Space-Time Reduced Basis Methods for Time-Periodic Parametric Differential Equations

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SPACE-TIME REDUCED BASIS METHODS FOR TIME-PERIODIC PARAMETRIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. We consider space-time methods for time-periodic problems and discuss well-posedness of the variational formulation as well as the use of Reduced Basis Methods (RBM) in a parameterized setting. We propose a possible discretization enforcing periodic boundary conditions in time and provide lower bounds for the space-time inf-sup constant. Rigorous RBM space-time a-posteriori error bounds for state and output are derived and it is shown that this approach yields results similar to those in stationary elliptic settings.

This is contrasted with the more common time-stepping approach which requires fixed-point methods with often long transient phases to obtain periodicity. We discuss RBM in this context and derive the corresponding a-posteriori bounds.

A convection-diffusion-reaction example is numerically investigated with regard to the inf-sup constant as well as the performance of both space-time and fixed-point RBM. We show the reliable representation of the stability by the space-time inf-sup constant and observe the advantage of space-time approaches in the online phase of the RBM.

1. Introduction

Time-periodic partial differential equations arise e.g. when considering rotators or propellers, often in parametric settings where some output functional (like the efficiency of an propeller) has to be optimized over a given parameter $\mu \in \mathcal{D}$ that may represent some design or steering property. Their computational cost is even larger than that of common initial value problems, rendering them particularly eligible for treatment with model reduction methods.
Reduced Basis Methods (RBM) construct a low-dimensional approximation of the solution space in an *offline phase* and solve only the Galerkin projection onto this basis in the time-critical *online phase*.

RBM for evolution equations have been considered e.g. in [4, 5, 6, 7, 13]. The common approach is to use time-stepping methods which have the disadvantage in time-periodic settings that long transient phases – or equivalently many fixed point iterations – are necessary to obtain periodic solutions. Moreover, error bounds in this setting can only be formulated for discrete spatio-temporal norms and involve sums of residual dual norms over all time steps, hence growing in time. Additional difficulties are introduced by time-variant operators, as then the construction of the reduced basis requires either the storage of additional information at each time point or additional computational effort to separate time and space [5]. A space-time approach for initial value problems has been treated in [13], where primal-dual

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output bounds in a discontinuous Galerkin approach have been derived. Recently, the advantage of space-time inf-sup constants over classical energy estimates for long-time integration has been observed in [16].

We discuss a general space-time variational formulation for time-periodic problems and its well-posedness for linear and quadratic problems in Section 2 and also obtain lower bounds for the inf-sup constant in the linear setting . Reliable space-time a-posteriori error bounds for both state and output involving dual norms of residuals as well as coercivity and inf-sup constant are derived in Section 3 for norms in different function spaces. In Section 4, the RBM space-time approach is outligned and contrasted with the fixed-point approach, where e.g. error bounds can only be derived in a discrete approximation of the norm in $L_2(0,T)$. Section 5 contains numerical results for a convection-diffusion-reaction example.

1.1. **Setting and notation.** We start by fixing some notation: Set $I := [0,T]$, $T > 0$, $\Omega \subset \mathbb{R}^d$ and let $V \hookrightarrow H := L_2(\Omega) \hookrightarrow V'$ be some Sobolev space incorporating boundary conditions on Ω (e.g., $V = H_0^1(\Omega)$). We abbreviate

$$[f,g] := \int_I \langle f(t), g(t)\rangle \, dt$$

and denote by $\langle\cdot,\cdot\rangle$ the inner product in H, also inducing the duality pairing of V' and V. Further, let $a(t;\cdot,\cdot)$ be a semilinear form (i.e., linear w.r.t. to the last argument), define $\mathcal{C}(t) : V \to V'$ by $\langle \mathcal{C}(t)\phi, \psi\rangle := c(t;\phi,\psi)$ for ϕ,$\psi \in V$ and consider the time-periodic problem

$$\dot{u}(t) + \mathcal{C}(t)u(t) = f(t) \quad \text{in } V',\ t \in (0,T) =: I, \tag{1.1a}$$

$$u(0) = u(T) \quad \text{in } H, \tag{1.1b}$$

where $f \in L_2(I;V')$ is given.

Since we are interested in time-averages of the time-periodic solution, we define the time-average for $g \in L_1(I;H)$ as $\bar{g} := \frac{1}{T}\int_I g(t)\,dt \in H$. Let $J \in V'$ be a linear functional, then we are interested in the quantity

$$s(u) := J(\bar{u}) = \frac{1}{T}\int_I J(u(t))\,dt. \tag{1.2}$$

2. Space-time formulation of time-periodic problems

Defining the spaces $\mathcal{X} := L_2(I;V) \cap H^1_{\mathrm{per}}(I;V')$ and $\mathcal{Y} := L_2(I;V)$, where $H^1_{\mathrm{per}}(I;V') := \{u \in \mathcal{Y} : \dot{u} \in L_2(I,V'), u(0) = u(T) \text{ in } H\}$ and $\|u\|^2_{\mathcal{X}} := \|u\|^2_{L_2(I;V)} + \|\dot{u}\|^2_{L_2(I;V')}$, then integration w.r.t. time yields the variational formulation of (1.1):

$$\text{Find } u \in \mathcal{X}: \quad b(u,v) = f(v) \qquad \forall\, v \in \mathcal{Y}, \tag{2.1}$$

where $b(\cdot,\cdot) : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ and $f(\cdot) : \mathcal{Y} \to \mathbb{R}$ are defined by

$$b(w,v) := [\dot{w}, v] + \mathcal{C}[w,v], \quad f(v) := [f,v], \quad w \in \mathcal{X}, v \in \mathcal{Y},$$

where we set $\mathcal{C}[w,v] := \int_0^T c(t;w(t),v(t))dt$.

