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# A new error bound for Reduced Basis approximation of parabolic partial differential equations

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## Abstract

We consider a space-time variational formulation for linear parabolic partial differential equations. We introduce an associated Petrov-Galerkin truth finite element discretization with favorable discrete inf-sup constant  $\beta_\delta$ :  $\beta_\delta$  is bounded from below by unity for the heat equation;  $\beta_\delta$  grows only linearly in time for non-coercive (asymptotically stable) convection operators. The latter in turn permits effective long-time *a posteriori* error bounds for reduced basis approximations, in sharp contrast to classical exponentially growing energy estimates.

## Résumé

Nous considérons une formulation variationnelle espace-temps pour les équations différentielles paraboliques linéaires. Nous y associons une discrétisation par éléments finis de Petrov-Galerkin pour laquelle la constante de stabilité inf-sup  $\beta_\delta$  possède des propriétés agréables :  $\beta_\delta$  est bornée inférieurement par la constante unité pour l'équation de la chaleur;  $\beta_\delta$  a une croissance seulement linéaire en temps pour des opérateurs de convection non-coercifs (asymptotiquement stables). Dans le cadre des approximations par bases réduites, cette dernière propriété permet d'obtenir des bornes efficaces pour l'erreur *a posteriori* en temps long, en net contraste avec les estimateurs d'erreur en énergie classiques qui présentent une croissance exponentielle.

*Keywords:* parabolic equations, space-time formulation, inf-sup stability, reduced basis, a posteriori estimation

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**Version française abrégée**

Soit l'équation aux dérivées partielles parabolique (1) et sa formulation variationnelle espace-temps :  $u \in \mathcal{X} := L_2(I; V) \cap H^1(I; V')$  vérifie (2) pour l'espace des fonctions test  $\mathcal{Y} := L_2(I; V)$ ; ici  $b(w, v) := \int_I [\langle \dot{w}(t), v(t) \rangle_H + a(w(t), v(t))] dt$  et  $f(v) := \int_I \langle g(t), v(t) \rangle_H dt$ . Les normes sur  $\mathcal{X}$  et  $\mathcal{Y}$  sont définies par  $\|w\|_{\mathcal{X}}^2 := \|w\|_{L_2(I; V)}^2 + \|\dot{w}\|_{L_2(I; V')}^2$  et  $\|v\|_{\mathcal{Y}} := \|v\|_{L_2(I; V)}$ . On considère  $V = H_0^1(\Omega)$  (opérateurs spatiaux de 2e ordre), où  $\Omega$  est le domaine spatial. On montre (Proposition 1.1 basée sur [4]) que pour des problèmes coercifs, la constante inf-sup  $\beta$  de  $b$  est positive et bornée inférieurement.

Nous introduisons ensuite une discrétisation par éléments finis de Petrov-Galerkin : soit  $u_\delta \in \mathcal{X}_\delta$  vérifiant  $b(u_\delta, v_\delta) = f(v_\delta), \forall v_\delta \in \mathcal{Y}_\delta$ ; avec  $\mathcal{X}_\delta := S_{\Delta t} \otimes V_h$  et  $\mathcal{Y}_\delta = Q_{\Delta t} \otimes V_h$ ,  $\delta = (\Delta t, h)$ , où  $S_{\Delta t}, V_h$  et  $Q_{\Delta t}$  sont respectivement des éléments finis linéaire par morceaux en temps (de pas  $\Delta t$ ) et en espace (de diamètre  $h$ ), et constants en temps. Cette méthode coïncide avec une discrétisation de Crank-Nicolson en temps, ainsi nos résultats s'appliquent directement à ce schéma standard. Nous définissons une norme modifiée (équivalente) sur  $\mathcal{X}$ ,  $\|\cdot\|_{\mathcal{X}}$  où la partie  $\int_I \|w\|_V^2 dt$  de  $\|\cdot\|_{\mathcal{X}}$  est remplacée par une somme des moyennes de  $w$  sur chaque élément temporel. Nous montrons (Proposition 2.1) que pour l'équation de la chaleur, la constante inf-sup discrète  $\beta_\delta$  est bornée inférieurement par la constante unité.

Nous considérons maintenant une approximation par bases réduites  $u_N$ , comme dans [1]. Nous pouvons construire à la Proposition 3.1 des bornes sur l'erreur *a posteriori* pour le champ solution  $u_N$  et pour la sortie scalaire  $s_N$ . L'utilité de ces bornes est très liée à la dépendance de  $\beta_\delta$  aux paramètres du problème et au temps final  $T$ . La procédure de calcul de ces bornes est similaire à la décomposition hors ligne-en ligne des bases réduites standard.

