The fundamental definition of the Solvency Capital Requirement in Solvency II

Marcus Christiansen und Andreas Niemeyer

Fakultät für Mathematik und Wirtschaftswissenschaften

UNIVERSITÄT ULM

The fundamental definition of the Solvency Capital Requirement in Solvency II

Marcus C. Christiansen

Institut für Versicherungswissenschaften

Universität Ulm

D-89069 Ulm, Germany

Andreas Niemeyer

Institut für Versicherungswissenschaften

Universität Ulm

D-89069 Ulm, Germany

February 2, 2012

Abstract: It is essential for insurance regulation to have a clear picture of the risk measures that are used. We compare different mathematical interpretations of the Solvency Capital Requirement (SCR) definition that can be found in the literature. We introduce a mathematical modeling framework that allows us to make a mathematically rigorous comparison. The paper shows similarities, differences, and properties such as convergence of the different SCR interpretations. Moreover, we generalize the SCR definition to future points in time based on a generalization of the value at risk. This allows for a sound definition of the Risk Margin. Our study helps to make the Solvency II insurance regulation more consistent.

Keywords: Solvency II; Solvency Capital Requirement; Risk Margin; dynamic value at risk; minimal SCR

1. Introduction

Solvency II is the new regulation framework of the European Union for insurance and reinsurance companies. It will replace the Solvency I regime and is planned to become effective in 2013. One main aspect of Solvency II is the calculation of the Solvency Capital Requirement (SCR), which is the amount of own funds that an insurance company is required to hold. For the calculation of the market values of the liabilities, Solvency II suggests using a cost-of-capital method and defines the Risk Margin (RM). For calculating the SCR, each company can choose between setting up its own internal model and using a provided standard formula. The calculation standards were defined in the documents of the Committee of European Insurance and Occupational Pensions Supervisors (CEIOPS; the successor is EIOPA), but they are mainly described verbally. To our knowledge, truly mathematical definitions only currently exist for the standard formula.

Since Solvency II will have a significant impact on the European insurance industry, a large number of papers have already been published on that topic. For example, Devolder (2011) studies the capital requirement under different risk measurements, Eling et al. (2007) outline the characteristics of Solvency II, Doff (2008) makes a critical analysis of the Solvency II proposal, Steffen (2008) gives an overview of the project, Filipović (2009) analyzes the aggregation in the standard formula and

Holzmüller (2009) focuses on the relation between the United States risk-based capital standards, Solvency II and the Swiss Solvency Test. Only a few papers give a mathematically substantiated definition of the SCR, e.g. Barrieu et al. (2010), Bauer et al. (2010), Devineau and Loisel (2009), and Kochanski (2010). They all define the SCR only at time 0, except for Ohlsson and Lauzeningks (2009), who define the SCR for any point in time, but only within a chain ladder framework. Another problem is that different mathematical definitions are used. The reason is that the directive of the European Parliament and the Council (2009) describes the SCR only verbally, and from a mathematical point of view there is room for interpretation. This paper yields the first mathematical analysis of similarities and differences of the various interpretations of the SCR.

The RM is supposed to enable the calculation of the liabilities' market values. It is less discussed in the literature. For example, Floreani (2011) studies conceptual issues relating to the RM in a one period model, Kriele and Wolf (2007) consider different approaches for a RM and Salzmann and Wüthrich (2010) analyze the RM in a chain ladder framework. Generally, the RM is defined by a cost-of-capital approach and is based on future SCRs. However, no current broad definitions for the future of SCRs currently exist in the literature, which subsequently lacks a mathematically correct definition of the RM. This paper fills this gap and presents therefore a definition of a dynamic value at risk. Moreover we show that the circularity of the RM definition can generally be solved.

The paper is structured as follows. In Section 2 we present different interpretations of the fundamental SCR definition. Section 3 introduces a general modeling framework on which we base our analyses. Section 4 compares the different definitions. In Sections 5 and 6, we study convergence properties of the SCR definitions, and we discuss the SCR for insurance groups. With the help of the generalized SCR definitions of Section 3, we present a sound definition of the RM in Section 7. Section 8 gives an overview of the main findings and points out open problems.

2. The regulatory framework

In this section we discuss the fundamental definition of the SCR taking into account regulatory requirements. In the directive of the European Parliament and the Council (2009), which is the binding framework for Solvency II, we find the following two definitions of the SCR:

- Article 101 of the directive requires that the SCR "shall correspond to the Value-at-Risk of the basic own funds of an insurance or reinsurance undertaking subject to a confidence level of 99.5% over a one-year period."

- At the beginning of the directive an enumeration of remarks is given that has been attached to the directive. Remark 64 of the directive (European Parliament and the Council, 2009, page 24) says that "the Solvency Capital Requirement should be determined as the economic capital to be held by insurance ... undertakings in order to ensure ... that those undertakings will still be in a position with a probability of at least 99.5%, to meet their obligations to policy holders and beneficiaries over the following 12 months."

However, from a mathematical point of view there is room for interpretation and we have to clarify the fundamental mathematical definition of the SCR. The following definitions are possible interpretations of the Solvency II framework.

(a) Let N_t be the net value of assets minus liabilities at time t, and let $v(0, t)$ be a discount factor for the time period $[0, t]$. Then a possible interpretation of Article 101 is

$$SCR_0 := \text{VaR}_{0.995}\Big(N_0 - v(0,1)N_1\Big). \tag{2.1}$$

The proper choice of the discount factor is unclear. Article 101 does not give a definite answer.

(a1) Let $v^{rl}(0, t)$ be the discount factor that corresponds to a riskless interest rate. Then a possible specification of definition (a) is

$$SCR_0 := \text{VaR}_{0.995}\Big(N_0 - v^{rl}(0,1)N_1\Big). \tag{2.2}$$

Such a definition can be found e.g. in Bauer et al. (2010), Devineau and Loisel (2009), Floreani (2011) and Ohlsson and Lauzeningks (2009). In practice there are various ways to obtain a riskless interest rate. It can be theoretically derived from a model, or it can simply be defined as the returns on government bonds or real bank accounts. Note that the discount factor $v^{rl}(0,t)$ is usually random.

(a2) Let $v^{co}(0,t)$ be a discount factor that relates to the real capital gains that the insurance company really earns on its assets in the time period $[0,t]$. Then another possible specification of definition (a) is

$$SCR_0 := \text{VaR}_{0.995}\Big(N_0 - v^{co}(0,1)N_1\Big). \tag{2.3}$$

We are not aware of such a definition in the literature, although it has advantageous properties as we will see later on.

(b) Assuming the existence of a martingale measure Q that allows for a risk-neutral valuation of assets and liabilities, some authors, e.g. Barrieu et al. (2010) and Kochanski (2010), define the SCR according to Article 101 as

$$SCR_0 := \text{VaR}_{0.995}\Big(\mathbb{E}_Q\big(v^{rl}(0,1)N_1\big) - v^{rl}(0,1)N_1\Big). \tag{2.4}$$

(c) A mathematical interpretation of Remark 64 of the directive leads to

$$SCR_0 := \inf\big\{N_0 \in \mathbb{R} : \text{P}\,(N_\epsilon \geq 0, \varepsilon \in [0,1]) \geq 0.995\big\}.$$

In practice, N_t is not calculated continuously in t but only on a discrete time grid. In case of a yearly basis, we simplify the above definition to

$$SCR_0 := \inf\big\{N_0 \in \mathbb{R} : \text{P}\,(N_1 \geq 0) \geq 0.995\big\}. \tag{2.5}$$

Bauer et al. (2010) state that this is the intuitive definition of the SCR, while (a1) is an approximation of it. With this it is to be considered that N_1 depends on N_0, such that it is really a minimization problem.

The different interpretations of the SCR lead us to the following questions.

(1) Are (some of) the different definitions equivalent? If not, can we find additional conditions that make (some of) them equivalent?

(2) Are Article 101 and Remark 64 consistent? If not, which additional assumptions do we need to make them consistent?

(3) If the different definitions cannot be harmonized, are there other arguments that support or disqualify some versions?

So far we only have discussed the definition of a present SCR that gives the solvency requirement for today. However, for the calculation of the Risk Margin, which will be discussed in more detail in Section 7, we also have to define future SCRs that describe solvency requirements at future points in time.

(4) How can we mathematically define an SCR_s that describes the solvency requirement at a future time $s > 0$?

In the Solvency II standard formula the one year perspective is replaced by shocks that happen instantaneously. Consequently, there is no discount factor and so the standard formula does not answer the questions. We start with a small example that illustrates the SCR definitions. The example is kept very simple in order to make the differences between the definitions more clear.
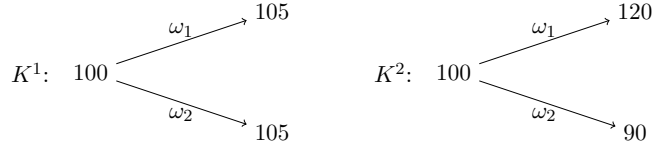
Figure 2.1: development of the assets

Example 2.1 (SCR of a riskless insurer). We consider a time horizon of one year and a financial market with two assets, a riskless bond K^1 and a stock K^2, which both have a price of $K_0^1 = K_0^2 = 100$ at time 0. Two scenarios $\Omega = \{\omega_1, \omega_2\}$ may occur, see Figure 2.1. Both scenarios shall have the same probability, $\mathrm{P}(\{\omega_1\}) = \mathrm{P}(\{\omega_2\}) = 0.5$. We consider a simplified insurance company that is closed to new business and which has an asset portfolio with no bonds and two stocks, $(H_0^1, H_0^2) = (0, 2)$. The insurance portfolio consists of just one unit-linked life-insurance with a sum insured of K_1^2 at time 1. Given that no assets are traded during the whole year, i.e. $(H_t^1, H_t^2) = (H_0^1, H_0^2)$ for all $t \in [0, 1]$, we obtain $N_0 = 100$ and $N_1 = K_1^2$. In the following we calculate the SCR according to the different definitions.

