu_+^j))$. With these quantities at hand, for $\mu \in \mathcal{D}$ we define

$$\widehat\beta^N(\mu) = \left(\sqrt{\min\{\mathcal{F}(0,\bar\mu^{\mathcal{S}\mu}), \mathcal{F}(t_*;\bar\mu^{\mathcal{S}\mu})\}} - \delta^N(\mu - \bar\mu^{\mathcal{S}\mu};\bar\mu^{\mathcal{S}\mu}) \right)^+, \tag{4.43}$$

where $\mathcal{S} : \mathcal{D} \to \{1,\dots,J\}$ is a mapping, such that $\mu \in \mathcal{R}^{\mathcal{S}\mu}$ and

$$t_* := \begin{cases} \bar\mu_+^{\mathcal{S}\mu} - \bar\mu^{\mathcal{S}\mu}, & \text{if } \mu \geq \bar\mu^{\mathcal{S}\mu}, \\ \bar\mu_-^{\mathcal{S}\mu} - \bar\mu^{\mathcal{S}\mu}, & \text{otherwise.} \end{cases}$$

We summarize our findings as follows.

Proposition 4.6. *For all $\mu \in \mathcal{D}$ we have*

$$\beta^N(\mu) \geq \widehat\beta^N(\mu) \geq 0. \tag{4.44}$$

Proof. The result is an immediate consequence of Lemma 4.1, Lemma 4.2 and the definition of $\widehat\beta^N(\mu)$, i.e. (4.43). □

Note, that Proposition 4.6 only ensures that $\widehat\beta^N(\mu)$ is a lower bound for $\beta^N(\mu)$. To ensure a 'good' lower bound, the partition $\mathcal{P}^J$ should be sufficiently fine. Therefore, $\mathcal{P}^J$ is called ε_β-conforming (c.p. [16]), $\varepsilon_\beta \in (0,1)$, if for all $\mu \in \mathcal{D}$

$$\widehat\beta^{N^{\max}}(\mu) \geq \varepsilon_\beta\, \beta^{N^{\max}}(\bar\mu^{\mathcal{S}\mu}) > 0. \tag{4.45}$$

The online complexity for the evaluation of $\widehat\beta^N(\mu)$ is $\mathcal{O}(\log J)$ to determine j, s.t. $\mu \in \mathcal{R}^j$ and $\mathcal{O}(M_0^2) + \mathcal{O}(M_1^2) + \mathcal{O}\Big(M_1 (N^{\max})^2\Big)$ to evaluate $\delta^N(t;\bar\mu)$. Additionally, for the computation of the first summand of $\delta^N(t;\bar\mu)$ (c.p. (4.34)), i.e.,

$$\left\| \widehat h_0(\bar\mu + t) - \widehat h_0(\bar\mu) - t\partial_\mu \widehat h_0(\bar\mu) \right\|_{L_\infty(\Omega)}$$

we use (in analogy to (4.29)):

$$\begin{aligned}
&\left\|\widehat{h}_0(\bar{\mu}+t) - \widehat{h}_0(\bar{\mu}) - t\partial_\mu \widehat{h}_0(\bar{\mu})\right\|_{L_\infty(\Omega)} \\
= &\left\|\sum_{m=1}^{M_0} \left(\vartheta_m^0(\bar{\mu}+t) - \vartheta_m^0(\bar{\mu}) - \partial_\mu \vartheta_m^0(\bar{\mu})\right) \varphi_m^0\right\|_{L_\infty(\Omega)} \\
\leq &\sum_{m=1}^{M_0} \left|\vartheta_m^0(\bar{\mu}+t) - \vartheta_m^0(\bar{\mu}) - \partial_\mu \vartheta_m^0(\bar{\mu})\right|.
\end{aligned}$$

For further details on the construction itself refer to [16].

4.5.2. *Non-Affine Parameter Dependence.* Now, we turn back to the presence of non-affine coefficient functions, i.e., $h(\mu) \neq \widehat{h}(\mu)$. As denoted earlier, we will follow the idea presented (amongst others) in [9], i.e., using the proceeding presented in Section 4.5.1 for the case of affine parameter dependence and add a correction term to derive the desired lower bound for $\beta^N(\mu)$.

To do so, we first recall $\widehat{\beta}^N(\mu)$ to be the lower bound for $\beta(\widehat{u}^N(\mu); \widehat{h}(\mu))$ developed in Section 4.5.1. Furthermore, we define (recall (2.7)) the correction term $\beta_c^N(\mu) := \gamma\Big(\widehat{u}^N(\mu); h(\mu) - \widehat{h}(\mu)\Big)$ and therewith:

$$\overline{\beta}^N(\mu) := \widehat{\beta}^N(\mu) - \beta_c^N(\mu), \tag{4.46}$$

such that we get the following estimate.

Corollary 4.1. *For all $\mu \in \mathcal{D}$ we have*

$$\beta^N(\mu) \geq \overline{\beta}^N(\mu). \tag{4.47}$$

Proof. Recalling the definition of the supremizer $T_\mu^N w$ (c.p. (4.30) and note that it is defined w.r.t. $\widehat{h}(\mu)$) we find

$$\begin{aligned}
\beta^N(\mu) &= \inf_{w\in X}\sup_{v\in X} \frac{dg(w,v;h(\mu))\,[\widehat{u}^N(\mu)]}{\|w\|_X\,\|v\|_X} \\
&\geq \inf_{w\in X} \frac{dg\Big(w, T_\mu^N w; \widehat{h}(\mu) + \Big(h(\mu) - \widehat{h}(\mu)\Big)\Big)\,[\widehat{u}^N(\mu)]}{\|w\|_X\,\left\|T_\mu^N w\right\|_X} \\
&\geq \beta(\widehat{u}^N(\mu); \widehat{h}(\mu)) + \inf_{w\in X} \frac{dg\Big(w, T_\mu^N w; h(\mu) - \widehat{h}(\mu)\Big)\,[\widehat{u}^N(\mu)]}{\|w\|_X\,\left\|T_\mu^N w\right\|_X} \\
&\geq \beta(\widehat{u}^N(\mu); \widehat{h}(\mu)) - \sup_{w\in X} \frac{dg\Big(w, T_\mu^N w; h(\mu) - \widehat{h}(\mu)\Big)\,[\widehat{u}^N(\mu)]}{\|w\|_X\,\left\|T_\mu^N w\right\|_X}.
\end{aligned}$$

Next, by the definition of the continuity constant (2.7), we have

$$\begin{aligned}
&\sup_{w\in X} \frac{dg\Big(w, T_\mu^N w; h(\mu) - \widehat{h}(\mu)\Big)\,[\widehat{u}^N(\mu)]}{\|w\|_X\,\left\|T_\mu^N w\right\|_X} \\
\leq &\sup_{w\in X}\sup_{v\in X} \frac{dg\Big(w, v; h(\mu) - \widehat{h}(\mu)\Big)\,[\widehat{u}^N(\mu)]}{\|w\|_X\,\|v\|_X} \\
= &\ \gamma\Big(\widehat{u}^N(\mu); h(\mu) - \widehat{h}(\mu)\Big).
\end{aligned}$$

In view of Proposition 4.6 this completes the proof. □

Note that concerning the estimation of $\beta_c^N(\mu)$ we find by (2.11) and (2.14):

$$\begin{aligned}\beta_c^N(\mu) &\leq \rho_0 \left\| h_0(\mu) - \widehat{h}_0(\mu) \right\|_{L\infty(\Omega)} \\ &\quad + 2\rho_1 \left\| h_1(\mu) - \widehat{h}_1(\mu) \right\|_{L\infty(\Omega)} \left\| \widehat{u}^N(\mu) \right\|_X \\ &\leq \varepsilon_{\mathrm{emp}} \big(\rho_0 + 2\rho_1 \left\| \widehat{u}^N(\mu) \right\|_X \big) ,\end{aligned}$$

so that we have computable versions for all involved constants at hand.