Enfin, nous présentons des résultats numériques pour la constante inf-sup discrète  $\beta_\delta$ . Pour des opérateurs *non-coercifs* mais asymptotiquement stables (e.g. convection), on observe que le paramètre inf-sup croît seulement en  $\mu_1 T$ , où  $\mu_1$  est la vitesse de déplacement et  $T$  le temps final. L'exemple est un domaine spatial unidimensionnel  $(0, 1)$  avec l'opérateur  $\mu_1(x - 1/2)u_x - u_{xx}$  pour  $\mu_1 > 2\pi^2$  ( $\mu_1 < 2\pi^2$  entraîne un problème coercif). Ces estimateurs sont beaucoup plus précis que ceux classiques en énergie et prédisant une croissance exponentielle en  $e^{\mu_1 T}$ , inutilisables en pratique.

## 1. Space-time formulation

We first formulate a general linear parabolic equation. Consider Hilbert spaces  $V \hookrightarrow H \hookrightarrow V'$  and an operator  $A \in \mathcal{L}(V, V')$ ,  $\langle Av, w \rangle_H = a(v, w)$  for  $v, w \in V$ . Setting  $I := [0, T]$ ,  $T > 0$  and given  $g \in L_2(I; V')$ , we look for  $u$  such that

$$\dot{u}(t) + Au(t) = g(t) \text{ in } V', \quad t \in I; \quad u(0) = 0, \quad (1)$$

and an associated output of the form  $s := \int_I \ell(u(t)) dt$  for some  $\ell \in V'$ . We restrict ourselves to LTI systems even though some of our results can be extended to the more general situation.

For the derivation of the space-time variational form of (1), we introduce the trial space  $\mathcal{X} := L_2(I; V) \cap H^1(I; V')$  with norm  $\|w\|_{\mathcal{X}}^2 := \|w\|_{L_2(I; V)}^2 + \|\dot{w}\|_{L_2(I; V')}^2$  and the test space  $\mathcal{Y} := L_2(I; V)$  with norm  $\|v\|_{\mathcal{Y}} := \|v\|_{L_2(I; V)}$ . Note that  $\mathcal{X} = (L_2(I) \otimes V) \cap (H^1(I) \otimes V')$  and  $\mathcal{Y} = L_2(I) \otimes V$  which will allow for a tensor product discretization. Then definitions  $b(w, v) := \int_I [\langle \dot{w}(t), v(t) \rangle_H + a(w(t), v(t))] dt$  and  $f(v) := \int_I \langle g(t), v(t) \rangle_H dt$  yield the space-time variational formulation

$$u \in \mathcal{X} : \quad b(u, v) = f(v), \quad \forall v \in \mathcal{Y}. \quad (2)$$

The well-posedness of (2) has been shown (under suitable assumptions) in [4, Theorem 5.1].

The approach of [4] can also yield an estimate for the inf-sup constant  $\beta := \inf_{w \in \mathcal{X}} \sup_{v \in \mathcal{Y}} \frac{b(w, v)}{\|w\|_{\mathcal{X}} \|v\|_{\mathcal{Y}}}$ . We define  $\varrho := \sup_{0 \neq \phi \in V} \frac{\|\phi\|_H}{\|\phi\|_V}$  and  $\beta_a^* := \inf_{\phi \in V} \sup_{\psi \in V} \frac{a(\psi, \phi)}{\|\phi\|_V \|\psi\|_V}$ ; we then claim

**Proposition 1.1.** *Assume that there exist  $M_a < \infty$ ,  $\alpha > 0$ , and  $\lambda \in \mathbb{R}$  such that  $|a(\psi, \phi)| \leq M_a \|\psi\|_V \|\phi\|_V$  (continuity) and  $a(\phi, \phi) + \lambda \|\phi\|_H^2 \geq \alpha \|\phi\|_V^2$  (Gårding) for all  $v, w \in V$ . Then,  $\beta \geq \beta_{LB} := \frac{(\alpha - \lambda \varrho^2) \min\{1, M_a^{-2}\}}{\max\{1, (\beta_a^*)^{-1}\} \sqrt{2}}$ .*

**Remark 1.2.** *Note that  $\beta_{LB}$  does not depend on the final time. However, the estimate is only meaningful if  $\alpha \geq \lambda \varrho^2$ , i.e., if the system is coercive. In the non-coercive case, (1) is often transformed via  $\hat{u}(t) := e^{-\lambda t} u(t)$  to obtain a coercive problem; however, this yields an inf-sup bound that behaves as  $e^{-\lambda T}$  — clearly unsuitable for long-time integration.*