(i) Since the bond is riskless, the riskless discount factor is $v^{rl}(0, 1) = 1.05^{-1}$, and by definition (a1) we obtain

$$SCR_0 = \mathrm{VaR}_{0.995}\Big(100 - 1.05^{-1}K_1^2\Big) = \frac{100}{7}\,.$$

(ii) Since $v^{co}(0, 1) = \frac{100}{K_1^2}$, the SCR according to definition (a2) is

$$SCR_0 = \mathrm{VaR}_{0.995}\Big(100 - \frac{100}{K_1^2}K_1^2\Big) = 0\,.$$

(iii) Since we can show that $\mathrm{Q}(\{\omega_i\}) = \mathrm{P}(\{\omega_i\}) = 0.5$ for $i = 1, 2$, we have $v^{rl}(0,1)\mathbb{E}_{\mathrm{Q}}(N_1) = 100 = N_0$, and consequently the SCR according to definition (b) is equal to the SCR from definition (a1).

(iv) Definition (c) requires to minimizing N_0. Suppose that the company holds $H_0^2 \in \mathbb{R}$ stocks at time 0. Then $N_0 = (H_0^2 - 1)100$ and $N_1 = (H_1^2 - 1)K_1^2 = (H_0^2 - 1)K_1^2$, and thus we get

$$SCR_0 = \inf\big\{(H_0^2 - 1)100 : \mathrm{P}((H_0^2 - 1)K_1^2 \geq 0) \geq 0.995\big\}$$
$$= \inf\big\{(H_0^2 - 1)100 : H_0^2 \geq 1\big\} = 0\,.$$

The numerical example shows that the different SCR definitions are not generally equivalent. Which SCR is adequate here, zero or greater than zero? Let us recall the fundamental intention of the Solvency II project. According to Remark 16 (European Parliament and the Council, 2009, page 8) "the main objective of insurance and reinsurance regulation and supervision is the adequate protection of policy holders and beneficiaries." Consequently, if the company holds one stock it has a perfect hedge for the liabilities, and the policy holder is sufficiently protected. Hence, it seems reasonable to set SCR_0 equal to zero. However, the SCR definitions that are most frequently used in the literature, namely definitions (a1) and (b), both lead to an SCR greater than zero.

3. A mathematical modeling framework

For a more detailed comparison of the different SCR definitions of the previous section, we have to establish a more detailed modeling framework. We keep the model as simple as possible without any loss of generality. The modeling takes some effort, but in turn we will gain mathematically rigorous results.

Definition 3.1 (assets and liabilities). Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space with filtration $(\mathcal{F}_t)_{t \geq 0}$, and let all events in $\mathcal{F}_0$ either have probability one or zero.

1. Let K_t^i be the capital accumulation function that gives the market value of the investment i at time t ($1 \leq i \leq m$) and let $K_t = (K_t^1, ..., K_t^m)$. We assume that $(K_t^i)_{t \geq 0}$ is an adapted positive semimartingale.

2. Let H_t^i be the units of asset K_t^i that the insurer holds at time t ($1 \leq i \leq m$) and let $H_t = (H_t^1, ..., H_t^m)$. We assume that $(H_t^i)_{t \geq 0}$ is an adapted finite-variation process.

3. We define $A_t := H_t \cdot K_t := \sum_{i=1}^m H_t^i K_t^i$ as the market value of the assets that an insurer holds at time t. For technical reasons we generally assume that $A_t = 0$ implies $H_t = (0, ..., 0)$.

4. If $A_t \in \mathbb{R} \setminus \{0\}$, we define $\theta_t^i := \frac{H_t^i K_t^i}{H_t \cdot K_t}$ as the proportion that the market value of investment i has on the total market value of the asset portfolio. We call $(\theta_t)_t := (\theta_t^1, ..., \theta_t^m)_t$ the asset strategy of the insurer.

5. Let Z_t denote the sum of all actuarial payments on the interval $[0, t]$. We assume that $(Z_t)_{t \geq 0}$ is an adapted finite-variation process.

6. Let L_t be the time t market value of the liabilities of the insurer. We assume that $(L_t)_{t \geq 0}$ is some adapted stochastic process.

7. Let Y_t denote the sum of all payments to and from the shareholder. We assume that $(Y_t)_{t \geq 0}$ is an adapted finite-variation process.

Defining $(K_t^i)_{t \geq 0}$ as a semimartingale, $(H_t^i)_{t \geq 0}$, $(Z_t)_{t \geq 0}$, and $(Y_t)_{t \geq 0}$ as finite-variation processes, and $(L_t)_{t \geq 0}$ as an arbitrary stochastic process comprises basically all modeling frameworks that are used in the actuarial literature. The class of semimartingales includes diffusion processes as well as Lévy processes and discrete investment returns. The class of finite-variation processes includes absolutely continuous and discrete payments. Consequently, with this framework we are able to model almost all insurance companies. However, this framework allows constellations of parameters that does not correspond to a reasonable insurance company.

Proposition 3.2 (product rule). *The change in the market value of the assets at time t can be split up into a sum of the return from the investments and the gain through purchase and sale of investments, i.e.*

$$\mathrm{d}A_t = H_{t-} \cdot \mathrm{d}K_t + K_t \cdot \mathrm{d}H_t. \tag{3.1}$$

The proposition is an application of the integration by parts formula for semimartingales (Protter, 2005, page 68 and Theorem 28 in section II).

Assumption. Purchases and sales of investments $K_t \cdot \mathrm{d}H_t$ happen due to premium and insurance benefit payments and expenses $\mathrm{d}Z_t$ and due to payments to and from the shareholder $\mathrm{d}Y_t$,

$$K_t \cdot \mathrm{d}H_t = \mathrm{d}Z_t + \mathrm{d}Y_t. \tag{3.2}$$

Premium payments and payments from the shareholder have a positive sign, whereas insurance benefit payments and payments to the shareholder get a negative sign. Let

$$\mathrm{d}\overline{\phi}_t := \sum_{i=1}^m \theta_{t-}^i \mathrm{d}\phi_t^i \quad \text{with } \mathrm{d}\phi_t^i := \frac{\mathrm{d}K_t^i}{K_{t-}^i} \tag{3.3}$$

in case of $A_{t-} \neq 0$ and let $\mathrm{d}\overline{\phi}_t := 0$ for $A_{t-} = 0$. For each investment i, $\mathrm{d}\phi_t^i$ is the interest intensity for this investment. Since the total return from the investments of the insurer equals

$$H_{t-} \cdot \mathrm{d}K_t = A_{t-} \, \mathrm{d}\overline{\phi}_t,$$

5

we can interpret $\mathrm{d}\overline{\phi}_t$ as the company specific average interest intensity. Applying definition (3.3) and (3.2), equation (3.1) can be rewritten as

$$\mathrm{d}A_t = A_{t-}\,\mathrm{d}\overline{\phi}_t + \mathrm{d}Z_t + \mathrm{d}Y_t\,. \tag{3.4}$$

This means that the change in the market value of the assets at time t equals the sum of the investment return on the assets (with average interest rate $\mathrm{d}\overline{\phi}_t$), the actuarial payments, and the payments to and from the shareholder at time t. The unique solution (Protter, 2005, Theorem 37 in section II) of

$$\mathrm{d}\overline{K}_t = \overline{K}_{t-}\,\mathrm{d}\overline{\phi}_t, \quad \overline{K}_0 = 1,$$

is a positive semimartingale and describes the capital accumulation function of a synthetic asset composed according to $(\theta_t)_t$ that yields exactly the same interest rate as the investment portfolio of the insurer. Thus, we define the company specific discount factor (cf. (2.3)) by

$$v^{co}(s,t) := \frac{\overline{K}_s}{\overline{K}_t}. \tag{3.5}$$

The discount factor $v^{co}(s,t)$ is always positive since $(\overline{K}_t)_t$ is a positive semimartingale. Similarly, we assume that $v^{rl}(s,t)$ has a representation of the form $v^{rl}(s,t) = K_s^{rl}/K_t^{rl}$ where $\mathrm{d}K_t^{rl} = K_{t-}^{rl}\,\mathrm{d}\phi_t^{rl}$ with the riskless interest rate $\mathrm{d}\phi_t^{rl}$.

Proposition 3.3. *For all $0 \leq s \leq t < \infty$ we have*

$$v^{co}(s,t)A_t - A_s = \int\limits_{(s,t]} v^{co}(s,u)\,\mathrm{d}(Z_u + Y_u).$$

Proof. Let $\overline{H}_t := \overline{K}_t^{-1}H_t \cdot K_t = \overline{K}_t^{-1}A_t$. Analogously to the proof of Theorem 37 in chapter II of Protter (2005), we can show that $\overline{K}_t^{-1}$ is a semimartingale, and, thus, $\overline{H}_t$ is a semimartingale. Applying the integration by parts formula (Protter, 2005, page 68 and Theorem 28 in chapter II), we get $\mathrm{d}(\overline{H}_t\overline{K}_t) = \overline{H}_{t-}\,\mathrm{d}\overline{K}_t + \overline{K}_{t-}\,\mathrm{d}\overline{H}_t + \mathrm{d}[\overline{H},\overline{K}]_t$, where the third addend is the so called quadratic variation. Since the definition of $\overline{K}_t$ implies that

$$\overline{H}_{t-}\,\mathrm{d}\overline{K}_t = \overline{H}_{t-}\,\overline{K}_{t-}\,\mathrm{d}\overline{\phi}_t = A_{t-}\,\mathrm{d}\overline{\phi}_t = H_{t-} \cdot \mathrm{d}K_t$$

and since $\mathrm{d}(\overline{H}_t\overline{K}_t) = \mathrm{d}A_t = \mathrm{d}(H_t \cdot K_t)$, we have $\overline{K}_{t-}\,\mathrm{d}\overline{H}_t + \mathrm{d}[\overline{H},\overline{K}]_t = K_t \cdot \mathrm{d}H_t$. The right hand side has finite variation, because H_t has finite variation. As the bracket process $\mathrm{d}[\overline{H},\overline{K}]_t$ also has finite variation, $\overline{K}_{t-}\,\mathrm{d}\overline{H}_t$ must have finite variation, too. As stochastic integrals with finite variation integrator and left-continuous integrand always have finite variation (Protter, 2005, page 63 and Theorem 17 in chapter II), we get that $\mathrm{d}\overline{H}_t = \overline{K}_t^{-1}(\overline{K}_{t-}\,\mathrm{d}\overline{H}_t)$ has finite variation, as well. Hence, we get

$$\overline{K}_t\,\mathrm{d}\overline{H}_t = \overline{K}_{t-}\,\mathrm{d}\overline{H}_t + \mathrm{d}[\overline{H},\overline{K}]_t = K_t \cdot \mathrm{d}H_t = \mathrm{d}Z_t + \mathrm{d}Y_t.$$

With the help of this equality we obtain

$$\frac{\overline{K}_s}{\overline{K}_t}A_t - A_s = \overline{K}_s\,(\overline{H}_t - \overline{H}_s) = \int\limits_{(s,t]} \overline{K}_s\,\mathrm{d}\overline{H}_u = \int\limits_{(s,t]} \frac{\overline{K}_s}{\overline{K}_u}\,\overline{K}_u\,\mathrm{d}\overline{H}_u = \int\limits_{(s,t]} v^{co}(s,u)\,\mathrm{d}(Z_t + Y_t).$$

$\square$

Definition 3.4. Let $N_t := A_t - L_t$ be the net value at time t, which is defined as the difference between the market value of the assets and the liabilities.