4.6. **Sampling Procedure.** Now we develop a procedure to determine appropriate samples and snapshots. First note, that compared to [16] we have now finished to incorporate possible non-affine coefficient functions. Hence, the sampling procedure to be presented here does not differ from the one presented in [16]. Nevertheless, for the sake of completeness we will briefly describe it.

We first construct the primal samples S^N and primal spaces W^N, respectively, for $1 \leq N \leq N^{\max}$. Afterwards, we select the dual samples $\widetilde{S}^{\widetilde{N}}$ and dual spaces $\widetilde{W}^{\widetilde{N}}$, respectively, for $1 \leq \widetilde{N} \leq \widetilde{N}^{\max}$. As the applied greedy procedure is very similar for both, we will detail only the first one.

Before we start, we select a large (random) test sample $\Xi \subset \mathcal{D}$, a 'smallest (energy) tolerance' $\varepsilon_{\mathrm{sp}} > 0$ and an initial sample S^1. Furthermore, since $\widehat{u}^{N^{\max}}(\mu)$ (and therewith $\overline{\beta}^N(\mu)$, $\tau^N(\mu)$ and $\Delta^N(\mu)$) is not available yet, we have to replace $\overline{\beta}^N(\mu)$ by a crude surrogate, say $\overline{\beta}_s^N(\mu)$, and define all involved quantities, namely $\tau^N(\mu)$ and $\Delta^N(\mu)$, w.r.t. this surrogate. The procedure reads as follows.

For $N = 1, 2, \ldots$, do

(1) compute $\mu^* := \arg\max_{\mu\in\Xi} \tau^N(\mu)$;
(2) if $\tau^N(\mu^*) >= 1$, update $S^{N+1} := S^N \cup \{\mu^*\}$ and continue;
(3) compute $\mu^{**} := \arg\max_{\mu\in\Xi} \Delta^N(\mu)$;
(4) if $\Delta^N(\mu^{**}) > \varepsilon_{\mathrm{sp}}$, update $S^{N+1} := S^N \cup \{\mu^{**}\}$ and continue;
(5) stop.

It is important to note, that $\Delta^N(\mu)$ (with $\overline{\beta}^N(\mu)$ replaced by its surrogate $\overline{\beta}_s^N(\mu)$) is an accurate surrogate for the true error, that can be calculated very efficiently in the limit of many queries. Only the selected snapshots must actually be computed, thus we may choose $\#\Xi$ very large. In summary, we can expect that the sequence of spaces W^N will provide rapidly certifiable (thanks to μ^*) and rapidly convergent (thanks to μ^{**}) approximations uniformly in $\mathcal{D}$. We will come back to this point later in the presentation of our numerical results, later.

5. An Application: A Rotating Propeller

As already pointed out earlier, one of our final aims is to treat flow problems around moving bodies. Thus, in an earlier work [14] we dealt with Reduced Basis Methods for solving parameter-dependent convection-diffusion problems around rigid bodies. Although, we successfully applied the Reduced Basis Method for solving the problem at hand, we were lacking any kind of a-posteriori error estimators both for quantifying the quality of the computed approximations (i.e., without computing the true solutions) and for an optimal basis assembling procedure. Now, we want to pick up the problem considered in [14] and apply the a-posteriori error estimators developed above. In the sequel we will briefly recapitulate the problem of interest.

For simplicity, we consider just a stationary convection–diffusion problem (even though the approach also applies for more realistic flow models) in a rectangle $\Box := [-1.5, 1.5]^2 \subset \mathbb{R}^2$ in which one or more rigid bodies are located in dependence of a parameter μ. We assume that the shape of the bodies are identical and fixed. The bodies can be interpreted as blades of a rotor or propeller and, in the case of only one body, $B(\mu)$ is the blade obtained by rotating a reference blade $B = B(0)$ around its center of symmetry by an angle $\mu \in \mathcal{D} = [0, \frac{\pi}{2}]$, see Figure 5.1. Here, we restrict ourselves to the case of one blade only and refer to [14] for the more realistic case of several blades. The geometry is shown in Figure 5.1.

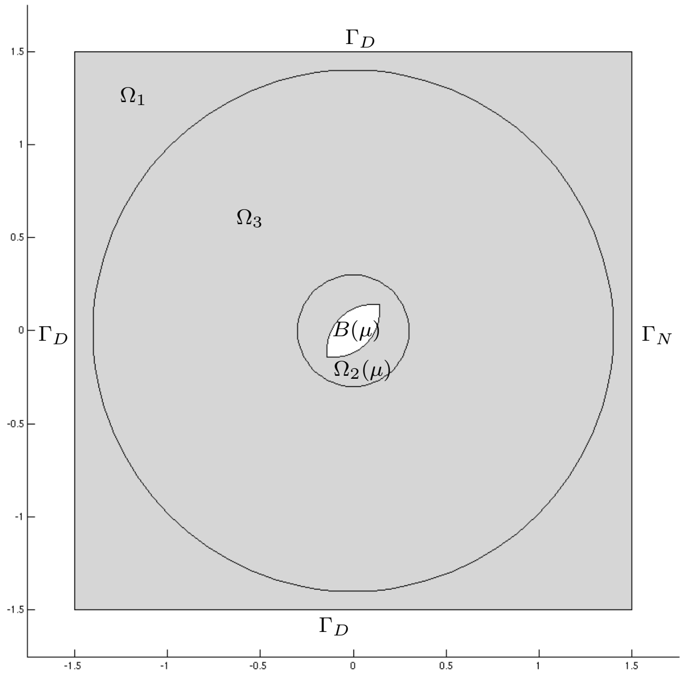

FIGURE 5.1. Geometry for one Blade, $\mu = \frac{\pi}{4}$

We subdivide $\Box \setminus B(\mu) =: \Omega(\mu)$ into three subdomains according to Figure 5.1, so that we obtain

$$\Omega(\mu) := \Omega_1 \cup \Omega_2(\mu) \cup \Omega_3.$$

Given coefficients $\underline{\varrho}$ and $\underline{\phi}$ (that may also be non-constant, $\underline{\phi}$ being a vector field), we consider the following convection-diffusion problem:

$$\text{(5.1)} \qquad \begin{cases} -\underline{\varrho}\,\Delta u + \big(\underline{\phi} \cdot \nabla u\big)\, u = 0\ , & \text{in } \Omega(\mu), \\ u = 0\ , & \text{on } \partial B(\mu), \\ u = g\ , & \text{on } \Gamma_D, \\ \frac{\partial u}{\partial n} = 0\ , & \text{on } \Gamma_N, \end{cases}$$

where $\Gamma_N := \partial\Box \cap \{x = 1.5\}$ is the Neumann part of the outer boundary $\partial\Box$ and $\Gamma_D := \partial\Box \setminus \Gamma_N$ the Dirichlet part of $\partial\Box$.

To apply a Reduced Basis Method to (5.1) and its variational formulation, respectively, we first have to transform the problem to a reference domain, say $\hat{\Omega} := \Omega(\hat{\mu})$, where $\hat{\mu} = 0$. Since in our particular problem the parameter μ represents a rotation, the transformation is obvious. In Ω_1, no mapping is applied (hence Ω_1 is independent of μ), $\Omega_2(\mu)$ is rotated by the angle $-\mu$, thus we have an affine transformation in this subdomain. In Ω_3 (which is also independent of μ as Ω_1), each point is rotated by an angle depending on its position, i.e., points close to the outer circle are almost not rotated, while points near the inner circle are rotated almost by the angle $-\mu$. Again, for more details, we refer to [14].