## 2. Petrov-Galerkin Truth Approximation

Let  $\mathcal{X}_\delta \subset \mathcal{X}$ ,  $\mathcal{Y}_\delta \subset \mathcal{Y}$  be finite dimensional subspaces and  $u_\delta \in \mathcal{X}_\delta$  the discrete approximation of (2), i.e.,  $b(u_\delta, v_\delta) = f(v_\delta) \forall v_\delta \in \mathcal{Y}_\delta$ ,  $s_\delta = \int_0^T \ell(u_\delta(t)) dt$ . Henceforth, we concentrate on the case  $H = L_2(\Omega)$ ,  $V = H_0^1(\Omega)$ . Let  $\mathcal{X}_\delta = S_{\Delta t} \otimes V_h$ ,  $\mathcal{Y}_h = Q_{\Delta t} \otimes V_h$ ,  $\delta = (\Delta t, h)$ , where  $S_{\Delta t}$ ,  $V_h$  are piecewise linear and  $Q_{\Delta t}$  piecewise constant finite elements with respect to triangulations  $\mathcal{T}_h^{\text{space}}$  in space and  $\mathcal{T}_{\Delta t}^{\text{time}} \equiv \{t^{i-1} \equiv (i-1)\Delta t < t \leq i\Delta t \equiv t^i, 1 \leq i \leq r\}$  in time for  $\Delta t := T/r$ . This coincides with the Crank–Nicolson (CN) scheme if a trapezoidal approximation of the right-hand side temporal integration is used; hence, we can derive error bounds for the CN scheme via our space-time formulation.

Towards that end we introduce a different norm on  $\mathcal{X}$ : For  $w \in \mathcal{X}$  and  $I^i := (t^{i-1}, t^i)$ , set  $\bar{w}^i := (\Delta t)^{-1} \int_{I^i} w(t) dt \in V$  and  $\bar{w} := \sum_{i=1}^r \chi_{I^i} \otimes \bar{w}^i \in L_2(I; V)$ , where  $\chi_{I^i}$  is the characteristic function on  $I^i$ ; then, set  $|||w|||_{\mathcal{X}}^2 := \|\dot{w}\|_{L_2(I; V')}^2 + \|\bar{w}\|_{L_2(I; V)}^2$ . It can be proven that  $\|\cdot\|_{\mathcal{X}}$  and  $|||\cdot|||_{\mathcal{X}}$  are equivalent. We may now demonstrate

**Proposition 2.1.** *Set  $Au := -\Delta u$ ; then we have the inf-sup lower bound  $\beta_\delta := \inf_{w_\delta \in \mathcal{X}_\delta} \sup_{v_\delta \in \mathcal{Y}_\delta} \frac{b(w_\delta, v_\delta)}{|||w_\delta|||_{\mathcal{X}} \|v_\delta\|_{\mathcal{Y}}} \geq 1$ .*

*Proof.* (Sketch) Note that  $\lambda = 0$ ,  $\alpha = 1$ , and  $M_a = 1$  for our choice  $\|w\|_V \equiv \|\nabla w\|_{L_2(\Omega)}$ . Let  $0 \neq w_\delta \in \mathcal{X}_\delta$ . Set  $z_\delta := A_h^{-1} \dot{w}_\delta$  (i.e.,  $a(z_\delta, \phi_h) = \langle \dot{w}_\delta, \phi_h \rangle_H$ ,  $\forall \phi_h \in V_h$ ) and  $v_\delta := z_\delta + \bar{w}_\delta$ . Then,  $b(w_\delta, v_\delta) \geq |||w_\delta|||_{\mathcal{X}}^2 + \|w_\delta(T)\|_H^2 \geq |||w_\delta|||_{\mathcal{X}} \|v_\delta\|_{\mathcal{Y}}$  since  $(z_\delta, \bar{w}_\delta)_{\mathcal{Y}} = \frac{1}{2} \|w_\delta(T)\|_H^2$  and  $\|v_\delta\|_{\mathcal{Y}}^2 = \int_I \|A_h^{-1} \dot{w}_\delta\|_V^2 dt + \Delta t \sum_{i=1}^r \|\bar{w}_\delta^i\|_V^2 + \|w_\delta(T)\|_H^2 = |||w_\delta|||_{\mathcal{X}}^2 + \|w_\delta(T)\|_H^2$ .  $\square$