The net value should not be confused with the net asset value, which is used in the official Solvency II documents. The net asset value is the market value of the assets minus the best estimate of the liabilities. The net value can be interpreted as the economic equity.

In definition (c) of the previous section we have to minimize the value of the asset portfolio at time zero. Because upsizing and downsizing of the asset portfolio can be disproportional to the existing portfolio, we have to extend our modeling framework. Suppose that we have an insurance company with asset portfolio $(H_t)_t$, liabilities $(L_t)_t$, actuarial payments $(Z_t)_t$, and shareholder payments $(Y_t)_t$. We assume that the asset portfolio may be shifted to $(H_t + \widetilde{H}_t)_t$ such that the market value of the assets at time t changes to $A_t + \widetilde{A}_t = H_t \cdot K_t + \widetilde{H}_t \cdot K_t$. Accordingly, the new net value at time t is

$$\widetilde{N}_t = A_t + \widetilde{A}_t - L_t \,,$$

since the liabilities do not change, i.e. $\widetilde{L}_t = 0$. We interpret $(\widetilde{H}_t)_t$ as an additional asset portfolio that evolves according to

$$\mathrm{d}\widetilde{A}_t = \widetilde{A}_{t-}\, \mathrm{d}\widetilde{\phi}_t,$$

see (3.4), where the average interest rate $\mathrm{d}\widetilde{\phi}_t$ and the asset strategy $\widetilde{\theta}_t$ of the asset portfolio $(\widetilde{H}_t)_t$ are defined analogously to (3.3). Note, that $\widetilde{A}_t$ and $\widetilde{H}_t$ can be negative. Again, $\widetilde{A}_t = 0$ implies $\widetilde{H}_t = (0, ..., 0)$. We assume that L_t is independent of $\widetilde{A}_t$. Accordingly, we can define a discount factor for the additional asset portfolio

$$v^{ad}(s,t) := \frac{\widetilde{K}_s}{\widetilde{K}_t} \quad \text{with } \mathrm{d}\widetilde{K}_t = \widetilde{K}_{t-}\, \mathrm{d}\widetilde{\phi}_t, \ \widetilde{K}_0 = 1.$$

Since the differential equations for $\widetilde{A}_t$ and $\widetilde{K}_t$ are equal up to the initial condition, we have

$$v^{ad}(s,t) = \frac{\widetilde{A}_s}{\widetilde{A}_t}. \tag{3.6}$$

Theorem 3.5. *Given that $\widetilde{N}_0$ is deterministic, we have*

$$\inf \left\{ \widetilde{N}_0 \in \mathbb{R} : \mathrm{P}(\widetilde{N}_1 \geq 0) \geq 0.995 \right\} = \mathrm{VaR}_{0.995}\!\left(N_0 - v^{ad}(0,1)N_1 \right). \tag{3.7}$$

Proof. From (3.6) we get $\widetilde{A}_{s+1} = v^{ad}(s, s+1)^{-1}\widetilde{A}_s$, which leads to

$$\widetilde{N}_{s+1} = N_{s+1} + v^{ad}(s, s+1)^{-1}\widetilde{A}_s = N_{s+1} + v^{ad}(s, s+1)^{-1}\left(\widetilde{N}_s - N_s \right),$$

and since $v^{ad}(s, s+1)$ is always positive, we obtain

$$\{\widetilde{N}_{s+1} \geq 0\} = \{v^{ad}(s, s+1)\, \widetilde{N}_{s+1} \geq 0\} = \{N_s - v^{ad}(s, s+1)\, N_{s+1} \leq \widetilde{N}_s\} \tag{3.8}$$

for all $s \geq 0$. If $\widetilde{N}_0$ is deterministic, the left hand side of (3.7) is well-defined and equals the right hand side of (3.7) because

$$\mathrm{P}\!\left(\widetilde{N}_1 \geq 0 \right) = \mathrm{P}\!\left(N_0 - v^{ad}(0,1)\, N_1 \leq \widetilde{N}_0 \right).$$

$\square$

The corollary allows us to substitute definition (2.5) with

$$SCR_0^c := \mathrm{VaR}_{0.995}\!\left(N_0 - v^{ad}(0,1)N_1 \right). \tag{3.9}$$

The assumption that $\widetilde{N}_0$ is deterministic means that $\widetilde{N}_0$ is known at present time zero, which meets with reality. The equality (3.7) is at first view surprising, since the left- and right-hand side are different in structure. This difference is especially important when Monte-Carlo simulations are used. In order to calculate the left-hand side, a starting level of $\widetilde{N}_0$ is needed before $\widetilde{N}_1$ can be simulated.

This simulation only approximates the ruin probability for this starting level. Consequently, we need methods such as nested intervals and the simulation has to be run over and over again until the desired ruin probability is reached. In contrast, the right-hand side can be calculated with one run of simulations.

For the definition of a dynamic value at risk we need the following proposition. The proof can be found in the appendix. In a discrete time setting, a dynamic value at risk was already introduced by Kriele and Wolf (2012). We give a more general definition for continuous time intervals.

Proposition 3.6. *Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space with random variables $X_{[0,s]} : (\Omega, \mathcal{F}) \to (\Omega', \mathcal{F}'_s)$ and $Y : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathfrak{B}(\mathbb{R}))$, where $(\Omega', \mathcal{F}'_s)$ is some measurable space. Then the function $h : \Omega' \to \mathbb{R}$ defined by*

$$h_{Y,\alpha}(x) := \inf\{y \in \mathbb{R} : \mathrm{P}(Y \leq y | X_{[0,s]} = x) \geq \alpha\}$$

is $\mathcal{F}'_s$-$\mathfrak{B}(\mathbb{R})$-measurable.

Definition 3.7 (dynamic value at risk)**.** Suppose that the assumptions of Proposition 3.6 hold, and let $\mathcal{F}_s = \sigma(X_{[0,s]}) \subset \mathcal{F}$, that is, $\mathcal{F}_s$ is generated by $X_{[0,s]}$. Then we define for $\alpha \in (0,1)$

$$\mathrm{VaR}_\alpha\big(Y \big| \mathcal{F}_s\big) := h_{Y,\alpha}(X_{[0,s]}).$$

With the help of Definition 3.7 we can generalize the SCR definitions (2.1), (2.2), (2.3), (2.4) and (3.9) to future points in time by replacing the values at risk by dynamic values at risk.

Definition 3.8 (present and future SCR)**.** The SCR at time s is defined as

(a) $SCR_s^a := \mathrm{VaR}_{0.995}\Big(N_s - v(s, s+1)\, N_{s+1} \Big| \mathcal{F}_s\Big),$

(a1) $SCR_s^{a1} := \mathrm{VaR}_{0.995}\Big(N_s - v^{rl}(s, s+1)\, N_{s+1} \Big| \mathcal{F}_s\Big),$

(a2) $SCR_s^{a2} := \mathrm{VaR}_{0.995}\Big(N_s - v^{co}(s, s+1)\, N_{s+1} \Big| \mathcal{F}_s\Big),$

(b) $SCR_s^b := \mathrm{VaR}_{0.995}\Big(v^{rl}(s, s+1)\big(\mathbb{E}_{\mathrm{Q}}(N_{s+1}|\mathcal{F}_s) - N_1\big) \Big| \mathcal{F}_s\Big),$

(c) $SCR_s^c := \mathrm{VaR}_{0.995}\Big(N_s - v^{ad}(s, s+1)\, N_{s+1} \Big| \mathcal{F}_s\Big).$

4. Comparison of the different SCR definitions

In a next step we want to learn if and when the different SCR definitions are equivalent. We start with an example.

Example 4.1. We consider a simplified insurance company that is closed to new business and has to fulfill insurance liabilities of 105 in one year (e.g. an endowment insurance without surplus participation and a remaining term of one year). We consider the same financial market as in Example 2.1. The company holds one riskless bond K^1 and one stock K^2, i.e. $H_0^1 = H_0^2 = 1$. At time 0 the insurer has assets of $A_0 = 200$ and liabilities of $L_0 = \frac{105}{1.05} = 100$. Consequently, we have $N_0 = 100$ and $N_1 = K_1^2$, since $K_1^1 = 105$. In the following we calculate the SCR at time 0 according to the different SCR definitions.