We have built the periodicity requirement (1.1b) into the trial space $\mathcal{X}$ here. One could, however, also view this as a constraint. This leads to an alternative form. Let $\tilde{\mathcal{X}} := W(I) := \{u \in L_2(I;V) : \dot{u} \in L_2(I;V')\}$ with the $\mathcal{X}$-norm and

$\tilde{\mathcal{Y}} := L_2(I;V)\times H$ with the standard graph norm. Note that $\mathcal{X}$ is a closed subspace of $\tilde{\mathcal{X}}$. For $w \in \tilde{\mathcal{X}}$, $v = (v_1, v_2) \in \tilde{\mathcal{Y}}$, we set

$$\tilde{b}(w,(v_1,v_2)) := [\dot{w}, v_1] + \mathcal{C}[u, v_1] + \langle u(0) - u(T), v_2\rangle, \quad \tilde{f}(v) := [f, v_1]. \tag{2.2}$$

The variational form then reads

$$\text{Find } \tilde{u} \in \tilde{\mathcal{X}}: \quad \tilde{b}(\tilde{u}, \tilde{v}) = \tilde{f}(\tilde{v}) \qquad \forall\, \tilde{v} \in \tilde{\mathcal{Y}}. \tag{2.3}$$

It can easily be seen in that (2.3) and (2.1) are equivalent. Hence, one may chose either of them both for analysis as well as for the design of a numerical scheme.

2.1. **The linear case.** Let us first assume that $\mathcal{C}(t) = A(t)$, where $A(t) \in \mathcal{L}(V, V')$ is a linear operator induced by a bilinear form a which implies that b is bilinear also. The following assumptions on the form a are standard:

$$|a(t;\phi,\psi| \le M_a \|\psi\|_V \|\psi\|_V \quad \forall\, \phi,\psi \in V, t \in [0,T] \text{ a.e.}, \qquad \text{(continuity)} \tag{2.4}$$

$$a(t;\phi,\phi) \ge \alpha_a \|\phi\|_V^2 \qquad \forall\, \phi \in V, t \in [0,T] \text{ a.e..} \qquad \text{(coercivity)} \tag{2.5}$$

We want to follow some of the arguments in [14, Theorem 5.1] to investigate the well-posedness.

Remark 2.1. *Note that unlike in the initial value problems investigated in* [14, 16] *the coercivity condition* (2.5) *cannot be relaxed to a weaker Gårding inequality. In fact, a standard approach for the investigation of initial value problems is a transformation $\hat{u}(t) := e^{-\lambda t}u(t)$ with Gårding constant λ. Even though this keeps any initial value unchanged, this technique destroys periodicity and thus does not work in the time-periodic case.*

Proposition 2.2 (cf. [14], A.1)**.** *The form b is bounded, i.e.,*

$$M_b := \sup_{0\neq w\in\mathcal{X}, 0\neq v\in\mathcal{Y}} \frac{|b(w,v)|}{\|w\|_{\mathcal{X}}\|v\|_{\mathcal{Y}}} \le M_{UB} =: \sqrt{2}\max\{1, M_a\} < \infty.$$

Proof. Follows from (2.4), the definitions of $\|\cdot\|_{\mathcal{X}}$, $\|\cdot\|_{\mathcal{Y}}$ as well as Cauchy-Schwarz's, Hölder's and Young's inequalities. □

Proposition 2.3 (cf. [14], A.2)**.** *The form b satisfies an inf-sup condition, i.e.,*

$$\beta := \inf_{0\neq w\in\mathcal{X}} \sup_{0\neq v\in\mathcal{Y}} \frac{|b(w,v)|}{\|w\|_{\mathcal{X}}\|v\|_{\mathcal{Y}}} \ge \beta_{LB} := \frac{\alpha_a \min\{1, M_a^{-2}\}}{\sqrt{2}\max\{1, \beta_a^{-1}\}} > 0, \tag{2.6}$$

with $\beta_a(t) := \inf_{\phi\in V}\sup_{\psi\in V} \frac{a(t;\psi,\phi)}{\|\psi\|_V\,\|\phi\|_V}$ and $\beta_a := \inf_{t\in I}\beta_a(t)$.

Proof. We follow the argumentation in [14]. Let $0 \neq w \in \mathcal{X}$ be given and define $z_w(t) := (A(t)^*)^{-1}\,\dot{w}(t)$ for the adjoint $A^*(t)$ of $A(t)$, i.e. the operator $A^*: V' \to V$ with $\langle A^*(t)\psi, \phi\rangle = a(t;\phi,\psi)$. The bound $\|(A^*(t))^{-1}\| \le \beta_a^{-1}$ then yields for $v_w(t) := z_w(t) + w(t)$ that

$$\|v_w\|_{\mathcal{Y}} \le \sqrt{2}\max\{1, \beta_a^{-1}\}\|w\|_{\mathcal{X}} < \infty.$$

By definition of z_w and (2.5), $\langle \dot{w}(t), z_w(t)\rangle = a(t; z_w(t), z_w(t)) \ge \alpha_a \|z_w(t)\|_V^2 \ge \frac{\alpha_a}{M_a^2}\|\dot{w}(t)\|_{V'}^2$. Moreover, the periodicity of $w \in \mathcal{X}$ and $a(t; w(t), z_w(t)) = \langle w, \dot{w}(t)\rangle$

implies $[\dot{w}, w] + A[w, z_w] = [\dot{w}, w] + [w, \dot{w}] = \int_0^T \frac{d}{dt}\|w(t)\|_H^2 dt = \|w(T)\|_H^2 - \|w(0)\|_H^2 = 0$, so that

$$\begin{aligned} b(w, v_w) &= [\dot{w}, z_w] + [\dot{w}, w] + A[w, z_w] + A[w, w] \\ &\geq \alpha_a \min\{1, M_a^{-2}\}\|w\|_{\mathcal{X}}^2 \geq \frac{\alpha_a \min\{1, M_a^{-2}\}}{\sqrt{2}\max\{1, \beta_a^{-1}\}}\|w\|_{\mathcal{X}} x \|v_w\|_{\mathcal{Y}} > 0. \end{aligned}$$