## 3. The Reduced Basis Method (RBM)

Now, let  $\mu = (\mu_1, \mu_2) \in \mathcal{D} := \mathbb{R}^2$  be a parameter vector and  $A = A(\mu) := -\Delta u + \mu_1 \beta(x) \cdot \nabla u + \mu_2 u$ , i.e., a diffusion-convection-reaction operator with convection field  $\beta$ . We then introduce a standard Reduced Basis (RB) approximation for the Crank–Nicolson interpretation of our discrete problem,  $u_N(\mu) \in \mathcal{X}_{\Delta t, N} = S_{\Delta t} \otimes V_N$ ; here  $V_N$  is an RB space provided for example by the POD-Greedy procedure of [1]. The RB output is then given by  $s_N = \int_0^T \ell(u_N(t)) dt$ . (It is possible, alternatively, to consider a space–time RB approximation as well [5].) It is then simple [3] to demonstrate

**Proposition 3.1.** *The RB error satisfies  $\| \|u - u_N\| \|_{\mathcal{X}} \leq \|r_N\|_{\mathcal{Y}'} / \beta_\delta^{\text{LB}}$ , where  $r_N(v) := f(v) - b(u_N, v), \forall v \in \mathcal{Y}$ , and  $\beta_\delta^{\text{LB}}$  is a lower bound for  $\beta_\delta$  defined in Proposition 2.1. Furthermore,  $|s_\delta - s_N| \leq \|\ell\|_{V'} \sqrt{T} \|r_N\|_{\mathcal{Y}'} / \beta_\delta^{\text{LB}}$ .*

The utility of these *a posteriori* error bounds is critically dependent on the dependence of  $\beta_\delta$  as a function of the parameter  $\mu$  and final time  $T$ ,  $\beta_\delta(\mu; T)$ .

We briefly comment on the computational implications of this space-time error bound. As always, our model problem (as is the case here) must honor the “affine-in-functions-of-parameter” condition to permit effective offline–online computation of the reduced basis approximation and *a posteriori* error bounds. The current formulation introduces the further complication of the space–time norms. In fact, calculation of the dual norm  $\| \cdot \|_{\mathcal{Y}'}$  (through the corresponding Riesz representation [3]) does not couple different time steps/elements, and hence the additional online difficulty is relatively minor. (The offline stage for computation of the inf-sup lower bound by the Successive Constraint Method [3, 6] does require special treatment, in particular due to the norm  $\| \| \cdot \| \|_{\mathcal{X}}$ .)

#### 4. Numerical Results

We report numerical results for the Crank–Nicolson scheme for various choices of the parameters  $\mu_1, \mu_2$  as well as for different time steps  $\Delta t$  and uniform mesh sizes  $h$ . For simplicity, we consider the univariate case (in space)  $\Omega = (0, 1)$  and choose  $\beta(x) = x - \frac{1}{2}$ . Let us denote by  $\beta_\delta(\mu; T)$  the numerical value for the truth inf-sup corresponding to parameter  $\mu$  and final time  $T$ . We start by confirming the quality of the estimate in Proposition 2.1. Thus, we choose  $\mu_1 = \mu_2 = 0$ ; for several values of  $T, h$  and  $\Delta t$  we invariantly obtain 1.000, indicating that our estimate is in fact sharp.

Next, we investigate the case of convection,  $\mu_2 = 0$ , in which case  $a$  is coercive only for  $\mu_1 < 2\pi^2$ . We get  $\beta_\delta((50, 0); 1.0) = 0.052$ ,  $\beta_\delta((50, 0); 2.0) = 0.028$  and  $\beta_\delta((100, 0); 1.0) = 0.020$ ,  $\beta_\delta((100, 0); 2.0) = 0.010$ . These numbers are mostly invariant for sufficiently small  $h$  and  $\Delta t$ . The overall behavior is  $\beta_\delta((\mu_1, 0); T) \sim (\mu_1 T)^{-1}$ . Note  $T = \mathcal{O}(1)$  is effectively a “long time” in convective units,  $1/\mu_1$ . We emphasize that although the problem is non-coercive, the problem is asymptotically stable in the sense that all eigenvalues  $\sigma$  of  $-a(\psi, \phi) = \sigma \langle \psi, \phi \rangle_H$  lie in the left-hand plane; this stability is reflected in the inf-sup behavior. In contrast, a standard energy approach [2] gives effective inf-sup constants on the order of  $e^{\mu_1 T}$  (here about  $10^{-10}$ ). Hence,

the traditional method fails whereas our new bound reflects the true *time-coupled* properties of the system.

Finally, we consider the case  $\mu_1 = 0$  which gives rise to an asymptotically unstable (and non-coercive) system for  $\mu_2 < -\pi^2$ . This means that any error estimate *must* grow exponentially with the final time  $T$ . We observe this for our estimator as well, e.g.  $\beta_\delta((0, -20); 1.0) = 5.62 \cdot 10^{-5}$  (order of  $e^{-\mu_2 T}$ ).

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