(i) The SCR calculated with definition (a1) is

$$SCR_0^{a1} = \mathrm{VaR}_{0.995}\Big(N_0 - v^{rl}(0,1)N_1\Big) = \mathrm{VaR}_{0.995}\Big(100 - \frac{K_1^2}{1.05}\Big) = \frac{100}{7}\,.$$

(ii) The SCR calculated with definition (a2) is

$$SCR_0^{a2} = \text{VaR}_{0.995}\Big(N_0 - v^{co}(0,1)N_1\Big) = \text{VaR}_{0.995}\Big(100 - \frac{200}{K_1^2 + 105}K_1^2\Big) = \frac{100}{13}\,.$$

(iii) Since $v^{rl}(0,1)\mathbb{E}_Q(K_1^2) = 100 = N_0$, we obtain by definition (b)

$$SCR_0^b = SCR_0^{a1} = \frac{100}{7}\,.$$

(iv) Equation (3.7) gives us two possibilities to calculate the SCR, both of which we want to demonstrate. We assume that the insurer has additional assets $\widetilde{A}_0$ and that $\widetilde{\theta}_0^1 \in [0,1]$ is the percentage invested in the bond. Consequently, $\widetilde{\theta}_0^2 = 1 - \widetilde{\theta}_0^1$ is invested in the stock. In this example, we do not allow short sales, and the units of the additional assets shall not change, i.e. $\widetilde{H}_0^1 = \widetilde{H}_1^1$ and $\widetilde{H}_0^2 = \widetilde{H}_1^2$. We obtain $\widetilde{N}_0 = 100 + \widetilde{A}_0$ and $\widetilde{N}_1 = K_1^2 + \widetilde{A}_0\, v^{ad}(0,1)^{-1}$ with

$$v^{ad}(0,1) = \frac{\widetilde{A}_0}{\widetilde{A}_1} = \Big(\widetilde{\theta}_0^1 1.05 + (1 - \widetilde{\theta}_0^1)\frac{K_1^2}{100}\Big)^{-1}.$$

The left-hand side of (3.7) is

$$SCR_0^c = \inf\big\{\widetilde{N}_0 \in \mathbb{R} : \text{P}(\widetilde{N}_1 \geq 0) \geq 0.995\big\} = \inf\big\{100 + \widetilde{A}_0 : \widetilde{A}_0 \geq -600(\widetilde{\theta}_0^1 + 6)^{-1}\big\}$$
$$= 100 - 600(\widetilde{\theta}_0^1 + 6)^{-1}\,.$$

For the right-hand side of (3.7) we get

$$SCR_0^c = \text{VaR}_{0.995}\Big(N_0 - v^{ad}(0,1)N_1\Big) = 100 - 600(\widetilde{\theta}_0^1 + 6)^{-1}\,,$$

which is of course equal to the value above. The resulting SCR varies considerably depending on the choice of $\widetilde{\theta}_0^1$. One possibility for $\widetilde{\theta}_0^1$ would be to minimize over this parameter. This would lead to $SCR_0^c = 0$. The initial ratio of bonds θ_0^1 was $\frac{1}{2}$. We can choose $\widetilde{\theta}_0^1$ such that definitions (a2) and (c) are equal,

$$SCR_0^c = SCR_0^{a2} \Leftrightarrow 100 - 600(\widetilde{\theta}_0^1 + 6)^{-1} = \frac{100}{13} \Leftrightarrow \widetilde{\theta}_0^1 = \frac{1}{2} = \theta_0^1\,.$$

To summarize, we found that

- depending on the choice of $(\widetilde{\theta}_t)_t$, SCR_0^c can be equal to any of the other SCR definitions,

- SCR_0^{a2} equals SCR_0^c if $(\widetilde{\theta}_t)_t = (\theta_t)_t$,

- SCR_0^{a1} equals SCR_0^b.

We will see that all three facts do not only hold for our specific example but are generally true.

Theorem 4.2. *We have $SCR_s^a = SCR_s^c$ for all financial markets $(K_t)_t$, actuarial functions $(Z_t)_t$, and liability market values $(L_t)_t$ if and only if $v(s, s+1) = v^{ad}(s, s+1)$ almost surely.*

Proof. If $v(s, s+1) = v^{ad}(s, s+1)$ almost surely, then the definitions of SCR_s^a and SCR_s^c are equivalent. Suppose now that $\text{P}(v(s, s+1) \neq v^{ad}(s, s+1)) > 0$. Without loss of generality let $\text{P}(v(s, s+1) > v^{ad}(s, s+1)) \leq \text{P}(v(s, s+1) < v^{ad}(s, s+1))$. We define a disjoint decomposition $M_0 \cup M_1 \cup M_2 = \Omega$ by $\{v(s, s+1) > v^{ad}(s, s+1)\} \subset M_0 \in \mathcal{F}$, $\{v(s, s+1) < v^{ad}(s, s+1)\} = M_1 \in \mathcal{F}$, and $\{v(s, s+1) = v^{ad}(s, s+1)\} \supset M_2 \in \mathcal{F}$ with $P(M_2) < 0.005$. Since $P(M_0) \leq 0.5$, we have $P(M_1 \cup M_2) > 0.005$, and by setting $N_s = 0$ and

$$N_{s+1} = -\frac{1}{v^{ad}(s, s+1)}\mathbf{1}_{M_1 \cup M_2}$$

we obtain $SCR_s^c \geq 1$ since $\text{P}(N_s - v^{ad}(s, s+1)\,N_{s+1} \geq 1) = \text{P}(M_1 \cup M_2) > 0.005$ and $SCR_s^a < 1$ since $\text{P}(N_s - v(s, s+1)\,N_{s+1} < 1) = \text{P}(M_0 \cup M_1) > 0.995$. That means that $SCR_s^a \neq SCR_s^c$. □

Remark 4.3 (Invariance of SCR with respect to the initial capital). Since (3.6) yields that $v^{ad}(s, s+1) \widetilde{A}_{s+1} = \widetilde{A}_s$, we generally have

$$\widetilde{N}_s - v^{ad}(s, s+1)\widetilde{N}_{s+1} = N_s - v^{ad}(s, s+1)N_{s+1},$$

which implies that

$$SCR_s^c = \mathrm{VaR}_{0.995}\Big(N_s - v^{ad}(s, s+1)N_{s+1}\Big|\mathcal{F}_s\Big) = \mathrm{VaR}_{0.995}\Big(\widetilde{N}_s - v^{ad}(s, s+1)\widetilde{N}_{s+1}\Big|\mathcal{F}_s\Big). \qquad (4.1)$$

Originally, we motivated the definition of SCR_0^c by the net asset value minimization (2.5), which is by definition invariant with respect to the initial net asset value. Equation (4.1) says that this invariance property remains true for $s > 0$, given that additional assets are invested according to $(\widetilde{\theta}_t)_t$. By setting $v^{ad}(s, s+1) = v^{rl}(s, s+1)$ and $v^{ad}(s, s+1) = v^{co}(s, s+1)$, we analogously get that SCR_s^{a1} and SCR_s^{a2} are invariant with respect to the initial net asset value if additional capital is invested risklessly and proportionally to the existing asset portfolio, respectively. Hence, we can say that definitions (a1) and (a2) implicitly assume that redundant capital is invested riskless and proportional to the existing portfolio, respectively.

The next proposition analyzes the relationship between the discount factors, θ_t, and H_t.

Theorem 4.4. *Given that θ_t and $\widetilde{\theta}_t$ exist, we have $\theta_t = \widetilde{\theta}_t$ if and only if $H_t = \Psi_t \widetilde{H}_t$ for some real-valued and $\mathcal{F}_t$-measurable random variable $\Psi_t \in \mathbb{R} \setminus \{0\}$. We have $v^{co}(s, s+1) = v^{ad}(s, s+1)$ for all financial markets $(K_t)_t$ if and only if $\theta_t = \widetilde{\theta}_t$ almost surely on $(s, s+1]$.*

Proof. Let $\Omega = B_1 \cup ... \cup B_m$ be a disjoint and $\mathcal{F}_t$-measurable decomposition of Ω such that $\theta_t^j(\omega) > 0$ for $\omega \in B_j$, which implies that $H_t^j(\omega) \neq 0$ for $\omega \in B_j$. Such a decomposition always exists since we supposed that θ_t exists and since θ_t is $\mathcal{F}_t$-measurable. By solving the system of linear equations $\theta_t^i = \frac{H_t^i K_t^i}{H_t \cdot K_t}$, $i \in \{1, ..., m\}$, we obtain that the definition of θ_t is equivalent to

$$\frac{H_t^i(\omega)}{H_t^j(\omega)} = \frac{\theta_t^i(\omega)}{\theta_t^j(\omega)} \frac{K_t^j(\omega)}{K_t^i(\omega)}, \quad \omega \in B_j, \ i \in \{1, ..., m\} \setminus \{j\}.$$

An analogous result holds for $\widetilde{\theta}_t$ and $\widetilde{H}_t$, and if we assume that $\theta_t = \widetilde{\theta}_t$, we may set $B_j = \widetilde{B}_j$ and obtain

$$\frac{H_t^i(\omega)}{H_t^j(\omega)} = \frac{\widetilde{H}_t^i(\omega)}{\widetilde{H}_t^j(\omega)}, \quad \omega \in B_j, \ i \in \{1, ..., m\} \setminus \{j\}.$$

Thus, we can define Ψ_t by $\Psi_t(\omega) := H_t^j(\omega)/\widetilde{H}_t^j(\omega)$ for $\omega \in B_j$, which is non-zero and $\mathcal{F}_t$-measurable as $H_t^j(\omega)$ and $\widetilde{H}_t^j(\omega)$ are non-zero and $\mathcal{F}_t$-measurable and $B_j \in \mathcal{F}_t$, $j \in \{1, ..., m\}$. On the other hand, if we assume that $H_t = \Psi_t \widetilde{H}_t$, then the definitions of θ_t and $\widetilde{\theta}_t$ yield $\theta_t \frac{\Psi_t}{\Psi_t} = \widetilde{\theta}_t$.

Suppose now that there exists an $i_0 \in \{1, ..., m\}$ and a $t_0 \in (s, s+1]$ for which $\mathrm{P}(\theta_{t_0}^{i_0} \neq \widetilde{\theta}_{t_0}^{i_0}) > 0$. By defining $K_t^{i_0}(\omega) := 1 + \mathbf{1}_{[t_0, \infty)}(t)$ and $K_t^j(\omega) := 1$ for all $j \neq i_0$, from (3.5) and (3.3) we get

$$v^{co}(s, t) = \frac{1}{1 + \theta_{t_0-}^{i_0}}, \quad v^{ad}(s, t) = \frac{1}{1 + \widetilde{\theta}_{t_0-}^{i_0}}.$$

Thus, we obtain $\mathrm{P}(v^{ad}(s, s+1) = v^{co}(s, s+1)) = \mathrm{P}(\theta_{t_0}^{i_0} = \widetilde{\theta}_{t_0}^{i_0}) < 1$. On the other hand, if $(\theta_t)_t = (\widetilde{\theta}_t)_t$ almost surely, then we also have $(\overline{\phi}_t)_t = (\widetilde{\phi}_t)_t$ almost surely for the corresponding cumulative interest intensities according to (3.3) and, thus, $v^{co}(s, t) = v^{ad}(s, t)$ almost surely. $\qquad \square$

Remark 4.5. SCR definition (c) can change considerably depending on the choice of $v^{ad}(s, s+1)$. While in definitions (a1), (a2), and (b) the discount factor is largely determined by pre-existing circumstances, the discount factor $v^{ad}(s, s+1)$ is mainly a management decision that the insurer has to make by appointing an asset strategy $(\widetilde{\theta}_t)_t$.