As $w \in \mathcal{X}$ was arbitrary, the inf-sup condition is fulfilled. □

Proposition 2.4. *Let the bilinear form a satisfy* (2.4) *and* (2.5)*. Then, the problem* (2.1) *is well-posed.*

Proof. We need to verify the conditions of the Babuška-Aziz theorem. Since continuity and the inf-sup-condition have been shown already in Propositions 2.2 and 2.3, it only remains to verify the subjectivity, namely $\sup_{0 \neq u \in \mathcal{X}} |b(u, v)| > 0$ for all $0 \neq v \in \mathcal{Y}$. Let $0 \neq v \in \mathcal{Y}$. If we can find some $z \in \mathcal{X}$ with $\langle \dot{z}(t), w\rangle + a(t, z(t), w(t)) = a(t, v(t), w(t))$ for all $w \in \mathcal{Y}$, and t a.e. on $(0, T)$, we obtain $b(z, v) = [\dot{z}, v] + A[z, v] = A[v, v] \geq \alpha_a \|v\|_{\mathcal{Y}} > 0$, so that then the surjectivity condition is fulfilled. We are now going to construct such a $z \in \mathcal{X}$ in four steps.

(1) Faedo-Galerkin approximation of an initial value problem. Let $\{\phi_i : i \in \mathbb{N}\}$ be a basis for V, $V_n := \operatorname{span}\{\phi_i, i = 1, \ldots, n\}$ and $z_n(t) := \sum_{i=1}^n z_i^{(n)}(t)\phi_i$. Consider for some (arbitrary) $z_0 \in H$ the problem

$$\langle \dot{z}_n(t), w_n\rangle + a(t, z_n(t), w_n) = a(t, v(t), w_n), \quad z_n(0) = z_{n0}, \tag{2.7}$$

for all $w_n \in V_n$ a.e. on I, where z_{n0} is the orthogonal projection of z_0 onto V_n. This is a linear system of ODEs of the form

$$M^{(n)} \frac{d}{dt} z^{(n)}(t) + A^{(n)}(t) z^{(n)}(t) = f^{(n)}(t), t \in I \text{ a.e.}, \quad z_n(0) = z_{n0},$$

and has a solution $z_n \in C(I; V_n)$ with derivatives $\dot{z}_n \in L_2(I; V_n)$.

(2) A-priori estimates. Let $w_n = z_n(t)$ in (2.7). Then, using (2.4), (2.5) and Young's inequality with some $\varepsilon < \frac{\alpha_a}{M_a}$, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|z_n(t)\|_H^2 + \alpha_a\|z_n(t)\|_V^2 &\leq \langle \dot{z}_n(t), z_n(t)\rangle + a(t; z_n(t), z_n(t)) \\ &= a(t; v(t), z_n(t)) \leq M_a\|v(t)\|_V\|z_n(t)\|_V \\ &\leq M_a\varepsilon\|z_n(t)\|_V^2 + \frac{M_a}{4\varepsilon}\|v(t)\|_V^2 \end{aligned}$$

and hence

$$\frac{1}{2}\frac{d}{dt}\|z_n(t)\|_H^2 + (\alpha_a - M_a\varepsilon)\|z_n(t)\|_V^2 \leq \frac{M_a}{4\varepsilon}\|v(t)\|_V^2 \tag{2.8}$$

As $\alpha_a - M_a\varepsilon \geq 0$, integration over $[0, s]$, $s \in [0, T]$, yields

$$\|z_n(s)\|_H^2 - \|z_n(0)\|_H^2 \leq \frac{M_a}{2\varepsilon}\int_0^s \|v(t)\|_V^2 dt, \tag{2.9}$$

hence $\sup_{s \in [0,T]}\|z_n(s)\|_H^2 \leq c\|z_0\|_H^2 + \frac{M_a}{2\varepsilon}\|v\|_{\mathcal{Y}}^2 < \infty$ for some $c > 0$. Thus $\{z_n\}_{n \in \mathbb{N}}$ is uniformly bounded in $L_\infty(I; H)$. Similarly, we conclude from (2.8) with $s = T$ that $2(\alpha_a - M_a\varepsilon)\|z_n\|_{L_2(I;V)} \leq \|z_n(0)\|_H^2 - \|z_n(T)\|_H^2 + \frac{M_a}{2\varepsilon}\|v\|_{\mathcal{Y}}^2 < \infty$, so that $\{z_n\}_{n \in \mathbb{N}}$ is also uniformly bounded in $L_2(I; V)$.

(3) Periodicity. Abbreviating $\bar{c} := \frac{M_a}{4\varepsilon}$, $\bar{\alpha} := 2\frac{(\alpha_a - M_a\varepsilon)}{c_1} > 0$ with $\|\cdot\|_H \leq c_1\|\cdot\|_V$ and multiplying the inequality (2.8) by $e^{\bar{\alpha}t}$ yields

$$\frac{d}{dt}\left(e^{\bar{\alpha}t}\,\|z_n(t)\|_H^2\right) = e^{\bar{\alpha}t}\frac{d}{dt}\|z_n(t)\|_H^2 + e^{\bar{\alpha}t}\bar{\alpha}\,\|z_n(t)\|_H^2 \leq e^{\bar{\alpha}t}\bar{c}\,\|v(t)\|_V^2.$$

