Sharpe Ratio for Skew-Normal Distributions: A Skewness-Dependant Performance Trade-Off

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Abstract: Main academic criticism on the Sharpe ratio concerns its lack in incorporating skewness in performance evaluation. In this note we rewrite the classical Sharpe ratio for skew normal distributions. This new skew-normal Shape ratio consistently moves with skewness and no distorted information on performance is provided. An empirical investigation illustrates skew-normality of mutual and hedge fund returns. When investors are concerned about skewness, the use of the skew-normal Sharpe ratio thus seems a proper choice for making performance rankings.

Keywords: Optimal Asset Allocation; Sharpe Ratio; Azzalini's skewness coefficient, Skew-Normal returns.

JEL Classification: G10, G11, G19, G32.

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1. Introduction

An evergreen question among academics and practitioners is whether the classical Sharpe ratio may be a suitable performance index for ranking financial products. Criticisms from academia are known: a trade-off ratio based only on the mean and the standard deviation, although fully compatible with normally distributed returns (or, in general, elliptical returns), may guide to misleading evaluations when returns exhibit skewed distributions and fat tails. Many empirical studies have illustrated the presence of skewness and kurtosis in financial markets (see, e.g., Peiró, 1999; Leland, 1999). Consequently numerous alternative performance ratios based on downside risk and higher moments have been introduced (see, e.g., Farinelli *et al.*, 2008). However, the more the ratio becomes sophisticated, the more complex and time-consuming is the implementation.

The aim of this short note is to introduce an easy to use alternative to the classical Sharpe ratio that consistently reflects skewness. We therefore rewrite the classical Sharpe ratio using the skew-normal distribution. The skew-normal distribution introduced by Azzalini (1985) has recently been applied to financial problems such as modelling of volatility (De Luca *et al.*, 2006) and portfolio selection (Adcock, 2006).

This note is structured as follows: In Section 2, we illustrate by an academical counter-example, the deficiencies of the classical Sharpe ratio: The ratio might be unable to grasp a beneficial probability mass shift and instead of signalling a performance improvement just shows the worsening. Then, in Section 3 we set up the Sharpe ratio for skew-normal distributions and illustrate its skewness-consistency. In Section 4, fitting tests are carried out to check skew-normality of return distributions of hedge funds and mutual funds. We conclude in Section 5.

2. Sharpe ratio may lead to distorted performance information

A performance ratio should get better as the performance scenario gets better. If distributions are skewed, however, a favourable shift in probability mass may result in a lower Sharpe ratio. Let us consider the following scenario as an example:

$$X_a = \begin{cases} 0 & , with \quad p = 0.9 \\ 100 & , with \quad p = 0.05 \\ 100 + a & , with \quad p = 0.05 \; and \; a \in \Re^+ . \end{cases}$$

Since $E(X_a) = 10 + 0.05a$ and standard deviation $Dev(X_a) = \sqrt{9a + 0.0475a^2 + 900}$, the Sharpe ratio becomes $\Phi_{Sharpe}(X_a) = \dfrac{E(X_a)}{Dev(X_a)} = \dfrac{10 + 0.05a}{\sqrt{9a + 0.0475a^2 + 900}}$. For $a \geq 0$, $\Phi_{Sharpe}(X_a)$ is a decreasing function of *a*:

1. If $a = 0$, then $E(X_0) = 10$, $Dev(X_0) = 30$, and the Sharpe ratio becomes $\Phi(X_0) = 0.33333$. Pearson's skew coefficient $\gamma = \dfrac{\mu_3}{\mu_2^{3/2}} = 2.6667$ where μ_i is the *i*-th central moment, signals a positive skewness.
2. Suppose, now, that probability mass shifts on the right, e.g., $a = 5$. Then, $E(X_5) = 10.25$, $Dev(X_5) = 30.76$ and Pearson's skew coefficient increases to $\gamma = \dfrac{\mu_3}{\mu_2^{3/2}} = 2.670$. The performance scenario seems to get better, but contrary to expectation $\Phi(X_5) = 0.33322$. The Sharpe ratio decreases.

The explanation of this counter-intuitive effect is simple. As the probability mass goes to the right, the mean always increases, but this positive effect can be out weighted by an increase in variance. In conclusion, although the scenario is getting better, the Sharpe ratio decreases. An important question resulting from this example is whether such distorted situations also happen when real data are involved. Or can they be considered as anomalous cases that are easy to identify and simply require a special treatment in performance measurement models?

3. Sharpe ratio for skew-normal returns

An alternative to the classical Sharpe ratio might be to rewrite this performance ratio for the skew-normal distribution. Thanks to the possibility to incorporate skewness, these distributions seem to fit well returns in financial modelling (see Adcock, 2007; and Aas and Haff, 2006; Bacmann and Massi Benedetti, 2009). The original skew-normal definition is due to Azzalini (1985), but we use an alternative description more suitable in financial context. A continuous random variable *Y* is skew-normally distributed if and only if the following representation holds:

$$Y = \xi + \omega\left(\delta\left|Z_1\right| + \sqrt{1-\delta^2}\,Z_2\right),$$