By considering both Theorem 4.2 and Theorem 4.4, we get that $SCR_s^{a2} = SCR_s^c$ for all financial markets $(K_t)_t$, actuarial functions $(Z_t)_t$, and liability market values $(L_t)_t$ if and only if $\theta_t = \widetilde{\theta}_t$ almost surely on $(s, s+1]$. This property was already indicated by Example 4.1. The same example also indicates that (a1) and (b) are equal, which we can actually prove by using the following proposition.

Theorem 4.6. *We assume a risk-neutral measure* Q *and a discount factor* v^{rl} *such that*

$$K_t = \mathbb{E}_Q(v^{rl}(t, u)\, K_u | \mathcal{F}_t)$$

for all $u \geq t \geq 0$ *and such that*

$$L_t = \int_{(t,\infty)} \mathbb{E}_Q(v^{rl}(t, u)\, \mathrm{d}(-Z_u - Y_u) | \mathcal{F}_t)$$

for all $t \geq 0$. *Then we have*

$$N_s = \mathbb{E}_Q(v^{rl}(s, s+1)\, N_{s+1} | \mathcal{F}_s)$$

for all $s \geq 0$.

Proof. From the martingale property $K_t = \mathbb{E}_Q(v^{rl}(t, u)\, K_u | \mathcal{F}_t)$ we can deduce $\overline{K}_t = \mathbb{E}_Q(v^{rl}(t, u)\, \overline{K}_u | \mathcal{F}_t)$, which implies

$$\mathbb{E}_Q\big(v^{rl}(t, u)(v^{co}(t, u))^{-1} | \mathcal{F}_t\big) = 1$$

for all $u \geq t \geq 0$. Then, by applying Proposition 3.3, we get

$$
\begin{aligned}
\mathbb{E}_Q(v^{rl}(s, s+1)\, A_{s+1} | \mathcal{F}_s) &= \mathbb{E}_Q\Big(v^{rl}(s, s+1)\, (v^{co}(s, s+1))^{-1} A_s \\
&\quad + \int_{(s,s+1]} v^{rl}(s, s+1)\, (v^{co}(u, s+1))^{-1}\, \mathrm{d}(Z_u + Y_u) \Big| \mathcal{F}_s\Big) \\
&= A_s + \int_{(s,s+1]} \mathbb{E}_Q\big(v^{rl}(s, s+1)\, (v^{co}(u, s+1))^{-1}\, \mathrm{d}(Z_u + Y_u) \big| \mathcal{F}_s\big).
\end{aligned}
$$

As $Z_u + Y_u$ is of finite variation and has a representation of the form $\mathrm{d}(Z_u + Y_u) = (z_u + y_u)\, \mathrm{d}u + \Delta(Z_u + Y_u)$, we have

$$
\begin{aligned}
&\mathbb{E}_Q\big(v^{rl}(s, s+1)\, (v^{co}(u, s+1))^{-1}\, \mathrm{d}(Z_u + Y_u) \big| \mathcal{F}_s\big) \\
&= \mathbb{E}_Q\Big(v^{rl}(s, u)\, \mathbb{E}_Q\big(v^{rl}(u, s+1)\, (v^{co}(u, s+1))^{-1} \big| \mathcal{F}_u\big)\, (z_u + y_u) \Big| \mathcal{F}_s\Big)\, \mathrm{d}u \\
&\quad + \mathbb{E}_Q\Big(v^{rl}(s, u)\, \mathbb{E}_Q\big(v^{rl}(u, s+1)\, (v^{co}(u, s+1))^{-1} \big| \mathcal{F}_u\big)\, \Delta(Z_u + Y_u) \Big| \mathcal{F}_s\Big) \\
&= \mathbb{E}_Q\big(v^{rl}(s, u)\, (z_u + y_u) \big| \mathcal{F}_s\big)\, \mathrm{d}u + \mathbb{E}_Q\big(v^{rl}(s, u)\, \Delta(Z_u + Y_u) \big| \mathcal{F}_s\big),
\end{aligned}
$$

and, thus, we obtain

$$\mathbb{E}_Q(v^{rl}(s, s+1)\, A_{s+1} | \mathcal{F}_s) = A_s + \int_{(s,s+1]} \mathbb{E}_Q\big(v^{rl}(s, u)\, \mathrm{d}(Z_u + Y_u) \big| \mathcal{F}_s\big).$$

On the other hand, we have

$$
\begin{aligned}
\mathbb{E}_Q(v^{rl}(s, s+1)\, L_{s+1} | \mathcal{F}_s) &= \mathbb{E}_Q\Big(L_s - \int_{(s,s+1]} v^{rl}(s, u)\, \mathrm{d}(-Z_u - Y_u) \Big| \mathcal{F}_s\Big) \\
&= L_s + \int_{(s,s+1]} \mathbb{E}_Q\big(v^{rl}(s, u)\, \mathrm{d}(Z_u + Y_u) \big| \mathcal{F}_s\big).
\end{aligned}
$$

Hence, we get

$$\mathbb{E}_Q(v^{rl}(s, s+1)\, N_{s+1} | \mathcal{F}_s) = \mathbb{E}_Q(v^{rl}(s, s+1)\, A_{s+1} - v^{rl}(s, s+1)\, L_{s+1} | \mathcal{F}_s) = A_s - L_s = N_s.$$

$\square$

Under the assumptions of Theorem 4.6 we always have

$$\mathrm{VaR}_{0.995}\Big(N_s - v(s, s+1)\, N_{s+1}\Big|\mathcal{F}_s\Big)$$

$$= \mathrm{VaR}_{0.995}\Big(\mathbb{E}_Q(v^{rl}(s, s+1)\, N_{s+1}|\mathcal{F}_s) - v(s, s+1)\, N_{s+1}\Big|\mathcal{F}_s\Big)$$

for any choice of $v(s, s+1)$. In particular, for $v(s, s+1) = v^{rl}(s, s+1)$ we get

$$SCR_s^{a1} = \mathrm{VaR}_{0.995}\Big(\mathbb{E}_Q(v^{rl}(s, s+1)\, N_{s+1}|\mathcal{F}_s) - v^{rl}(s, s+1)\, N_{s+1}\Big|\mathcal{F}_s\Big) = SCR_s^b.$$

for all $s \geq 0$. Because of this equivalence of SCR_s^{a1} and SCR_s^b, it suffices to study definitions (a1), (a2), and (c) only.
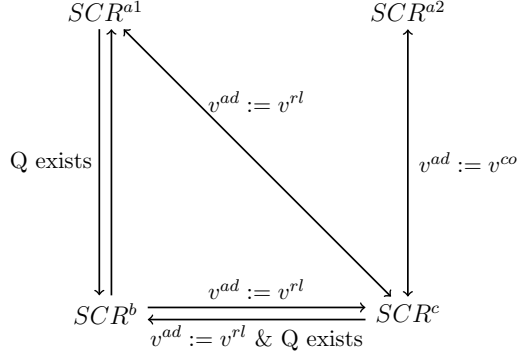

Figure 4.1: Relation between the SCR definitions

5. Convergence of SCR definitions

Example 5.1 (Convergence of SCR definitions (a1) and (a2) to (c))**.** We pick up example 4.1 but set $\widetilde{\theta}_0^1 = 0.4$, which has the effect that definitions (a2) and (c) are not equal anymore. We still have $SCR_0^{a1} = \frac{100}{7}$ and $SCR_0^{a2} = \frac{100}{13}$, but $SCR_0^c = \frac{25}{4}$. Suppose that the insurer uses SCR definition (a1) and aims at minimizing the asset portfolio. At time zero, there is a net value of $N_0 = 100$, but the regulatory requirement is just $y^{(1)} = SCR_0^{a1} = \frac{100}{7}$, so the insurer reduces $A_0 = 200$ to $100 + \frac{100}{7}$. The reduction of A_0 follows strategy $\widetilde{\theta}_0$, which means that bonds worth $\frac{240}{7}$ and stocks worth $\frac{360}{7}$ are paid out. However, the change of A_0 has an effect on SCR_0^{a1}, and a recalculation of the SCR (for $N_0 = \frac{100}{7}$) yields $y^{(2)} = \frac{340}{49}$. Thus, the insurer has to adapt the asset portfolio A_0 again. By repeating this procedure n-times, we get $y^{(n)} = \frac{25}{4} + (\frac{3}{35})^n \frac{375}{4}$, which converges for $n \to \infty$ to $\frac{25}{4} = SCR_0^c$. The same procedure for SCR definition (a2) leads to $y^{(n)} = \frac{25}{4} + (\frac{1}{65})^n \frac{375}{4}$, which converges also to SCR_0^c for $n \to \infty$. It is essential here that in each iteration step strategy $\widetilde{\theta}_0$ is used. If, for example, the company cuts the bonds only but keeps all stocks, SCR_0^{a1} would not have changed.

In fact, we can show that SCR^a always converges to SCR^c. Before we put this into a theorem, we introduce the following setting. An insurer has a net value of $y^{(0)} := N_s$ and calculates the SCR

$$y^{(1)} := SCR_s^a = \mathrm{VaR}_{0.995}\Big(N_s - v(s, s+1)\, N_{s+1}\Big|\mathcal{F}_s\Big).$$

In a next step the company reduces the asset portfolio by $\widetilde{A}_s$ such that the new net value is $\widetilde{N}_s = N_s + \widetilde{A}_s = y^{(1)}$. According to (3.6) we have $\widetilde{A}_{s+1} = v^{ad}(s, s+1)^{-1}\widetilde{A}_s$, and thus we get $\widetilde{N}_{s+1} = N_{s+1} + (y^{(1)} - N_s)v^{ad}(s, s+1)^{-1}$. With $y^{(2)}$ we denote the SCR that corresponds to the altered net value. As $y^{(2)}$ is not necessarily equal to $y^{(1)}$, the asset portfolio is again re-organized such that $\widetilde{N}_s = N_s + \widetilde{A}_s = y^{(2)}$. By repeating this procedure n-times, we obtain

$$y^{(n)} = \mathrm{VaR}_{0.995}\Big(y^{(n-1)} - v(s, s+1)\Big(N_{s+1} + (y^{(n-1)} - N_s)v^{ad}(s, s+1)^{-1}\Big)\Big|\mathcal{F}_s\Big). \tag{5.1}$$

Theorem 5.2. *The random variable SCR_s^c is the (almost surely) unique fix-point of iteration (5.1). If there exits an $\epsilon \in (0,1)$ such that*

$$\epsilon < \frac{v(s, s+1)}{v^{ad}(s, s+1)} < 2 - \epsilon, \tag{5.2}$$

then $\lim_{n \to \infty} y^{(n)} = SCR_s^c$, where $y^{(n)}$ is defined as in (5.1).