By integration over $[0,T]$, we obtain $e^{\bar{\alpha}T}\|z_n(T)\|_H^2 - e^{\bar{\alpha}0}\|z_n(0)\|_H^2 \leq \int_0^T e^{\bar{\alpha}t}\bar{c}\|v(t)\|_V^2 dt$, or rather

$$\|z_n(T)\|_H^2 \;\leq\; e^{-\bar{\alpha}T}\|z_n(0)\|_H^2 + \bar{c}\,e^{-\bar{\alpha}T}\int_0^T e^{\bar{\alpha}t}\|v(t)\|_V^2\,dt. \tag{2.10}$$

Set $M := \{z \in V_n : \|z\|_H \leq R := \sqrt{\frac{K}{1-e^{-\bar{\alpha}T}}}\}$ with $K := \bar{c}e^{-\bar{\alpha}T}\int_0^T e^{\bar{\alpha}t}\|v(t)\|_V^2 dt$. M is a convex and compact set in V_N. Moreover if $z_n(0) \in M$, (2.10) implies that $\|z_n(T)\|_H^2 \leq e^{-\bar{\alpha}T}R^2 + K \leq R$, i.e. $z_n(T) \in M$. Since by Gronwall's lemma the mapping $S : M \to M$, $z_n(0) \mapsto z_n(T)$, is continuous, the existence of a fixed point $S(\bar{z}_n) = \bar{z}_n \in M$ follows from Brouwer's fixed point theorem. By the a-priori estimates, the sequence $\{\bar{z}_n\}_{n\in\mathbb{N}}$ is bounded in H, so that there exists a subsequence (also denoted $\{\bar{z}_n\}$) converging weakly to some $\bar{z} \in H$.

(4) Convergence. Consider the periodic solution $z_n(t)$ from (3), i.e. the solution of (2.7) with initial value $z_{n0} = \bar{z}_n$. From the a-priori estimates, we have that $\{z_n\}$ is uniformly bounded in the separable space $L_2(I;V)$, so that there exists a subsequence (also denoted $\{z_n\}$) converging weakly to some z in $L_2(I;V)$.

Consider (2.7) for $w_n := \theta(t)\phi_j$, $\theta(t) \in C^1(I)$, and integrate over I. Integration by parts of the first term then yields for all $j = 1,\ldots,n$

$$\begin{aligned}-[z_n, \theta'\phi_j] &= \langle z_n(0), \theta(0)\phi_j\rangle - \langle z_n(T), \theta(T)\phi_j\rangle + A[v - z_n, \theta\phi_j]\\ &= \langle \bar{z}_n, \theta(0)\phi_j - \theta(T)\phi_j\rangle + A[v - z_n, \theta\phi_j].\end{aligned}$$

As $z_n \rightharpoonup z$ in $L_2(I;V)$ and $\bar{z}_n \rightharpoonup \bar{z}$ in H, we can pass to the limit $n \to \infty$ and obtain

$$-[z, \theta'\phi_j] = \langle \bar{z}, \theta(0)\phi_j - \theta(T)\phi_j\rangle + A[v - z, \theta\phi_j]. \tag{2.11}$$

This particularly holds true for all $\theta \in \mathcal{D}(I)$, so that $z_t = A(\cdot)(v - z)$ in the distribution sense and $z_t \in L_2(I;V')$ as $A : L_2(I;V) \to L_2(I;V')$. Moreover, (2.11) implies that for $w \in C^1(I;V)$, we have $-[z,\dot{w}] - \langle \bar{z}, w(0) - w(T)\rangle = A[v - z, w] = [\dot{z}, w] = -[z,\dot{w}] + \langle z(T), w(T)\rangle - \langle z(0), w(0)\rangle$, so that indeed $\bar{z} = z(0) = z(T)$ in H and hence $z \in \mathcal{X}$. With this z, the surjectivity condition is fulfilled. □

2.2. **The quadratic nonlinear case.** Let us now consider the case of a quadratic nonlinear operator $\mathcal{C}(t) = A(t) + N$, where $A(t) \in \mathcal{L}(V,V')$ as above and $N : V \times V' \to \mathbb{R}$ is defined by

$$\langle N(\phi), \psi\rangle := n(\phi,\phi,\psi), \qquad \phi,\psi \in V,$$

with $n : V \times V \times V \to \mathbb{R}$ being a trilinear form. We use the abbreviation $N[w,v] := \int_0^T n(w(t), w(t), v(t))\,dt$. For simplicity of the presentation, we assume that N is time-invariant. Most of what is said, however, can easily be extended to the instationary case as well.

We make the following assumptions:

$$n(\phi,\psi,\eta) \le M_n \|\phi\|_V \, \|\psi\|_V \, \|\eta\|_V, \qquad \forall\, \phi,\psi,\eta \in V, \tag{2.12}$$

$$n(\phi,\phi,\phi) \ge 0, \qquad \forall\, \phi \in V, \tag{2.13}$$

$$N[w_n,w_n,v] \to N[w,w,v] \quad \text{for all } w_n, v \in L_2(I,V) \text{ with } w_n \rightharpoonup w \text{ in } L_2(I;V),\ w_n \overset{*}{\rightharpoonup} w \text{ in } L_\infty(I;H) \text{ as } n \to \infty. \tag{2.14}$$

$$N(w) \in L_2(I,V') \text{ for } w \in L_2(I,V). \tag{2.15}$$

We are now ready to prove an existence result.

Proposition 2.5. *Let the bilinear form a fulfill* (2.4) *and* (2.5) *and assume that* (2.12)-(2.15) *hold. Then there exists a solution* $u \in \mathcal{X}$ *of* (2.1).