For $g \in H^{1/2}(\Gamma_D)$ let

$$\hat{V}(g) := \left\{ v \in H^1(\hat{\Omega}) : v = g \text{ on } \Gamma_D, v = 0 \text{ on } \partial\hat{B} \right\},$$

where $\hat{B}$ denotes $B(\hat{\mu})$. Then, the variational formulation for (5.1) on the reference domain takes the form: Find $\hat{u}(\mu) \in \hat{V}(g)$, such that

$$b(\hat{u}(\mu), \hat{v}; \mu) + c(\hat{u}(\mu), \hat{u}(\mu), \hat{v}; \mu) = 0, \quad \hat{v} \in \hat{V}(0), \tag{5.2}$$

for the bilinear form b and trilinear form c, respectively, that take the form

$$\begin{aligned} b(\hat{u}, \hat{v}; \mu) &= \sum_{n=1}^{3} \int_{\hat{\Omega}_n} \nabla \hat{u}(\hat{x}) \cdot (\underline{\underline{T}}^{(n)}(\hat{x}; \mu) \, \nabla \hat{v}(\hat{x})) \, d\hat{x}, \\ c(\hat{u}, \hat{w}, \hat{v}; \mu) &= \sum_{n=1}^{3} \int_{\hat{\Omega}_n} (\underline{t}^{(n)}(\hat{x}; \mu) \cdot \nabla \hat{u}(\hat{x})) \, \hat{w}(\hat{x}) \, \hat{v}(\hat{x}) \, d\hat{x}, \end{aligned} \tag{5.3}$$

where the matrix function $\underline{\underline{T}}^{(n)}$ and the vector field $\underline{t}^{(n)}$ are obtained in a straightforward way by the change of variable $\Omega_n \ni x \mapsto \hat{x} \in \hat{\Omega}_n$ and the chain rule.

The last step is to reduce the problem to homogeneous boundary conditions. For this purpose, we choose $\hat{u}_H \in H^1(\hat{\Omega}_1)$, such that $\hat{u}_H = g$ on Γ_D and $\hat{u}_H = 0$ on $\partial\hat{\Omega}_1 \cap \partial\hat{\Omega}_3$. Then, we can reformulate (5.2) in the form of (2.2) as follows:
Find $\hat{u}(\mu) \in \hat{V}(0)$, such that

$$a_0(\hat{u}(\mu), \hat{v}; \mu) + a_1(\hat{u}(\mu), \hat{u}(\mu), \hat{v}; \mu) = f(\hat{v}), \quad \hat{v} \in \hat{V}(0), \tag{5.4}$$

where

$$\begin{aligned} a_0(\hat{u}, \hat{v}; \mu) &:= b(\hat{u}, \hat{v}; \mu) + c(\hat{u}, \hat{u}_H, \hat{v}; \mu) + c(\hat{u}_H, \hat{u}, \hat{v}; \mu), \\ a_1(\hat{u}, \hat{w}, \hat{v}; \mu) &:= c(\hat{u}, \hat{w}, \hat{v}; \mu), \\ f(\hat{v}; \mu) &:= -b(\hat{u}_H, \hat{v}; \mu) - c(\hat{u}_H, \hat{u}_H, \hat{v}; \mu). \end{aligned} \tag{5.5}$$

Note that (unlike indicated) $f(\hat{v}; \mu)$ does not depend on μ due to the particular choice of $\hat{u}_H$ ($\operatorname{supp} \hat{u}_H = \hat{\Omega}_1$ and the fact that the mapping on $\hat{\Omega}_1$ is the identity).

Setting $X^e := \hat{V}(0)$ and $X \subset X^e$ a (suitable fine) finite-element space, we have transformed (5.1) such that the theory developed in the previous sections can be applied (c.p. (2.2) and (2.3), respectively), although now we have the sum of several bilinear and trilinear forms, respectively, each endowed with its own (partly non-affine) coefficient function. In the case of several blades, one obtains more subdomains, a more complicated transformation and also more linear combinations of bi- and trilinear forms. But still the problem can be transformed into a version that allows the application of the theory presented above.

Finally, we define the output of interest $s(\mu) := \ell(\hat{u}(\mu))$, where

$$\ell(\hat{v}) := \int_{\partial\hat{B}} \frac{\partial \hat{v}(\hat{x})}{\partial \hat{n}(\hat{x})} \, d\hat{x} \tag{5.6}$$

and $\hat{n}(\hat{x})$ denotes the outward normal vector in $\hat{x}$.

6. Numerical Results

This section is devoted to the description of several numerical experiments. Let us start with one critical observation. In the offline/online decomposition for the computation of the error terms in Sections 4.1 and 4.2, we had always to take square roots. For example, in order to compute $R^N(\mu) = \left\|\mathcal{R}^N(\mu)\right\|_X$, we used the offline/online decomposition to compute the *square* $\left\|\mathcal{R}^N(\mu)\right\|_X^2$. Taking the square root results in a loss of accuracy of half of the digits. Hence, the maximum accuracy we can expect is the square root of the machine accuracy. The same remark holds for the other error quantities $E^N(\mu)$, $\widetilde{R}^{N,\widetilde{N}}(\mu)$ and $\widetilde{E}^{N,\widetilde{N}}(\mu)$.

For this reason, we investigate the influence of this effect by first computing directly the error quantities (without the offline/online decomposition). This means that e.g. for $R^N(\mu)$ we solve the following linear problem

$$R^N(v;\mu) = \left(\mathcal{R}^N(\mu), v\right)_X, \quad v \in X.$$

Furthermore, for reasons we point out later, we use the true inf-sup constant $\beta^N(\mu)$ rather than its lower bound $\overline{\beta}^N(\mu)$.

In the second part we give the quantitative result (including computational savings), i.e., we compute all terms independent of $\mathcal{N}$ $(= \dim X)$ by the offline/online-decomposition. Moreover we use the fast computable lower bound $\overline{\beta}^N(\mu)$ to $\beta^N(\mu)$ developed in Section 4.5.

We consider the convection-diffusion problem around one rotating blade described in Section 5 above. For the fine FE-solution X we use $\mathbb{P}^1$-finite-elements, where the triangulation is shown in Figure 6.2. The empirical interpolation is carried out with a tolerance of $\varepsilon_{\text{emp}} = 1e^{-10}$, see (2.14).

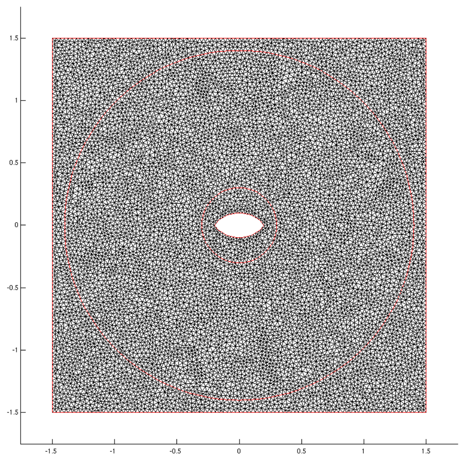

Figure 6.2. Mesh on Reference Domain $\hat{\Omega}$ ($\mathcal{N} = 10211$).

6.1. **Direct Computation of Residuals.**

As already mentioned, we first describe the results for computing the error terms directly without the offline/online decomposition.