Proof. If y is a fix-point of (5.1), then

$$y = \text{VaR}_{0.995}\left(y - v(s, s+1)\left(N_{s+1} + (y - N_s)v^{ad}(s, s+1)^{-1}\right)\Big|\mathcal{F}_s\right)$$

$$\Leftrightarrow 0 = \text{VaR}_{0.995}\left(\frac{v(s, s+1)}{v^{ad}(s, s+1)}\left(N_s - v^{ad}(s, s+1)N_{s+1} - y\right)\Big|\mathcal{F}_s\right)$$

$$\Leftrightarrow 0 = \text{VaR}_{0.995}\left(N_s - v^{ad}(s, s+1)N_{s+1} - y\Big|\mathcal{F}_s\right)$$

$$\Leftrightarrow y = \text{VaR}_{0.995}\left(N_s - v^{ad}(s, s+1)N_{s+1}\Big|\mathcal{F}_s\right) = SCR_s^c.$$

In the third line, we use that $\frac{v(s,s+1)}{v^{ad}(s,s+1)}$ is greater than zero such that we can omit it, see Proposition A.1 in the appendix. The equivalences yield that SCR_s^c is always a fix-point and that all fix-points equal SCR_s^c.

By multiplying equation (5.2) with -1, adding 1, and multiplying the result with $y^{(n)} - SCR_s^c$ separately for $y^{(n)} - SCR_s^c \geq 0$ or $y^{(n)} - SCR_s^c < 0$, we get

$$-(1 - \epsilon)\big|y^{(n)} - SCR_s^c\big| \leq (y^{(n)} - SCR_s^c)\left(1 - \frac{v(s, s+1)}{v^{ad}(s, s+1)}\right) \leq (1 - \epsilon)\big|y^{(n)} - SCR_s^c\big|.$$

From this inequality and equation (5.1) we obtain that

$$y^{(n+1)} = SCR_s^c + \text{VaR}_{0.995}\left(\frac{v(s, s+1)}{v^{ad}(s, s+1)}\left(N_s - v^{ad}(s, s+1)N_{s+1} - SCR_s^c\right)\right.$$

$$\left. + \left(1 - \frac{v(s, s+1)}{v^{ad}(s, s+1)}\right)(y^{(n)} - SCR_s^c)\Big|\mathcal{F}_s\right)$$

has the upper bound $SCR_s^c + (1 - \epsilon)\big|y^{(n)} - SCR_c^c\big|$ and the lower bound $SCR_s^c - (1 - \epsilon)\big|y^{(n)} - SCR_c^c\big|$. By induction we can show that

$$\big|y^{(n+1)} - SCR_s^c\big| \leq (1 - \epsilon)^{n+1}\big|y^{(0)} - SCR_s^c\big| \to 0 \quad (n \to \infty).$$

Hence, $\lim_{n \to \infty} y^{(n)} = SCR_s^c$. $\qquad\square$

Setting $v(s, s+1) = v^{rl}(s, s+1)$ and $v(s, s+1) = v^{co}(s, s+1)$, we get that iterative calculations of SCR_s^{a1} and SCR_s^{a2} converge to SCR_s^c. If the probability space Ω is countable, condition (5.2) can be relaxed to $0 < \frac{v(s,s+1)}{v^{ad}(s,s+1)} < 2$. Since the capital accumulation function is positive, the discount factors are positive and $\frac{v(s,s+1)}{v^{ad}(s,s+1)} < 2$ if and only if the average interest rate i^{ad} of the additional assets is smaller than $1 + 2i$, or $i^{ad} < 1 + 2i$. This restriction is usually met in practice. The proof also shows that for a discount ratio $\frac{v(s,s+1)}{v^{ad}(s,s+1)} \geq 2$ the sequence $y^{(n)}$ never converges. This fact is illustrated in Example 5.3. If the discount ratio is random and takes values both less and greater than 2, a general convergence result is out of reach.

Example 5.3 (speed of convergence). We modify the payoff of the stock from example 4.1, such that it earns a deterministic interest rate. We use this variable to analyze different discount ratios $\rho := \frac{v(s,s+1)}{v^{ad}(s,s+1)}$. With a starting point of $N_0 = 100$ we calculate the iteration (5.1) with $\widetilde{\theta}_t^i = \theta_t^i$, such that $v^{ad}(s, s+1) = v^{co}(s, s+1)$ and $SCR_s^c = SCR_s^{a2}$. The results for the first 15 steps are shown in Figure 5.1. For $\rho = 0.93$ the convergence is rapid. This is still the case for $\rho = 1.07$, even though
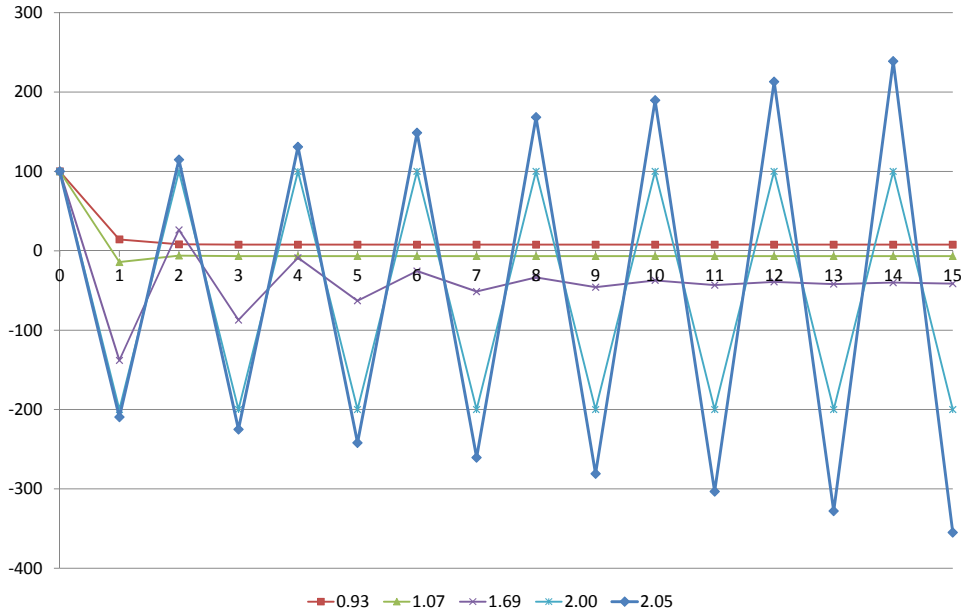
Figure 5.1: iteration (5.1) for different discount ratios

the iteration is not monotonous anymore. If the discount ratio is equal to $\rho = 2.00$, the iteration has two accumulation points, and the iteration jumps between them. For values larger than 2 we see a divergent behavior. It should be mentioned that these examples are quite extreme. For example, in case of $\rho = 2.00$ the stock performance has to be at least 215% in 99.5% of all events, given that the riskless interest rate is 5% and the stock ratio is 50%. In practice condition (5.2) is hardly a restriction.

6. Insurance groups

Insurance groups have the possibility to shift money between its subsidiary undertakings up to a certain extent. It may be sensible that such a shift should not change the SCR. Under certain conditions we show in the following that this is the case for SCR^c.

Suppose we have an insurance group that consists of n insurance companies with asset portfolios $(^iH_t)_t$, liabilities $(^iL_t)_t$, actuarial payments $(^iZ_t)_t$, and shareholder payments $(^iY_t)_t$ ($1 \leq i \leq n$). Let $(^i\widetilde{\theta}_t)_t$ be the asset strategy of company i for additional assets. We assume that the total assets of the n insurance companies are reallocated at time s and that $^i\widehat{H}_s$ are the units of assets that insurer i delivers or receives (depending on the sign) at time s. Let $^i\widehat{\theta}_s^j = \frac{^i\widehat{H}_s^j K_s^j}{^i\widehat{H}_s \cdot K_s}$ be the corresponding proportions of transferred assets, and let $^iSCR_s^c$ and $^i\widehat{SCR}_s^c$ be the SCRs for company i according to definition (c) before and after the asset transfer. As the n companies may have different asset strategies, the exchange of assets at time s occasionally comes with some trading on the capital markets in order to harmonize the received assets with the asset strategies $(^i\widetilde{\theta}_t)_t$.

Proposition 6.1 (invariance property). *We have $^i\widehat{SCR}_s^c = {}^iSCR_s^c$ for all $1 \leq i \leq n$. If we ban trading when the assets of an insurance group are reallocated, we have $^i\widehat{SCR}_s^c = {}^iSCR_s^c$ for all $1 \leq i \leq n$ and all capital markets $(K_t)_t$ if and only if $^i\widehat{\theta}_t = {}^i\widetilde{\theta}_t$ for all $1 \leq i \leq n$.*

Proof. If we allow for trading, any reallocation of assets conforms with the asset strategies $(^i\widetilde{\theta}_t)_t$, and the Corollary can be proven by following the arguments of Remark 4.3. If we ban trading, then condition $^i\widehat{\theta}_t = {}^i\widetilde{\theta}_t$ is sufficient and necessary because of Theorem 4.4. $\qquad\square$

As stated in the directive (European Parliament and the Council, 2009, Chapter II, Section 1), the SCR of a group should be calculated on the basis of the consolidated accounts (default method)

or by aggregating the stand-alone SCRs (alternative method). The default method is similar to the calculation of a single SCR. In the technical specifications to QIS 5 (CEIOPS, 2010, section 6) the alternative method is basically described as

$$SCR_{\text{group}} = \sum {}^i SCR_{\text{solo-adjusted}} \, ,$$

where ${}^i SCR_{\text{solo-adjusted}}$ is the SCR of company i adjusted according to some group effects. Consequently, if we use the SCR definition (c), the SCR of the group is invariant with respect to reallocation of capital, given that the asset strategies are consistent.