Proof. The proof is in part similar to the proof of Proposition 2.4 and we refer the reader to some details given there. We start again by a Faedo-Galerkin approximation of an initial value problem (IVP) similar to (2.7). In the quadratic case, it reads

$$\langle \dot{u}_n(t), w_n\rangle + a(t,u_n(t),w_n) + n(u_n(t),u_n(t),w_n) = \langle f(t), w_n\rangle,\ u_n(0) = u_{n0}, \tag{2.16}$$

for all $w_n \in V_n$ for $t \in I$ a.e. Due to our assumptions, this finite-dimensional nonlinear IVP has a solution u_n on some interval $[0,t_n] \subseteq I$. In view of (2.13), we can follow step (2) with right hand side $f \in L_2(I,V')$ in the proof of Proposition 2.4 to show that $\{u_n\}$ is uniformly bounded in $L_2(I;V)$ so that we can extend u_n to $[0,T] = I$. Again, we also obtain uniform boundedness in $L_\infty(I,H)$, so that due to (2.13) we can follow step (3) in the proof of Proposition 2.4 to show that there exists a periodic solution u_n in H for all $n \in \mathbb{N}$.

For these periodic solutions, we obtain again from the a-priori estimates the existence of a subsequence $\{u_n\}$ converging weakly in $L_2(I,V)$ to some $\tilde{u}$ in $L_2(I,V)$ and to $\bar{u} \in L_\infty(I,H)$ in the weak-* sense. It is not difficult to show that on $L_2(I,V) \cap L_\infty(I,H)$ both limits coincide. Finally, we have to show that this limit solution u solves (2.1). After integration by parts in (2.16), assumption (2.14) and the above considerations allow for any $\varphi \in C^1(I)$, $w \in V$, to pass to the limit:

$$-[u, \dot{\varphi} w] + A[u,\varphi w] + N[u,\varphi w] = \langle \bar{u}, (\varphi(0) - \varphi(T)) w\rangle + \langle f, \varphi w\rangle \tag{2.17}$$

with $\bar{u}_n = u_n(0) = u_n(T)$ for all $w_n \in V_n$. Taking $\varphi \in C_0^\infty(I)$ shows that $u_t + Au + N(u) = f$ in the sense of distributions, so that it follows from Au, $N(u)$, $f \in L_2(I,V')$ that $u_t \in L_2(I;V')$, which implies $u \in \mathcal{X}$. Using $v \in C^1(I;V) \hookrightarrow L_2(I;V)$ as test function in (2.17) shows $b(u,v) = f(v)$ for all $v \in \mathcal{Y}$.

Additionally, $u \in \mathcal{X}$ implies $u \in C(I,H)$, so that the periodicity condition in H is correctly posed. $\square$

We can relax assumption (2.15) to $N(w) \in L_1(I,V')$ for $w \in L_2(I,V)$, which is trivially fulfilled if (2.12) holds:

$$\|N(w)\|_{L_1(I;V')} = \int_I \sup_{\phi \in V} \frac{n(w(t),w(t),\phi)}{\|\phi\|_V} dt \le M_n \|w\|^2_{L_2(I;V)}.$$

In that case, a solution u of (1.1) is only in $C(I,V')$, so that the periodicity conditions can only be enforced in V'. This is summarized in the following existence result.

Corollary 2.6. *Assume* (2.4),(2.5) *and* (2.12)-(2.14). *Then there exists a solution* $u \in L_2(I,V) \cap L_\infty(I,H)$ *with* $\dot u \in L_1(I,V')$ *to the problem*

$$\tilde b(u,v) = \tilde f(v) \qquad \forall\, v = (v_1, v_2) \in \hat{\mathcal{Y}} := \mathcal{Y} \times V, \tag{2.18}$$

where $\tilde b$, $\tilde f$ *as in* (2.2). *This solution is in* $C(I,V')$.

Proof. As in the proof of Proposition 2.5. With $N(u) \in L_1(I,V)$ we have $u_t \in L_1(I,V')$, so that $u \in C(I,V')$ (cf. [15, p. 169]). □

Remark 2.7 (Local Uniqueness)**.** *We cannot expect to obtain globally unique solutions for the nonlinear problem. However, similar to* [2],[17]*, the Brezzi-Rappaz-Raviart theory may allow the construction of a-posteriori estimates that ensure local well-posedness.*

3. Error analysis

It is an obvious advantage of space-time variational methods that we obtain a variational problem that can be approximated in terms of a Petrov-Galerkin scheme and analyzed almost as in the elliptic case. The price to be paid, however, is the obvious increase of the dimension.

3.1. **Discretization and a priori estimates.** Let $\mathcal{X}_\mathcal{N} := S_{\Delta t} \otimes V_h \subset \mathcal{X}$, $\mathcal{Y}_\mathcal{N} := Q_{\Delta t} \otimes V_h \subset \mathcal{Y}$ be finite-dimensional trial and test spaces of dimension $\mathcal{N}$, where $S_{\Delta t} \subset H^1_{\mathrm{per}}(I)$, $Q_{\Delta t} \subset L_2(I)$ and $V_h \subset V$ are e.g. finite element spaces with respect to triangulations $\mathcal{T}^{\mathrm{time}}_{\Delta t} := \{t^k := k\,\Delta t, k = 0,\dots,r\}$, $\Delta t := \frac{T}{r}$, $r \in \mathbb{N}$, in time and $\mathcal{T}^{\mathrm{space}}_h := \{T_i : i = 1,\dots,n_{\mathrm{space}}\}$ in space. We could think of $S_{\Delta t}$ as being piecewise linear periodic finite elements, where periodicity is enforced by a coupling of the corresponding degrees of freedom, and $Q_{\Delta t}$ as being piecewise constant finite elements with respect to the same mesh. If $V_h = \mathrm{span}\{\phi_1,\dots,\phi_q\}$, we get $\dim \mathcal{X}_\mathcal{N} = \dim \mathcal{Y}_\mathcal{N} = r\,q$.