Primal Problem. We start by investigating the error for the *Primal Problem.* In Table 6.1 and Figure 6.3, we show the average values of $e^N(\mu)$ (c.p. (4.1)), $\Delta^N(\mu)$ (c.p. (3.7)), $\eta^N(\mu)$ (c.p. (4.2)) and $\tau^N(\mu)$ (c.p. (3.6)) for five representative values of the parameter (in fact we use the same values as in [14] for comparability) in dependence of the number N of snapshots. Here and in the sequel we use 'NaN' to indicate that a quantity is not computable since the condition for the corresponding proximity indicator is not fulfilled (e.g. for the computation of $\Delta^N(\mu)$ we need $\tau^N(\mu) \leq 1$, which turns out to be not the case for $N \leq 3$).

N	$\|e^N(\mu)\|_X$	$\Delta^N(\mu)$	$\eta^N(\mu)$	$\tau^N(\mu)$
1	3.21e-01	NaN	NaN	2.06e+02
3	1.60e-02	NaN	NaN	2.64e+00
5	9.72e-04	1.97e-03	2.44e+00	1.14e-01
7	5.72e-06	1.34e-05	2.17e+00	1.71e-03
9	5.95e-07	1.58e-06	2.03e+00	2.57e-04
11	5.03e-08	8.44e-08	2.01e+00	6.02e-06
13	2.35e-09	4.34e-09	2.01e+00	3.98e-07
15	3.09e-10	5.64e-10	2.18e+00	4.81e-08
17	1.44e-11	4.80e-11	4.38e+00	6.72e-09

TABLE 6.1. Error quantities for the primal problem with direct computation.

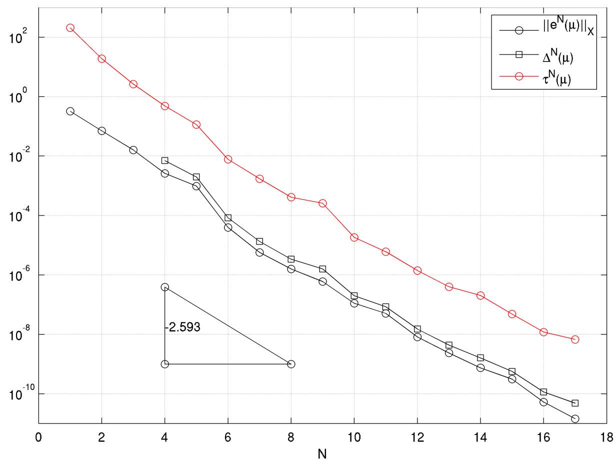

FIGURE 6.3. Semi-logarithmic plot of $e^N(\mu)$, $\Delta^N(\mu)$ and $\tau^N(\mu)$ over N.

We observe exponential decay of the error $e^N(\mu)$ w.r.t. N, as expected. This error is (thanks to the greedy sampling procedure in Section 4.6) rapidly certifiable, i.e., $\tau^N(\mu) \leq 1$ for $N \geq 4$. Thus, for $N \geq 4$ we can compute the error bound $\Delta^N(\mu)$, which turns out to be very close to the true error, i.e., we obtain effectivities $\eta^N(\mu)$ of approximately 2. Only for N close to $N^{\max} = 17$, the effectivity $\eta^N(\mu)$ is slightly increasing. This can be explained in view of Proposition 4.1 by considering $R^N(\mu)$ (c.p. (3.1)) and $E^N(\mu)$ (c.p. (3.2)) in Table 6.2. When $R^N(\mu)$ gets very close to $E^N(\mu)$ the efficiency is rising, obviously. Moreover, since for $N = 18$ our 'Safety-Condition' (4.12) is no longer valid, we chose $N^{\max} = 17$. If we would use more snapshots, the values $\tau^N(\mu)$ and $\Delta^N(\mu)$ would be dominated by the error $E^N(\mu)$ introduced by the EIM, i.e., the corresponding curves in Figure 6.3 would pass into a plateau. This is the typical behavior using the EIM (c.p. amongst others [9]).

N	$\eta^N(\mu)$	$R^N(\mu)$	$E^N(\mu)$	$E^N(\mu)/R^N(\mu)$
13	2.01e+00	2.19e-10	4.34e-13	1.98e-03
14	2.06e+00	6.72e-11	4.34e-13	6.46e-03
15	2.18e+00	2.89e-11	4.34e-13	1.50e-02
16	2.49e+00	4.96e-12	4.34e-13	8.75e-02
17	4.38e+00	1.38e-12	4.34e-13	3.15e-01
(18)	8.76e+00	5.31e-13	4.34e-13	8.18e-01

TABLE 6.2. Error terms and effectivities for increasing values of N.

Before we consider the *Dual Problem*, we investigate the (simple) output approximation $\widehat{s}_1^N(\mu)$ defined in (4.20) using only the approximation of the solution of the primal problem and the error estimator $\Delta_{s_1}^N(\mu)$ (c.p. (4.21)). For the linear functional ℓ, we use again the above described application (5.6). The results are shown in Table 6.3 where we give the values for the error $e_{s_1}^N(\mu) := |s(\mu) - \widehat{s}_1^N(\mu)|$, the estimator $\Delta_{s_1}^N(\mu)$ and the effectivity $\eta_{s_1}^N(\mu) := \Delta_{s_1}^N(\mu)/e_{s_1}^N(\mu)$.

N	$e_{s_1}^N(\mu)$	$\Delta_{s_1}^N(\mu)$	$\eta_{s_1}^N(\mu)$
1	4.96e-03	NaN	NaN
3	1.68e-04	NaN	NaN
5	5.73e-06	1.23e-02	5.04e+03
7	2.34e-08	8.33e-05	4.03e+03
9	2.51e-09	9.88e-06	1.22e+04
11	7.79e-11	5.26e-07	1.58e+04
13	4.77e-12	2.71e-08	8.31e+03
15	6.22e-13	3.52e-09	6.64e+03
17	1.05e-13	3.00e-10	2.42e+03

TABLE 6.3. Error, estimator and effectivity for the output computation using only the primal problem.

As expected, we observe again exponential decay of the error with a rough error bound (i.e., large effectivities), since the error is the dual norm of the output functional multiplied by $\Delta^N(\mu)$ (c.p. (4.21) and Proposition 4.3, respectively). We have depicted $e_{s_1}^N(\mu)$ in Figure 6.5 below together with the output approximations and corresponding error bounds using the *Dual Problem*.

Dual Problem. Next, we use the *Dual Problem* for the computation of the output of interest. Table 6.4 and Figure 6.4 show the values for $\widetilde{e}^{N,\widetilde{N}}(\mu)$ (c.p. (4.15)), $\widetilde{\Delta}^{N,\widetilde{N}}(\mu)$ (c.p. (4.16)) and $\widetilde{\eta}^{N,\widetilde{N}}(\mu) := \widetilde{\Delta}^{N,\widetilde{N}}(\mu)/\left\|\widetilde{e}^{N,\widetilde{N}}(\mu)\right\|_X$ in dependence of $\widetilde{N}$ for fixed $N = N^{\max} = 17$.

Again, we observe exponential decay of the errors and the quality of the error estimator $\widetilde{\Delta}^{N,\widetilde{N}}(\mu)$. However, the greedy sampling procedure stops here already

$\widetilde{N}$	$\left\Vert \widetilde{e}^{N,\widetilde{N}}(\mu)\right\Vert_X$	$\widetilde{\Delta}^{N,\widetilde{N}}(\mu)$	$\widetilde{\eta}^{N,\widetilde{N}}(\mu)$	$\widetilde{R}^{N,\widetilde{N}}(\mu)$	$\widetilde{E}^{N,\widetilde{N}}(\mu)$
1	9.99e-01	3.69e+00	3.00e+00	1.34e-01	4.68e-12
3	4.90e-02	1.05e-01	2.48e+00	5.45e-03	4.76e-12
5	9.91e-04	2.00e-03	2.39e+00	1.07e-04	4.76e-12
7	1.73e-05	3.62e-05	2.26e+00	1.73e-06	4.76e-12
9	8.92e-07	1.56e-06	2.97e+00	8.15e-08	4.76e-12
11	5.34e-08	2.00e-07	6.01e+00	4.88e-09	4.76e-12

TABLE 6.4. Error, estimator, indicator and error parts using the dual problem for the computation of an output of interest.