Example 6.2 (invariance property for SCR definitions (a1), (a2) and (c)). We consider example 4.1 but add one more asset, which has price $K_0^3 := 100$ at time zero and payoff $K_1^3 := 210 - K_1^2$ after one year. We consider two insurance companies that belong to an insurance group and both have liabilities of $L_0 = 100$ and $L_1 = 105$. Before a transfer of assets takes place company 1 has only one unit of K^2, while company 2 has one unit of K^1 and one unit of K^3. The number of units are assumed to be constant over time. Consequently, we obtain ${}^1N_0 = 0$, ${}^2N_0 = 100$, ${}^1N_1 = K_1^2 - 105$ and ${}^2N_1 = K_1^3$. The resulting SCRs are shown in Table 1. Suppose that company 2 transfers one

	SCR_0^{a1}	SCR_0^{a2}	SCR_0^c
Company 1	$\frac{100}{7} \approx 14.3$	$\frac{50}{3} \approx 16.7$	$\frac{25}{2} = 12.5$
Company 2	$\frac{100}{7} \approx 14.3$	$\frac{100}{13} \approx 7.7$	0

Table 1: SCR before the transfer takes place

unit of investment 3 at time zero to company 1, ${}^1\widehat{H}_0 = {}^1\widetilde{H}_0 = (0,0,1)$ and ${}^2\widehat{H}_0 = {}^1\widetilde{H}_0 = (0,0,-1)$. We obtain ${}^1\widetilde{N}_0 = 100$, ${}^2\widetilde{N}_0 = 0$, ${}^1\widetilde{N}_1 = 105$ and ${}^2\widetilde{N}_1 = 0$. The corresponding SCRs after the transfer are given in Table 2. As expected, definition (c) is invariant with respect to the exchange of assets.

	$\widehat{SCR}_0^{a1}$	$\widehat{SCR}_0^{a2}$	$\widehat{SCR}_0^c$
Company 1	0	0	12.5
Company 2	0	0	0

Table 2: SCR after the transfer took place

The SCR definitions (a1) and (a2) are not invariant, not only individually for each company but also in total. The transfer of investment K^3 from company 2 to company 1 is reasonable, since K^3 is a perfect hedge for investment K^2. After the asset transfer, both companies have no longer any risk, and SCRs of zero seem to be appropriate. But why is ${}^1\widehat{SCR}_0^c > 0$? As ${}^1\widetilde{H}_0 = (0,0,1)$ and definition (c) implicitly assumes that redundant assets are paid out (cf. definition (2.5) and Remark 4.3), shares of K^3 are paid out and the perfect hedge is disrupted.

The example illustrates that an invariance property is not always desirable. However, if asset values are minimized, we always end up with a SCR according to definition (c), see Theorem 5.2.

7. Risk Margin

This section deals with the Risk Margin and its interaction with the SCR. Since there is no universally valid definition of the RM in the academic literature, we close this gap and present a definition of the RM that is consistent with the directive. The key is that we defined the SCR in Definition 3.8 also for future points in time. In order to keep this section generally valid for all SCR definitions discussed in previous sections, we use $v(s, s+1)$ for the discount factor in the SCR definition.

As stated in article 75 paragraph 1(b) (European Parliament and the Council, 2009), "liabilities shall be valued at the amount for which they could be transferred, or settled, between knowledgeable

willing parties in an arm's length transaction." Paragraph 1 of article 77 requires that the "value of technical provisions shall be equal to the sum of a best estimate and a risk margin". Thus, the risk margin is the difference between the market value and the best estimate of the liabilities. This leads to a very general definition of the RM.

Definition 7.1 (RM general version)**.** The RM at time s is defined as

$$RM_s = L_s - B_s,$$

where B_s is the best estimate of the liabilities at time s.

This is the correct definition of the RM, but it is not a constructive definition, and there is a lot of room for interpretation. The purpose of the RM is to decompose the calculation of the market value of the liabilities into $L_s = B_s + RM_s$. Article 77 paragraph 5 of the directive requires that the "risk margin shall be calculated by determining the cost of providing an amount of eligible own funds equal to the Solvency Capital Requirement necessary to support the insurance and reinsurance obligations over the lifetime thereof," i.e. the directive suggests a cost-of-capital approach with respect to the SCR. A possible implementation of these requirements is given in the technical specifications to the fifth Quantitative Impact Study (CEIOPS, 2010). Therein, the RM is defined as

$$RM = c \sum_{k \geq 0} \frac{SCR_k^{RU}}{(1 + r_{k+1})^{k+1}}, \tag{7.1}$$

where SCR^{RU} is the SCR of a reference-undertaking, c the cost-of-capital rate and r_t the risk-free rate for maturity t. In the Swiss Solvency Test the RM is defined similar, but the sum starts at time 1. The calculation of the RM is based on a transfer scenario, where the liabilities are taken over by an artificial insurance company, which minimizes the part of the SCR comming from the hedgeable risks and which had no other insurance contracts before the transfer takes place (compare CEIOPS, 2010). That shall establish a comparability of the RM between different insurance companies. The formula has three deficiencies.

- Formula (7.1) defines the RM only for time $s = 0$.

- Future SCRs are random and hence the RM at time $s = 0$ is also random.

- A precise mathematical definition of the SCR of a reference undertaking is missing.

The technical specifications require that the reference undertaking behaves similar to the original undertaking, but hedges all hedgeable risks. If $(\theta^*)_t$ is the trading strategy that minimizes SCR_s, the SCR at time s of a reference-undertaking SCR_s^{RU} is the SCR calculated with trading strategy $(\theta^*)_t$. However, the precise definition of $(\theta^*)_t$ and its calculation is not a trivial task, and is beyond the scope of this paper. Since it is extremely difficult to minimize over all possible trading strategies, in practice, the market risk is usually neglected. As the RM is intended for the calculation of the market value of the current liabilities, we exclude new business in SCR^{RU}. This assumption is also made in CEIOPS (2009).

We generalize (7.1) by the following definition.

Definition 7.2 (RM cost-of-capital version)**.** The RM at time s is defined as

$$RM_s := \sum_{k \geq s} \mathbb{E}_{Q/P}\big(c(k) \, v^{rl}(s, k+1) \, SCR_k^{RU} \big| \mathcal{F}_s\big),$$

where $c(k)$ is the cost-of-capital rate at time k. Since choice of the measure is unclear, we write Q/P to indicate that a mixture of both measures is likely.

In our opinion the conditional expected value is the most convincing way to make RM_s $\mathcal{F}_s$-measurable. However, it is also possible to take a dynamic value at risk or another risk measure. For the practical purpose of calculating the RM, this definition is usually too complex, so that simplifications are needed. Suppose that the cost-of-capital rate is constant, i.e. $c(t) \equiv c$, and that $v^{rl}(s, k+1)$ and SCR_k^{RU} are conditionally independent given $\mathcal{F}_s$. Note that these two requirements are often not fulfilled. Then we obtain the following, simplified version of the RM,

$$RM_s = c \sum_{k \geq s} \mathbb{E}_{\mathbb{Q}}\big(v^{rl}(s, k+1)\big|\mathcal{F}_s\big) \mathbb{E}_{\mathbb{Q}/\mathbb{P}}\big(SCR_k^{RU}\big|\mathcal{F}_s\big).$$

In practice further simplifications are used, which cannot be theoretically established.

We seem to have a circular reference in Definition 7.2 since SCR_s^{RU} depends on N_s, $N_s = A_s - B_s - RM_s$ depends on RM_s, RM_s equals the sum of

$$E_{s,t} := \mathbb{E}_{\mathbb{Q}/\mathbb{P}}\big(c(t)\, v^{rl}(s, t+1)\, SCR_t^{RU}\big|\mathcal{F}_s\big), \quad t \geq s,$$

and $E_{s,s}$ depends on SCR_s^{RU}. In the following we show that the circular reference can be solved. Ohlsson and Lauzeningks (2009) solve this problem for a non-life SCR and Kriele and Wolf (2007) solve this problem using an approximation of the SCR, while we do this exactly and in a general framework. In the standard formula the RM is not considered for the calculation of the SCR, so that there is no circular reference. This is also the case in the Swiss Solvency Test, since the first summand of the RM is skipped.

Theorem 7.3. *The SCR of a reference undertaking at time s can be calculated without a circular reference, more precisely*

$$SCR_s^{RU} = \frac{1}{1 + \mathbb{E}_{\mathbb{Q}/\mathbb{P}}\big(c(k)\, v^{rl}(s, s+1)\big|\mathcal{F}_s\big)} \mathrm{VaR}_{0.995}\left(\Lambda_s\big|\mathcal{F}_s\right),$$

where

$$\Lambda_s := (A_s - B_s) - v(s, s+1)(A_{s+1} - B_{s+1}) + v(s, s+1) \sum_{t \geq s+1} \big(E_{s+1,t} - E_{s,t}\big).$$

Proof. For simplification we just write v for $v(s, s+1)$. By definitions 3.8, 7.1 and 7.2, we get

$$\begin{aligned}
SCR_s^{RU} =&\, \mathrm{VaR}_{0.995}\Big(A_s - vA_{s+1} + vB_{s+1} - B_s + vRM_{s+1} - RM_s\Big|\mathcal{F}_t\Big) \\
=&\, \mathrm{VaR}_{0.995}\Big(A_s - vA_{s+1} + vB_{s+1} - B_s - \mathbb{E}_{\mathbb{Q}/\mathbb{P}}\big(c(s)\, v^{rl}(s, s+1)SCR_s^{RU}\big|\mathcal{F}_s\big) \\
&\, - v \sum_{k \geq s+1} \mathbb{E}_{\mathbb{Q}/\mathbb{P}}\big(c(k)\, v^{rl}(s, k+1)SCR_k^{RU}\big|\mathcal{F}_s\big) \\
&\, + v \sum_{k \geq s+1} \mathbb{E}_{\mathbb{Q}/\mathbb{P}}\big(c(k)\, v^{rl}(s+1, k+1)SCR_k^{RU}\big|\mathcal{F}_{s+1}\big)\Big|\mathcal{F}_s\Big) \\
=&\, \mathrm{VaR}_{0.995}\Big(\Lambda_s - \mathbb{E}_{\mathbb{Q}/\mathbb{P}}\big(c(s)\, v^{rl}(s, s+1)\big|\mathcal{F}_s\big)SCR_s^{RU}\Big|\mathcal{F}_s\Big).
\end{aligned}$$