Using this tensor product structure, we can derive the discrete system as follows. Let $u_\mathcal{N} = \sum_{k=1}^r \sum_{i=1}^q u_i^k\, \sigma^k \otimes \phi_i =: \mathbf{u}_\mathcal{N}^T \Sigma_{\Delta t} \otimes \Phi_h$, where $S_{\Delta t} = \mathrm{span}\{\sigma^k : \ k = 1,\dots,r\}$. Set $Q_{\Delta t} = \mathrm{span}\{\tau^k : \ k = 1,\dots,r\}$, then we get

$$\begin{aligned} b(u_\mathcal{N}, \tau^l \otimes \phi_j) &= [\dot u_\mathcal{N}, \tau^l \otimes \phi_j] + \mathcal{C}[u_\mathcal{N}, \tau^l \otimes \phi_j] \\ &= \sum_{k=1}^r \sum_{i=1}^q u_i^k (\dot\sigma^k, \tau^l)\ \langle \phi_i, \phi_j\rangle + \int_I c(t; u_\mathcal{N}(t), \tau^l(t) \otimes \phi_j)\, dt \\ &=: [(\mathbf{G}_{\Delta t} \otimes \mathbf{M}_h^{\mathrm{space}})\mathbf{u}_\mathcal{N}]_{l,j} + [\mathbf{C}(\mathbf{u}_\mathcal{N})]_{l,j}, \end{aligned}$$

with ($\mathbf{G}_{\Delta t} = [(\dot\sigma^k, \tau^l)]_{k,l}$, $\mathbf{M}_h^{\mathrm{space}} = [\langle \phi_i, \phi_j\rangle]_{i,j}$ and $\mathbf{C}$ depends on the form $c(\cdot;\cdot,\cdot)$. In the linear time-invariant case, we get

$$[\mathbf{C}(\mathbf{u}_\mathcal{N})]_{l,j} = \sum_{k=1}^r \sum_{i=1}^q u_i^k (\sigma^k, \tau^l)\, a(\phi_i, \phi_j) =: [(\tilde{\mathbf{M}}^{\mathrm{time}}_{\Delta t} \otimes \mathbf{A}_h)\mathbf{u}_\mathcal{N}]_{l,j}$$

with $\tilde{\mathbf{M}}^{\mathrm{time}}_{\Delta t} := [(\sigma^k, \tau^l)]_{k,l}$ and $\mathbf{A}_h = [a(\phi_i, \phi_j)]_{i,j}$. Standard a priori estimates similar to Cea's lemma can be derived. Here, we are more interested in a posteriori error estimates that are applicable in the RB context, i.e., estimates in terms of dual norms of an appropriate residual. We will describe later how to compute such norms in the RBM.

3.2. **A posteriori error analysis.** The space-time formulation does not restrict the representation of the solution to a discrete number of time steps and thus allows the treatment of evolution equations with arguments from the elliptic theory. Let $u_\mathcal{N} \in \mathcal{X}_\mathcal{N}$ denote the discrete solution, i.e.,

$$b(u_\mathcal{N}, v_\mathcal{N}) = f(v_\mathcal{N}), \qquad \forall v_\mathcal{N} \in \mathcal{Y}_\mathcal{N}, \tag{3.1}$$

then we obtain Galerkin orthogonality, i.e., $b(u - u_\mathcal{N}, v_\mathcal{N}) = 0$ for all $v_\mathcal{N} \in \mathcal{Y}_\mathcal{N}$. We define the residual

$$r(v) := f(v) - b(u_\mathcal{N}, v) = b(u - u_\mathcal{N}, v), \qquad v \in \mathcal{Y}, \tag{3.2}$$

and obtain the following error estimates for the state variable.

Proposition 3.1. *The following a posteriori error bounds hold:*

$$\|u - u_\mathcal{N}\|_\mathcal{Y} \leq \frac{1}{\alpha_a}\|r\|_{\mathcal{Y}'}, \quad \|u - u_\mathcal{N}\|_\mathcal{X} \leq \frac{1}{\beta}\|r\|_{\mathcal{Y}'}.$$

Proof. For the first estimate, we observe that $b(v,v) = \frac{1}{2}\left(\|v(T)\|_H^2 - \|v(0)\|_0^2\right) + \int_0^T a(t,v,v)dt \geq \alpha_a \|v\|_\mathcal{Y}^2$ by periodicity and (2.5). The claim then follows from (3.2). The second estimate follows directly from the definition of β and (3.2). □

For an error functional $s : \mathcal{X} \to \mathbb{R}$, $s \in \mathcal{X}'$, the dual problem reads

$$z \in \mathcal{Y} : \quad b(w, z) = s(w), \quad \forall w \in \mathcal{X}, \tag{3.3}$$

along with its discrete approximation $z_\mathcal{N} \in \mathcal{Y}_\mathcal{N}$. With the dual inf-sup constant

$$\beta^* := \inf_{0 \neq w \in \mathcal{Y}} \sup_{0 \neq v \in \mathcal{X}} \frac{b(v,w)}{\|v\|_\mathcal{X}\,\|w\|_\mathcal{Y}} \tag{3.4}$$

and the dual residual $r^*(w) := s(w) - b(w, z_\mathcal{N}) = b(w, z - z_\mathcal{N})$, $w \in \mathcal{X}$, we obtain the following primal-dual output estimates.