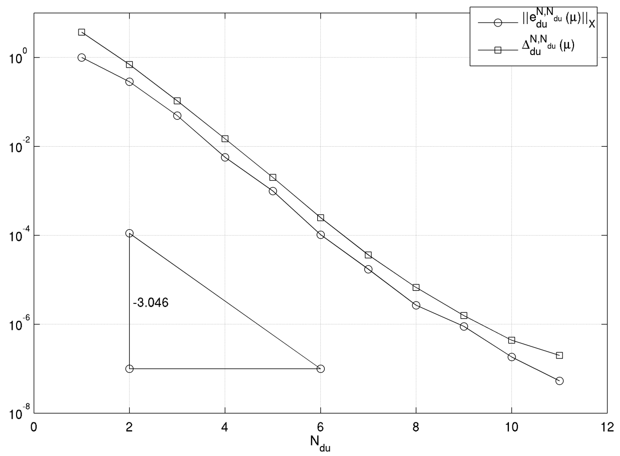

FIGURE 6.4. Semi-logarithmic plot of $\widetilde{e}^{N,\widetilde{N}}(\mu)$ and $\widetilde{\Delta}^{N,\widetilde{N}}(\mu)$ over N.

at $\widetilde{N}^{\max} = 11$ snapshots for the *Dual Problem.* In order to understand this phenomenon, we have also included the values for $\widetilde{R}^{N,\widetilde{N}}(\mu)$ (c.p. (4.13)) and $\widetilde{E}^{N,\widetilde{N}}(\mu)$ (c.p. (4.14)) in Table 6.4. We see that $\widetilde{E}^{N,\widetilde{N}}(\mu)$ is almost constant and $\widetilde{R}^{N,\widetilde{N}}(\mu)$ approaches $\widetilde{E}^{N,\widetilde{N}}(\mu)$ rapidly, so that both terms become small. Hence, the first term in the error bound $\widetilde{\Delta}^{N,\widetilde{N}}(\mu)$ in (4.16) is almost negligible, whereas the second term

$$\frac{1-\sqrt{1-\tau^N(\mu)}}{1+\sqrt{1-\tau^N(\mu)}}\left\Vert \widehat{\psi}^{N,\widetilde{N}}(\mu)\right\Vert_X$$

is dominating. Now, since $\tau^N(\mu)$ is bounded from below due to the error introduced by the empirical interpolation, also $\widetilde{\Delta}^{N,\widetilde{N}}(\mu)$ is bounded from below. In order to quantify this effect, Table 6.5 shows the same quantities as Table 6.4, but now in dependence of the number N of primal snapshots for fixed $\widetilde{N} = \widetilde{N}^{\max} = 11$. For $N \geq 11$ both the error $\widetilde{e}^{N,\widetilde{N}}(\mu)$ and the (dual norm of the) residuals $\widetilde{R}^{N,\widetilde{N}}(\mu)$ and $\widetilde{E}^{N,\widetilde{N}}(\mu)$, are (almost) constant (where $\widetilde{R}^{N,\widetilde{N}}(\mu) \gg \widetilde{E}^{N,\widetilde{N}}(\mu)$), while $\widetilde{\Delta}^{N,\widetilde{N}}(\mu)$ is still decreasing. This shows that one has to carefully balance N and $\widetilde{N}$.

Now, we finally investigate the error for the approximations of the output of interest $s(\mu)$ and their error bounds taking advantage of the *Dual Problem.* To this end, we compare the directly computed $e_{s_1}^N(\mu)$ along with the estimator $\widetilde{\Delta}_{s_1}^{N,\widetilde{N}}(\mu)$

N	$\left\|\widetilde{e}^{N,\widetilde{N}}(\mu)\right\|_X$	$\widetilde{\Delta}^{N,\widetilde{N}}(\mu)$	$\widetilde{\eta}^{N,\widetilde{N}}(\mu)$	$\widetilde{R}^{N,\widetilde{N}}(\mu)$	$\widetilde{E}^{N,\widetilde{N}}(\mu)$
1	3.12e-01	NaN	NaN	5.01e-02	4.59e-12
3	7.66e-03	NaN	NaN	1.55e-03	4.76e-12
5	4.56e-04	1.95e+00	1.27e+04	8.82e-05	4.76e-12
7	1.60e-06	2.65e-02	1.41e+04	3.14e-07	4.76e-12
9	1.64e-07	3.97e-03	1.49e+04	3.00e-08	4.76e-12
11	5.48e-08	9.30e-05	2.22e+03	5.41e-09	4.76e-12
13	5.34e-08	6.24e-06	2.11e+02	4.88e-09	4.76e-12
15	5.34e-08	8.37e-07	2.29e+01	4.88e-09	4.76e-12
17	5.34e-08	2.00e-07	6.01e+00	4.88e-09	4.76e-12

TABLE 6.5. Error, estimator, indicator and error parts using the dual problem for the computation of an output of interest in dependence of N for fixed $\widetilde{N} = \widetilde{N}^{\max} = 11$.

(c.p. (4.23)) (which involves the dual problem) with the quantities

$$e_{s_2}^{N,\widetilde{N}}(\mu) := |s(\mu) - \widehat{s}_2^{N,\widetilde{N}}(\mu)|,$$

$\widetilde{\Delta}_{s_2}^{N,\widetilde{N}}(\mu)$ defined in (4.27) using the *Dual Problem* (for $\widehat{s}_2^{N,\widetilde{N}}(\mu)$ itself c.p. (4.26)). We also indicate the corresponding effectivities in dependence of N. In Table 6.6 and the left part of Figure 6.5 we fixed $\widetilde{N} = 5$, while in Table 6.7 and right part of Figure 6.5 we fixed $\widetilde{N} = 10$.

N	$e_{s_1}^{N}(\mu)$	$\widetilde{\Delta}_{s_1}^{N,\widetilde{N}}(\mu)$	$\widetilde{\eta}_{s_1}^{N,\widetilde{N}}(\mu)$	$e_{s_2}^{N,\widetilde{N}}(\mu)$	$\widetilde{\Delta}_{s_2}^{N,\widetilde{N}}(\mu)$	$\widetilde{\eta}_{s_2}^{N,\widetilde{N}}(\mu)$
1	4.96e-03	NaN	NaN	3.57e-04	NaN	NaN
3	1.68e-04	NaN	NaN	8.75e-06	NaN	NaN
5	5.73e-06	3.85e-04	4.53e+01	1.40e-08	3.79e-04	2.54e+04
7	2.34e-08	4.25e-08	1.98e+00	2.64e-11	1.91e-08	8.95e+02
9	2.51e-09	3.18e-09	1.40e+00	5.42e-13	6.78e-10	1.37e+04
11	7.79e-11	9.07e-11	1.44e+00	7.61e-13	1.31e-11	1.64e+01
13	4.77e-12	5.48e-12	1.26e+00	2.34e-13	8.71e-13	7.75e+01
15	6.22e-13	1.20e-12	2.35e+00	2.57e-13	3.89e-13	2.84e+00
17	1.05e-13	6.28e-13	5.84e+00	2.54e-13	3.23e-13	2.03e+00

TABLE 6.6. Comparison of errors and indicators for the output of interest with the direct method and using the dual problem for fixed $\widetilde{N} = 5$.