Since $\mathbb{E}_{\mathbb{Q}/\mathbb{P}}\big(c(s)\, v^{rl}(s, s+1)\big|\mathcal{F}_s\big)SCR_s^{RU}$ is $\mathcal{F}_s$-measurable, it can be taken out of the dynamic value at risk and the theorem follows by solving for SCR_s^{RU}. $\qquad\square$

The above formula for SCR_s^{RU} depends only on $E_{s,t}$ with $t \geq s+1$, which in turn only depends on future SCRs at times $t \geq s+1$, so that the formula is a recursion formula that can be solved backwards. Since new business is not included and since the term of each considered business is finite, there exists a time point $n \in \mathbb{N}$ for which $SCR_m^{RU} = RM_m = 0$ for all $m \geq n$, and hence the recursion is finite. Unfortunately, the computing time is growing exponentially with the length of the contract term, which means that numerical calculations are very time-consuming. In case we have an insurance portfolio only consisting of contracts with term one, SCR_s^{RU} usually increases if we neglect the RM, since $(1 + \mathbb{E}_{\mathbb{Q}/\mathbb{P}}\big(c(k)\, v^{rl}(s, s+1)\big|\mathcal{F}_s\big))^{-1} < 1$ for reasonable interest rates. Hence, neglecting the RM leads to a prudent upper bound. If contract terms are greater than 1, we lose that monotony property since the SCR includes the difference between RM_s and RM_{s+1}.

8. Conclusion

We started the paper with a comparison of Article 101 and Remark 64 of the Solvency II directive and presented mathematical interpretations of both. Our main findings are:

- Remark 64 can be defined as a value at risk and, thus, the mathematical structure is similar to Article 101.

- Article 101 and Remark 64 are consistent if and only if the discount factor in Article 101 corresponds to the investment strategy of the additional assets.

- The alternative definition (b) is equivalent to Article 101 with a riskless discount factor.

- When assets are minimized iteratively by applying Article 101, the resulting SCRs converge to the SCR from Remark 64.

For the calculation of the market value of the liabilities, Solvency II suggests using a cost of capital method and calculating a Risk Margin. However, the definition of the RM depends on future SCRs, and we are not aware of a mathematical sound definition of future SCRs in the literature. Further problems are that the Solvency II definition of the RM is circular and that it ignores the fact that future SCRs are random.

- We showed how to define future SCRs based on a generalization of the value at risk to a dynamic value at risk.

- For the first time a general RM definition is given that takes into account the randomness of the future SCRs.

- We showed that the circularity of the RM definition can generally be solved.

An opportunity for future research is to find a more accurate definition of the SCR of a reference-undertaking. In this paper we assume the existence of a trading strategy $(\theta_t^*)_t$ that minimizes the SCR. The calculation and existence of such a strategy is an open problem.

A. Appendix

Proof of Proposition 3.6. As $\mathbb{R}$ is a Polish space (see Kechris, 1995), according to Bauer (1981, chapter 10) there exists a Markov kernel Q such that $x \mapsto Q(x, A)$ is a version of

$$\mathrm{P}(Y \in A | X_{[0,s]} = x), \quad A \in \mathfrak{B}(\mathbb{R}).$$

Setting $A = (-\infty, y]$, we get that the function $x \mapsto F_x(y) := Q(x, (-\infty, y]) = \mathrm{P}(Y \leq y | X_{[0,s]} = x)$ is $\mathcal{F}_s'$-$\mathfrak{B}(\mathbb{R})$-measurable for each $y \in \mathbb{R}$. For any fixed x, $F_x(y)$ is a cumulative distribution function, and we define the quantile function or generalized inverse as

$$F_x^{-1}(\alpha) = \inf\{y \in \mathbb{R} | F_x(y) \geq \alpha\}.$$

Since $x \mapsto F_x(y)$ is $\mathcal{F}_s'$-$\mathfrak{B}(\mathbb{R})$-measurable, it holds that

$$\{x \in \Omega : F_x(r) \geq \alpha\} \in \mathcal{F}_s', \quad \forall\, r \in \mathbb{R}.$$

From Milbrodt (2010, page 229) we know that $F(r) \geq \alpha \Leftrightarrow F^{-1}(\alpha) \leq r$ for all $r \in \mathbb{R}$ and $\alpha \in (0, 1]$ such that we obtain

$$\{x \in \Omega : F_x^{-1}(\alpha) \leq r\} = \{x \in \Omega : F_x(r) \geq \alpha\} \in \mathcal{F}, \quad \forall\, r \in \mathbb{R}.$$

Hence, the function

$$x \mapsto F_x^{-1}(\alpha) = \inf\{y \in \mathbb{R} : \mathrm{P}(Y \leq y | X_{[0,s]} = x) \geq \alpha\} = h_{Y,\alpha}(x)$$

is $\mathcal{F}_s'$-$\mathfrak{B}(\mathbb{R})$-measurable. $\qquad\square$

Proposition A.1. *Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space with random variables W and Z, where $Z(\omega) > 0$ for all $\omega \in \Omega$. For all $\alpha \in (0,1)$ $\mathrm{Var}_\alpha(W|\mathcal{F}_s) = 0$ if and only if $\mathrm{Var}_\alpha(WZ|\mathcal{F}_s) = 0$.*

Proof. Let $\mathcal{F}_s = \sigma(X_{[0,s]}) \subset \mathcal{F}$. Since $\mathrm{P}(W \leq 0|X_{[0,s]} = x) = \mathrm{P}(WZ \leq 0|X_{[0,s]} = x)$ almost surely and $\mathrm{P}(W < 0|X_{[0,s]} = x) = \mathrm{P}(WZ < 0|X_{[0,s]} = x)$ almost surely, we have

$$\inf\{w \in \mathbb{R} : \mathrm{P}(W \leq w|X_{[0,s]} = x) \geq \alpha\} = 0$$
$$\Leftrightarrow \inf\{w \in \mathbb{R} : \mathrm{P}(WZ \leq w|X_{[0,s]} = x) \geq \alpha\} = 0.$$

The claim follows by the definition of the dynamic value at risk. $\qquad\square$

References

Barrieu, P., Bensusan, H., Hillairet, C., Loisel, S., Ravanelli, C., Salhi, Y., et al. (2010). Understanding, modelling and managing longevity risk: key issues and main challenges. *Forthcoming in: Scandinavian Actuarial Journal*.

Bauer, D., Bergmann, D., and Reuss, A. (2010). On the Calculation of the Solvency Capital Requirement based on Nested Simulations. *Working paper, University of Ulm*.

Bauer, H. (1981). *Probability theory and elements of measure theory*. Academic Press London.

CEIOPS (2009). Final CEIOPS' Advice for Level 2 Implementing Measures on Solvency II: Technical Provisions – Article 86 (d) Calculation of the Risk Margin QIS 5 Technical Specifications. `https://eiopa.europa.eu/fileadmin/tx_dam/files/consultations/consultationpapers/CP42/CEIOPS-L2-Final-Advice-on-TP-Risk-Margin.pdf`.

CEIOPS (2010). QIS 5 Technical Specifications. `https://eiopa.europa.eu/fileadmin/tx_dam/files/consultations/QIS/QIS5/QIS5-technical_specifications_20100706.pdf`.

Devineau, L. and Loisel, S. (2009). Risk aggregation in Solvency II: How to converge the approaches of the internal models and those of the standard formula? *Bulletin Français d'Actuariat*, 9(18):107–145.

Devolder, P. (2011). Revised version of: Solvency requirement for long term guarantee: risk measure versus probability of ruin. *European Actuarial Journal*, 1(2):200–214.

Doff, R. (2008). A Critical Analysis of the Solvency II Proposals. *The Geneva Papers on Risk and Insurance - Issues and Practice*, 33(2):193–206.

Eling, M., Schmeiser, H., and Schmit, J. (2007). The Solvency II process: Overview and critical analysis. *Risk Management and Insurance Review*, 10(1):69–85.

European Parliament and the Council (2009). Directive of the European Parliament and of the Council on the taking-up and pursuit of the business of insurance and reinsurance (Solvency II).

Filipović, D. (2009). Multi-level risk aggregation. *Astin Bulletin*, 39(2):565–575.

Floreani, A. (2011). Risk margin estimation through the cost of capital approach: Some conceptual issues. *The Geneva Papers on Risk and Insurance - Issues and Practice*, 36(2):226–253.

Holzmüller, I. (2009). The United States RBC Standards, Solvency II and the Swiss Solvency Test: A Comparative Assessment. *The Geneva Papers on Risk and Insurance - Issues and Practice*, 34(1):56–77.

Kechris, A. (1995). *Classical descriptive set theory*. Springer.

Kochanski, M. (2010). Solvency capital requirement for german unit-linked insurance products. *German Risk and Insurance Review*, 6:33–70.

Kriele, M. and Wolf, J. (2007). On market value margins and cost of capital. *Blätter der DGVFM*, 28(2):195–219.

Kriele, M. and Wolf, J. (2012). *Wertorientiertes Risikomanagement von Versicherungsunternehmen.* Springer.

Milbrodt, H. (2010). *Wahrscheinlichkeitstheorie - Eine Einführung mit Anwendungen und Beispielen aus der Versicherungs- und Finanzmathematik.* Verlag Versicherungswirtschaft.

Ohlsson, E. and Lauzeningks, J. (2009). The one-year non-life insurance risk. *Insurance: Mathematics and Economics*, 45(2):203–208.

Protter, P. (2005). *Stochastic Integration and Differential Equations.* Springer.

Salzmann, R. and Wüthrich, M. (2010). Cost-of-capital margin for a general insurance liability runoff. *Astin Bulletin*, 40(2):415–451.

Steffen, T. (2008). Solvency II and the Work of CEIOPS. *The Geneva Papers on Risk and Insurance - Issues and Practice*, 33(1):60–65.