Proposition 3.2. *The output error is bounded by the a posteriori estimate*

$$|s(u) - s(u_\mathcal{N})| \leq \frac{M_b}{\beta \cdot \beta^*}\|r\|_{\mathcal{Y}'}\|r^*\|_{\mathcal{X}'}.$$

Proof. Using Galerkin orthogonality we have

$$\begin{aligned} |s(u) - s(u_\mathcal{N})| &= |b(u - u_\mathcal{N}, z)| = |b(u - u_\mathcal{N}, z - z_\mathcal{N})| \\ &\leq M_b \,\|u - u_\mathcal{N}\|_\mathcal{X}\, \|z - z_\mathcal{N}\|_\mathcal{Y} \leq \frac{M_b}{\beta \cdot \beta^*}\|r\|_{\mathcal{Y}'}\|r^*\|_{\mathcal{X}'} \end{aligned}$$

by Proposition 3.1 and the definitions of r^* and β^*. □

Remark 3.3 (Eigenvalue problems)**.** *The constants α_a, β, M_a and M_b can all be formulated in terms of Rayleigh quotients and can hence be calculated as solutions to generalized (space-time) eigenvalue problems, see e.g.* [12]. *For time-independent operators $A(t) \equiv A$, α_a and M_a can even be obtained by generalized eigenvalue problems in space only.*

4. Reduced Basis Methods

Now, let $\mu \in \mathcal{D} \subset \mathbb{R}^P$ be a parameter vector that the the form b (and possibly also the right-hand side f) is assumed to depend on. Both in *real-time* and/or *multi-query* contexts, reduced basis methods can reduce the computational effort of solving (3.1) for each considered parameter by constructing the basis of a reduced space $\mathcal{X}_N \subset \mathcal{X}_\mathcal{N}$, $N \ll \mathcal{N}$, that is a low-dimensional approximation of the solution manifold $\{u(\mu), \mu \in \mathcal{D}\}$.

Such a reduced basis $\{\zeta_1, \dots \zeta_N\}$ is typically determined in a *training* or *offline phase* by

- iteratively choosing a sample set of parameters $\{\mu_1, \dots, \mu_N\}$ e.g. by a (POD-)Greedy method based upon an a-posteriori error estimate,
- computing the solutions $u(\mu_1), \dots, u(\mu_N) \in \mathcal{X}_\mathcal{N}$ (also called *snapshots*) by using a sufficiently fine 'truth' discretization $\mathcal{X}_\mathcal{N}$, $\mathcal{Y}_\mathcal{N}$ of (2.1),
- possibly orthogonalizing the snapshots.

The reduced space is then defined as $\mathcal{X}_N := \operatorname{span}\{u(\mu_1), \dots, u(\mu_N)\}$ accompanied by some test space $\mathcal{Y}_N$ such that the discrete problem is uniformly stable. Reduced basis solutions $u_N(\mu) \in \mathcal{X}_N$ for other parameters are obtained in the *online phase* via the Galerkin projection of (3.1) onto $\mathcal{X}_\mathcal{N}$, $\mathcal{Y}_\mathcal{N}$, i.e.

$$b(u_N(\mu), v_N; \mu) = f(v_N; \mu) \qquad \forall v_N \in \mathcal{Y}_N.$$

A crucial requirement for the feasibility of the approach is the $\mathcal{N}$-independence of all online quantities. A necessary assumption is the existence of affine decompositions of all linear and bilinear forms, e.g.

$$b(w, v; \mu) = \sum_{q=1}^{Q} \theta_q(\mu)\, b_q(w, v), \qquad w \in \mathcal{X}, v \in \mathcal{Y}, \tag{4.1}$$

with parameter-dependent functions $\theta_q : \mathcal{D} \to \mathbb{R}$ and parameter-independent forms $b_q : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$, $1 \le q \le Q$. This allows the $\mathcal{N}$-dependent precomputation of all parameter-independent quantities in the offline phase and reduces e.g. the assembly for online computations to matrix-vector multiplications of dimension N. If necessary, *empirical interpolation methods* (EIM) can be employed to construct affine approximations for all involved forms [1].

4.1. **Space-Time approach.**

It is one of the nice features of the space-time approach that we can basically use the RBM techniques as developed for elliptic problems. Hence, we can be relatively short here and refer the reader for more details e.g. to [12].

4.1.1. *A-posteriori error estimates.* Allowing for the parameter-dependence of solution as well as continuity and stability constants, the a-posteriori estimates of Section 3.2 can easily be transferred. In order to obtain *efficient* (i.e. $\mathcal{N}$-independent) estimates, we additionally assume the existence of computable lower or upper bounds for all involved constants, e.g. $\beta_{LB}(\mu) \le \beta(\mu) := \inf_{w\in\mathcal{X}} \sup_{v\in\mathcal{Y}} \frac{b(w,v;\mu)}{\|w\|_\mathcal{X}\, \|v\|_\mathcal{Y}}$ with $\beta_{LB}(\mu)$ only N-dependent. Such bounds can either be derived by theoretical results as in Proposition 2.3 or by a so-called *Successive Constraint Method* (SCM) [8] that uses eigenvalue problem solutions computed offline to obtain online bounds through linear optimization procedures. With these minor modifications we directly obtain

Corollary 4.1. *Denote* $0 < \alpha_{LB}(\mu) < \alpha_a(\mu)$, $0 < \beta_{LB}(\mu) \leq \beta(\mu)$ *and* $M_{UB}(\mu) \geq M_b(\mu)$ *and well as the reduced output* $s_N(\mu) := s(u_N(\mu))$*. Then the following a-posteriori estimates hold*

$$\|u_{\mathcal{N}}(\mu) - u_N(\mu)\|_{\mathcal{Y}} \leq \frac{1}{\alpha_{LB}(\mu)}\|r_N(\mu)\|_{\mathcal{Y}'} =: \Delta_N^{\mathcal{Y},ST}(\mu),$$
$$\|u_{\mathcal{N}}(\mu) - u_N(\mu)\|_{\mathcal{X}} \leq \frac{1}{\beta_{LB}(\mu)}\|r_N(\mu)\|_{\mathcal{Y}'} =: \Delta_N^{\mathcal{X},ST}(\mu),$$
$$|s(\mu) - s_N(\mu)| \leq \frac{M_{UB}(\mu)}{\beta_{LB}(\mu)\cdot\beta^*_{LB}(\mu)}\|r_N(\mu)\|_{\mathcal{Y}'}\|r^*_N(\mu)\|_{\mathcal{X}'} =: \Delta_N^{s,ST}(\mu).$$