Recall that the error estimators are certifiable for $N \geq 4$ ($\tau^N(\mu) \leq 1$), only. First, we see that $\widetilde{\Delta}_{s_1}^{N,\widetilde{N}}(\mu)$ is decreasing quite fast to the true error $e_{s_1}^{N}(\mu)$, which results in effectivities of approximately 2 for $\widetilde{N} = 5$ and almost 1 for $\widetilde{N} = 10$, respectively. Then, it is increasing for N close to $N^{\max}$ as soon as the error introduced by the

N	$e^N_{s_1}(\mu)$	$\widetilde{\Delta}^{N,\widetilde{N}}_{s_1}(\mu)$	$\widetilde{\eta}^{N,\widetilde{N}}_{s_1}(\mu)$	$e^{N,\widetilde{N}}_{s_2}(\mu)$	$\widetilde{\Delta}^{N,\widetilde{N}}_{s_2}(\mu)$	$\widetilde{\eta}^{N,\widetilde{N}}_{s_2}(\mu)$
1	4.96e-03	NaN	NaN	8.90e-04	NaN	NaN
3	1.68e-04	NaN	NaN	4.66e-06	NaN	NaN
5	5.73e-06	3.84e-04	4.53e+01	1.33e-08	3.79e-04	7.46e+04
7	2.34e-08	4.15e-08	1.94e+00	5.44e-13	1.81e-08	2.59e+04
9	2.51e-09	3.11e-09	1.09e+00	2.57e-13	6.05e-10	7.77e+02
11	7.79e-11	7.86e-11	1.02e+00	2.54e-13	9.05e-13	1.07e+01
13	4.77e-12	4.90e-12	1.07e+00	2.55e-13	3.21e-13	1.99e+00
15	6.22e-13	1.13e-12	2.20e+00	2.54e-13	3.20e-13	1.99e+00
17	1.05e-13	6.25e-13	5.83e+00	2.55e-13	3.20e-13	1.98e+00

TABLE 6.7. Comparison of errors and indicators for the output of interest with the direct method and using the dual problem for fixed $\widetilde{N} = 10$.

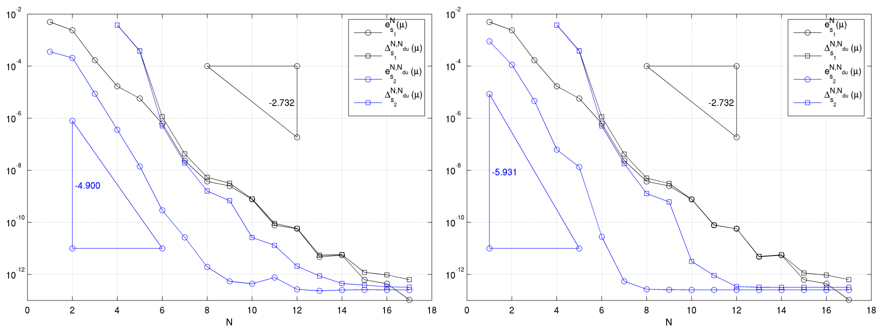

FIGURE 6.5. $e^N_{s_1}(\mu)$, $\widetilde{\Delta}^{N,\widetilde{N}}_{s_1}(\mu)$, $e^{N,\widetilde{N}}_{s_2}(\mu)$ and $\widetilde{\Delta}^{N,\widetilde{N}}_{s_2}(\mu)$ for $\widetilde{N} = 5$ (left) and $\widetilde{N} = 10$ (right)

empirical interpolation ($E^N(\widehat{\psi}^{N,\widetilde{N}}(\mu);\mu)$, c.p. (3.2)) is close to (or even larger than) the remaining summands in $\widetilde{\Delta}^{N,\widetilde{N}}_{s_1}(\mu)$.

The second observation is that for $e^{N,\widetilde{N}}_{s_2}(\mu)$ and $\widetilde{\Delta}^{N,\widetilde{N}}_{s_2}(\mu)$, we benefit from the 'square-effect' in $\widetilde{\Delta}^{N,\widetilde{N}}_{s_2}(\mu)$ as long as $E^N(\widehat{\psi}^{N,\widetilde{N}}(\mu);\mu)$ is smaller than the term $\big(R^N(\mu) + E^N(\mu)\big)\,\widetilde{\Delta}^{N,\widetilde{N}}(\mu)$. Recall that in $\widetilde{\Delta}^{N,\widetilde{N}}_{s_2}(\mu)$ the term $R^N(\widehat{\psi}^{N,\widetilde{N}}(\mu);\mu)$ (c.p. (3.1)) is shifted into the output approximation $\widehat{s}^{N,\widetilde{N}}_2(\mu)$. However, as soon as $E^N(\widehat{\psi}^{N,\widetilde{N}}(\mu);\mu)$ is dominating, both terms $e^{N,\widetilde{N}}_{s_2}(\mu)$ and $\widetilde{\Delta}^{N,\widetilde{N}}_{s_2}(\mu)$ pass into a plateau. This implies that the corresponding effectivity $\widetilde{\eta}^{N,\widetilde{N}}_{s_2}(\mu)$ is not increasing for N close to $N^{\max}$ as opposed to $\widetilde{\eta}^{N,\widetilde{N}}_{s_1}(\mu)$. The reason is that both the error $e^{N,\widetilde{N}}_{s_2}(\mu)$ and the bound $\widetilde{\Delta}^{N,\widetilde{N}}_{s_2}(\mu)$ are affected by the EIM, whereas $e^N_{s_1}(\mu)$ not.

In summary, we may conclude that $\widetilde{\Delta}^{N,\widetilde{N}}_{s_2}(\mu)$ is always smaller than $\widetilde{\Delta}^{N,\widetilde{N}}_{s_1}(\mu)$, but at the cost of higher effectivities. Furthermore, $e^{N,\widetilde{N}}_{s_2}(\mu)$ is smaller than $e^N_{s_1}(\mu)$ as long as it does not reach a plateau introduced by the error w.r.t. the EIM. This can be observed in Figure 6.5 for $N = 17$. Finally, Figure 6.6 visualizes both errors and error estimators in dependence of both N and $\widetilde{N}$.

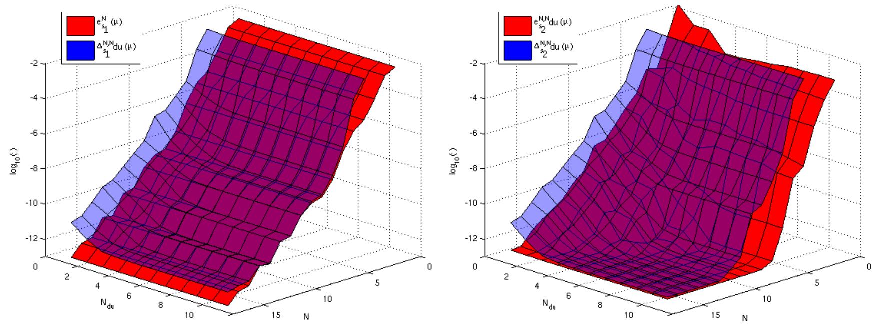

FIGURE 6.6. Error and estimators $e_{s_1}^{N}(\mu)$, $\widetilde{\Delta}_{s_1}^{N,\widetilde{N}}(\mu)$ (left) and $e_{s_2}^{N,\widetilde{N}}(\mu)$ and $\widetilde{\Delta}_{s_2}^{N,\widetilde{N}}(\mu)$ (right) in dependence of N and $\widetilde{N}$.