where $\delta \in [-1,1]$; Z_1 and Z_2 are independent standard Gaussian, and $|\ |$ stands for Half-Gaussian; ξ and ω (with $\omega \geq 0$) are the *locatio*n and the *scale,* respectively. The parameter δ, called the *Azzalini skewness parameter*, plays a key role in determining the skewness as it weights the presence of a Half–Gaussian $\left|Z_1\right|$, on a one-side tail of the return *Y*. The more δ is positive (negative), the more the skewness is pronounced on the right (on the left) tail. If $\delta = 0$, *Y* collapses in a Gaussian variable. The mean and the standard deviation are , respectively $E(Y) = \xi + \omega\sqrt{2/\pi}\,\delta$ and $Dev(Y) = \omega\sqrt{1-2\delta^2/\pi}$. Pearson's skew coefficient is $\gamma = \frac{4-\pi}{2}\frac{\left(\delta\sqrt{2/\pi}\right)^3}{\left(1-2\delta^2/\pi\right)^{3/2}}$. The Sharpe ratio for a skew-normal variable with reference to a benchmark *b* can be written as follows:

$$\Phi^{SN}(\xi,\omega,\delta;b) = \frac{(\xi-b)+\omega\sqrt{2/\pi}\,\delta}{\omega\sqrt{1-2\delta^2/\pi}}.$$

It is immediate to see that Φ^{SN} is a strictly increasing function of δ, as location ξ and scale ω are fixed. That perfectly matches with intuition: a beneficial probability mass shift signalled by an increase in δ (or in γ) produces always a favourable effect on Φ^{SN}. This result is also in line with decision theory: Investors like high odd moments (mean, skewness) and low even moments (standard deviation, kurtosis). In conclusion, skew-normality guarantees that a favourable skewness shift not only increases the mean, but also that its positive effect on the Sharpe ratio is never out weighted by an increasing standard deviation.

4. Skewness in Financial Returns and Goodness of Fit

A strong argument in favour of the skew normal Sharpe ratio would be that returns are well described by skew normal distributions. We thus analyze whether the return distributions of financial products are close of being *skew-normal*. To provide a robust answer to this question we consider different financial products. We first analyze 1437 mutual funds with monthly returns from January 1995 to December 2004. The data originate from the Datastream database. Secondly, we consider 2452 hedge funds reporting monthly net of fees returns for the period from January 1994 to July 2007. The data is provided by the Center for International Securities and Derivatives Markets (CISDM), a database widely used in hedge fund research. We selected hedge funds, because the returns of these funds are known to be very skewed and to have heavier tails than traditional mutual funds.

Descriptive statistics on the return distributions of the funds are presented in Table 1, i.e., the mean, standard deviation, skewness as well as the results of the Jarque-Bera (JB) Test. At a 1 percent significance level, the hypotheses of normally distributed returns is rejected for a test value of 9.21; as shown in the Table, hedge funds tend to have higher test values, which emphasizes that hedge fund returns tend to exhibit higher levels of skewness.

Table 1: Descriptive Statistics and Goodness of Fit Tests for Mutual Funds and Hedge Funds

Fund No.	Descriptive Statistics							Goodness of Fit	
	Empirical distribution				Skew-normal			Normal	Skewed Normal
	Mean (%)	St. Dev. (%)	Skew-Ness	JB-Test	Location ξ (%)	Scale ω (%)	Skew-coeff δ		
Panel A: 1437 Mutual Funds									
MF 1	0.69	5.08	-0.34	2.33	0.06	0.07	-0.87	187.91	189.60
MF 2	0.49	5.22	-0.22	1.25	0.05	0.07	-0.80	185.04	185.05
MF 3	0.29	3.85	-0.80	13.94	0.05	0.06	-0.96	220.99	228.95
MF 4	0.41	3.15	-0.84	17.58	0.04	0.05	-0.94	245.29	252.45
MF 5	0.50	3.69	-0.91	19.80	0.05	0.06	-0.96	226.13	235.08
…	…	…	…	…	…	…	…	…	…
Mean for 1437	0.51	5.01	-0.45	10.78	0.04	0.07	-0.65	206.37	209.76
Panel B: 2452 Hedge Funds									
HF 1	0.61	0.38	1.21	46.33	0.00	0.01	0.95	217.13	220.07
HF 2	0.93	2.67	1.40	157.17	-0.01	0.04	0.86	143.72	146.23
HF 3	2.10	10.50	0.75	22.77	-0.08	0.14	0.86	71.47	73.73
HF 4	0.66	6.27	0.86	17.59	-0.06	0.09	0.89	89.61	92.03
HF 5	1.79	2.25	2.75	268.23	0.00	0.03	0.98	131.04	144.26
…	…	…	…	…	…	…	…	…	…
Mean for 2452	1.09	4.21	0.06	19.98	0.00	0.05	0.05	175.76	179.03

Table 1 also reports parameter values for the skew-normal as well as the log likelihood value to compare the goodness of fit of different distributions. We only spotlight the most relevant results, i.e., we present goodness of fit for the normal and the skew-normal (results for other parametric distributions are available upon request). The skew-normal better fits both mutual fund as well as hedge fund data, e.g., with the hedge funds the goodness of fit of the skew-normal is on average 179.03 compared to 175.76 with the normal.

5. Conclusion

In this note, we first show how a favourable shift in skewness may lower the classical Sharpe ratio. Secondly, the Sharpe ratio for skew-normal distributions is introduced and its skewness-consistency is proved. Since goodness-of-fit tests illustrate that mutual funds and hedge funds returns can be better described by skew-normal distributions, we recommend considering the skew-normal Sharpe ratio as an alternative measure in performance evaluation.

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