We conclude that the method works well also for problems with non-affine coefficient functions that are approximated by the EIM. Furthermore, we have detailed the influence of the error introduced by the EIM into the developed a-posteriori error estimators, both in theory and by the numerical experiments.

6.2. **Computation of Residuals with offline/online Decomposition.** Finally, we investigate the performance of the fully developed scheme, namely using the offline/online decomposition presented in Sections 4.1 and 4.2 as well as the lower bound for the inf-sup constant derived in Section 4.5. As already pointed out earlier this will allow for an $\mathcal{N}$-independent method. We use the same data as in the previous section.

We start again by analyzing the *Primal Problem*. Table 6.8 and Figure 6.7 show the values of the error $e^N(\mu)$, the estimator $\Delta^N(\mu)$, effectivity $\eta^N(\mu)$ and proximity indicator $\tau^N(\mu)$ in dependence of N. As before we obtain exponential decay of the

N	$\|e^N(\mu)\|_X$	$\Delta^N(\mu)$	$\eta^N(\mu)$	$\tau^N(\mu)$	$R^N(\mu)$	$E^N(\mu)$
2	7.02e-02	NaN	NaN	Inf	8.42e-03	4.09e-13
4	2.60e-03	NaN	NaN	6.61e-01	2.95e-04	4.03e-13
6	3.91e-05	8.32e-05	2.34e+00	7.85e-03	4.12e-06	4.03e-13
8	1.60e-06	3.43e-06	2.04e+00	4.25e-04	1.48e-07	4.03e-13
10	1.08e-07	3.40e-07	5.03e+00	4.02e-05	1.49e-08	4.03e-13

TABLE 6.8. Error $e^N(\mu)$, the estimator $\Delta^N(\mu)$, effectivity $\eta^N(\mu)$ and proximity indicator $\tau^N(\mu)$ in dependence of N using the offline/online decomposition for the computation.

quantities. As already mentioned in Section 6 the value of $R^N(\mu)$ is bounded from below by the square root of the machine accuracy due to floating point arithmetic while computing $\|\mathcal{R}^N(\mu)\|_X^2$. This means, as opposed to the previous part, we can choose at most $N^{\max} = 10$ snapshots. Moreover, for $N = 4$ the proximity indicator $\tau^N(\mu)$ is less than one, but the error estimator $\Delta^N(\mu)$ is indicated to be not computable ('NaN'). In order to explain this, recall that all presented values are mean values for five choices of the parameter μ. Hence 'NaN' already occurs if the proximity indicator $\tau^N(\mu)$ is larger than one for just one choice of μ. In this case, the a-posteriori error estimator $\Delta^N(\mu)$ is not computable.

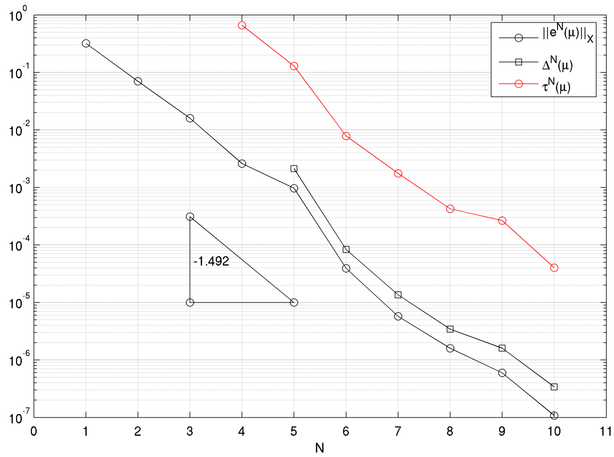

FIGURE 6.7. Semi-logarithmic plot of $e^N(\mu)$, $\Delta^N(\mu)$ and $\tau^N(\mu)$ versus N.

Finally, for $N \leq 3$ we obtain $\tau^N(\mu) = \infty$. This means, that at least for one of the five representative values of μ the second order correction $\delta^N(\mu - \bar{\mu}^{\mathcal{S}\mu}; \bar{\mu}^{\mathcal{S}\mu})$ (c.p. (4.34)) takes a larger value than $\sqrt{\min\{\mathcal{F}(0, \bar{\mu}^{\mathcal{S}\mu}), \mathcal{F}(t_*; \bar{\mu}^{\mathcal{S}\mu})\}}$ for $\widehat{\beta}^N(\mu)$ in (4.43), s.t. $\widehat{\beta}^N(\mu) = 0$ and hence $\tau^N(\mu) = \infty$.

In order to investigate the reason for this phenomenon, Figure 6.8 visualizes the numerical estimate for the inf-sup constant $\widehat{\beta}^N(\mu)$, $\mu \in [0, \frac{\pi}{2}]$, for different values of N. We have split the parameter space $\mathcal{D}$ into 40 intervals, indicated by the dotted vertical lines. We obtain that the second order correction is small enough for $N \geq 4$, s.t. $\widehat{\beta}^N(\mu) > 0$ for all $\mu \in \mathcal{D}$. For $N \geq 6$, the quality of the derived lower bound turns out to be quite good. We can also clearly see which parameter values μ are chosen by the sampling procedure.

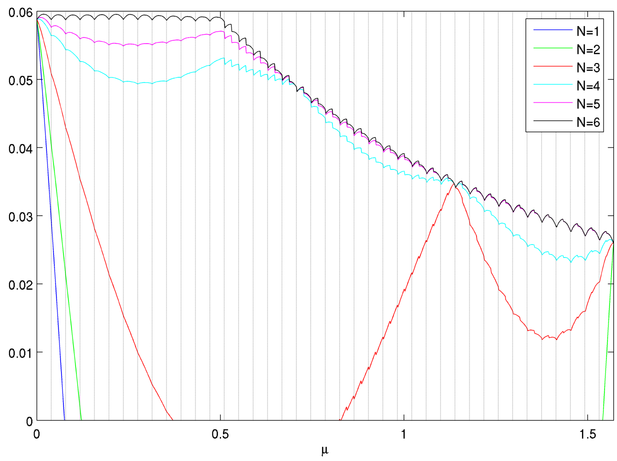

FIGURE 6.8. Computable inf-sup-estimate $\widehat{\beta}^N(\mu)$, $\mu \in [0, \frac{\pi}{2}]$, for different values of N

Next, we investigate the *Dual Problem*. Table 6.9 and Figure 6.9 show the values for the dual error $\widetilde{e}^{N,\widetilde{N}}(\mu)$, a-posteriori error estimate $\widetilde{\Delta}^{N,\widetilde{N}}(\mu)$ and the efficiency $\widetilde{\eta}^{N,\widetilde{N}}(\mu)$ in dependence on the number $\widetilde{N}$ of dual snapshots for fixed number $N = N^{\max} = 10$ of primal snapshots. As before the number of dual

$\widetilde{N}$	$\left\|\widetilde{e}^{N,\widetilde{N}}(\mu)\right\|_X$	$\widetilde{\Delta}^{N,\widetilde{N}}(\mu)$	$\widetilde{\eta}^{N,\widetilde{N}}(\mu)$
1	9.99e-01	3.73e+00	3.03e+00
2	2.84e-01	7.00e-01	2.59e+00
3	4.91e-02	1.07e-01	2.52e+00
4	5.69e-03	1.56e-02	2.58e+00
5	9.91e-04	2.64e-03	4.06e+00

TABLE 6.9. Dual error $\widetilde{e}^{N,\widetilde{N}}(\mu)$, a-posteriori error estimate $\widetilde{\Delta}^{N,\widetilde{N}}(\mu)$ and efficiency $\widetilde{\eta}^{N,\widetilde{N}}(\mu)$ in dependence of $\widetilde{N}$ for fixed $N = N^{\max} = 10$.

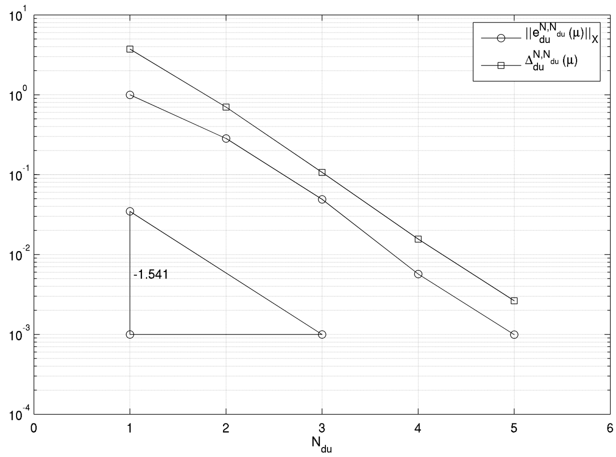

FIGURE 6.9. Semi-logarithmic plot of $\widetilde{e}^{N,\widetilde{N}}(\mu)$ and $\widetilde{\Delta}^{N,\widetilde{N}}(\mu)$ versus $\widetilde{N}$.

snapshots, that can be selected by the sampling procedure, is restricted by the second summand in the dual a-posteriori error estimate $\widetilde{\Delta}^{N,\widetilde{N}}(\mu)$, i.e. by the value of the proximity indicator $\tau^N(\mu)$ for $N = N^{\max}$, which in turns is bounded for the reasons already explained above. Thus, we can only select $\widetilde{N}^{\max} = 5$ snapshots for the *Dual Problem*.

Next, we investigate the error for the approximations of the output of interest $s(\mu)$ and their error bounds, respectively using the *Dual Problem*. In Table 6.10 and Figure 6.10 we compare the already presented values $e_{s_1}^N(\mu)$, (using the direct method) and $\widetilde{\Delta}_{s_1}^{N,\widetilde{N}}(\mu)$ with $e_{s_2}^{N,\widetilde{N}}(\mu)$, $\widetilde{\Delta}_{s_2}^{N,\widetilde{N}}(\mu)$ (using the *Dual Problem*) together with the corresponding effectivities in dependence of N for fixed $\widetilde{N} = \widetilde{N}^{\max} = 5$.

Although we were only able to select $N^{\max} = 10$ snapshots (for the reasons mentioned above), for the *Primal Problem* and $\widetilde{N}^{\max} = 5$ for the *Dual Problem*, the approximations for the output are still quite good. Of course, the basic observations from the first part are still valid here, i.e., $\widetilde{\Delta}_{s_2}^{N,\widetilde{N}}(\mu)$ is always smaller than $\widetilde{\Delta}_{s_1}^{N,\widetilde{N}}(\mu)$ at the cost of higher effectivities. Furthermore, as $N^{\max}$ is smaller as in the first part, the error $e_{s_1}^N(\mu)$ does not cross the 'EIM-plateau'. As we see, $e_{s_2}^{N,\widetilde{N}}(\mu)$ almost enters this plateau here, s.t. the application of $e_{s_2}^{N,\widetilde{N}}(\mu)$ in combination with the estimator $\widetilde{\Delta}_{s_2}^{N,\widetilde{N}}(\mu)$ is obviously advantageous.

N	$e_{s_1}^{N}(\mu)$	$\widetilde{\Delta}_{s_1}^{N,\widetilde{N}}(\mu)$	$\widetilde{\eta}_{s_1}^{N,\widetilde{N}}(\mu)$	$e_{s_2}^{N,\widetilde{N}}(\mu)$	$\widetilde{\Delta}_{s_2}^{N,\widetilde{N}}(\mu)$	$\widetilde{\eta}_{s_2}^{N,\widetilde{N}}(\mu)$
2	2.40e-03	NaN	NaN	2.08e-04	NaN	NaN
4	1.69e-05	NaN	NaN	3.60e-07	NaN	NaN
6	6.17e-07	1.14e-06	1.92e+00	2.86e-10	5.19e-07	2.45e+03
8	3.71e-09	5.40e-09	1.65e+00	1.97e-12	1.69e-09	8.56e+02
10	7.50e-10	7.89e-10	1.06e+00	4.33e-13	3.96e-11	1.66e+02

TABLE 6.10. Comparison of output of interest for the direct method (labeled s_1) and using the *Dual Problem* (indicated by s_2) for fixed $\widetilde{N} = \widetilde{N}^{\max} = 5$.

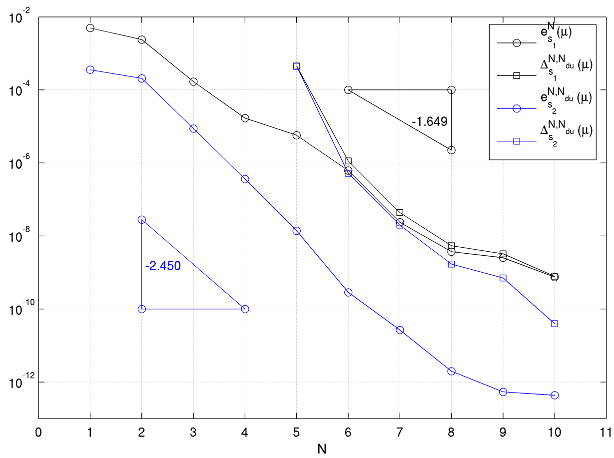

FIGURE 6.10. Semi-logarithmic plot of $e_{s_1}^{N}(\mu)$, $\widetilde{\Delta}_{s_1}^{N,\widetilde{N}}(\mu)$ (direct method) and $e_{s_2}^{N,\widetilde{N}}(\mu)$, $\widetilde{\Delta}_{s_2}^{N,\widetilde{N}}(\mu)$ (using the *Dual Problem*) for $\widetilde{N} = \widetilde{N}^{\max} = 5$.

Finally, we present the computational savings. Table 6.11 shows the ratio between the cpu-time needed for computing the output of interest $s(\mu)$. We compare the cpu-time for the FE-solution on the mesh visualized in Figure 6.3 and the time needed for approximating the output of interest $s(\mu)$ using the RBM. The line $\widetilde{N} = 0$ indicates that no dual problem is used, i.e., we use $\widehat{s}_1^N(\mu)$ in combination with $\Delta_{s_1}^N(\mu)$ (direct method). The numbers are interpreted as follows: E.g. for $N = N^{\max} = 10$ and $\widetilde{N} = \widetilde{N}^{\max} = 5$ the output approximation (including its error bound) can be obtained 223 times faster than computing $s(\mu)$ directly (which of course includes the computation of $u(\mu)$).

All computations are done with *Matlab 6.5* together with *Femlab 2.3* on an *AMD Opteron Processor 252* at 2.6 GHz. Note that the possibility of computing $R^N(\mu)$ and $\widetilde{R}^{N,\widetilde{N}}(\mu)$ at double precision would render the whole second part unnecessary, as in this case the results for using the proceeding presented in Sections 4.1 and 4.2 would be the same as in the first part.

	$N=2$	$N=4$	$N=6$	$N=8$	$N=10$
$\widetilde{N}=0$	385	364	346	328	311
$\widetilde{N}=1$	351	287	261	244	228
$\widetilde{N}=2$	349	285	258	240	224
$\widetilde{N}=3$	349	284	257	240	224
$\widetilde{N}=4$	349	284	257	240	223
$\widetilde{N}=5$	348	284	257	239	223

Table 6.11. Computational savings for the RBM-approach.

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