Remark 4.2 (Restriction to reduction error)**.** *It has to be emphasized that the error analysis in reduced basis methods is only concerned with the error the Galerkin projection onto* $\mathcal{X}_N$ *introduces with respect to the* $\mathcal{N}$*-dimensional truth system. Generally, it is assumed that* $u_{\mathcal{N}}(\mu) \to u(\mu)$ *for* $\mathcal{N} \to \infty$ *and* $u_{\mathcal{N}}$ *presents an acceptable approximation of the exact solution so as to justify the neglect of the discretization error in substituting* $\mathcal{X}$ *by* $\mathcal{X}_{\mathcal{N}}$*. Consequently, the residual in Corollary 4.1 is defined as*

$$r_N(v;\mu) := f(v) - b(u_N(\mu), v; \mu) = b(u_{\mathcal{N}} - u_N, v), \qquad v \in \mathcal{Y}_{\mathcal{N}}.$$

The evaluation of the above error bounds involves the dual norm of $r_N(\mu)$ wich can be obtained via the *Riesz representor* $\hat{\varepsilon}_N(\mu) \in \mathcal{Y}_{\mathcal{N}}$ of the residual, i.e. the solution of

$$(\hat{\varepsilon}_N(\mu), v)_{\mathcal{Y}} = r_N(v;\mu) \qquad \forall v \in \mathcal{Y}_{\mathcal{N}},$$

as $\|r_N(\mu)\|_{\mathcal{Y}'} = \|\hat{\varepsilon}_N(\mu)\|_{\mathcal{Y}}$. Not only can this Riesz representor be computed in the RB context, but moreover the affine decomposition of f and b can be exploited to divide this computation into an offline phase – the calculation of representors of each parameter-independent term and the precomputation of the inner products between those representors – and the fast and $\mathcal{N}$-independent online phase consisting only of the composition of those products, the parameter function evaluations and the current reduced basis solution $u_N(\mu)$ to obtain $\|\hat{\varepsilon}_N(\mu)\|_{\mathcal{Y}}$ directly, cf. [12]. Note that each offline Riesz representor calculation – in total $Q_f + NQ_b$ problems with Q_f, Q_b the number of affine terms of f and b – here implies the solution of a space-time problem while the online effort is not different from the elliptic case.

4.1.2. *Space-time reduced basis.* As mentioned above, a space-time reduced basis is contructed iteratively in the offline training phase by a Greedy procedure. In short summary, in each iteration the parameter for which $u_{\mathcal{N}}$ is worst approximated by the current reduced basis is chosen out of a training set Ξ_{train}. Instead of the infeasibly expensive calculation of $\arg\max_{\mu\in\Xi_{\text{train}}} \|u_{\mathcal{N}}(\mu) - u_N(\mu)\|$, the error is replaced by the corresponding efficient error estimate.

For each of those parameters, the *snapshot* $u_{\mathcal{N}}(\cdot,\cdot;\mu) \in \mathcal{X}_{\mathcal{N}}$ is determined as the space-time solution of a $\mathcal{N}$-dimensional *truth* problem (3.1). The orthogonalized snapshots form the basis $\{\xi_1, \dots, \xi_N\}$ of $\mathcal{X}_N$ and $\mathcal{Y}_N$, respectively.

Using this space-time basis, each online solution for a new parameter $\mu \in \mathcal{D}$ requires the solution of a N-dimensional (dense) equation system, i.e. a computational effort of $\mathcal{O}(N^3)$.

4.2. **Fixed point methods.** The usual approach to time-dependent problem is the use of time-stepping methods. Fixed-point approaches are an extension of such methods to obtain periodic or stationary solutions. Observing that periodic solutions are fixed points of the operator $\mathcal{S} : V \to V$, $u(0) \mapsto u(T)$, a sequence of initial value problems (IVP) is solved where in each iteration the initial value is taken as the final time solution of the previous iteration. For the ease of exposition, we restrict ourselves to the implicit Euler method and consider a uniform time discretization $\{t_k := k\Delta t\}_{k=0,\dots,K}$ with $t_K := T$, denoting the discrete solution at time t_k by $u_h^k \in V_h$. To obtain a periodic snapshot for a given parameter, one thus solves

$$\langle u_h^k(\mu), w_h\rangle + \Delta t\, a(t_k, u_h^k(\mu), w_h; \mu) = \langle u_h^{k-1}(\mu), w_h\rangle + \Delta t\, \langle f(t_k;\mu), w_h\rangle \quad \forall\, w_h \in V_h,$$

beginning with an (arbitrary) initial value $u^{0,(0)}(\mu) := u_h^{(0)}(0;\mu)$ and setting $u^{0,(\nu)}(\mu) = u^{K,(\nu-1)}(\mu)$ in subsequent iterations until $\|u^{0,(I)}(\mu) - u^{0,(I-1)}(\mu)\|_V \le tol$ for some prespecified tolerance. In the following, we denote by $I = I(tol)$ the required number of *fixed-point iterations* before convergence.

The reduced basis space in such a setting is then constructed as a *spatial* approximation $V_N \subset V_h$ and the Galerkin projection $u_N(\mu) := \{u_N^k(\mu), k = 1, \dots, K\}$ corresponds to a solution of the online fixed point problem

$$\begin{aligned}\langle u_N^k(\mu), w_N\rangle &+ \Delta t\, a(t_k, u_N^k(\mu), w_N; \mu)\\ &= \langle u_N^{k-1}(\mu), w_N\rangle + \Delta t\, \langle f(t_k;\mu), w_N\rangle \quad \forall\, w_N \in V_N, k = 1, \dots, K,\end{aligned}$$

with $u_N^0(\mu) = u_N^K(\mu)$.

Remark 4.3 (Online effort)**.** *Note that the above RB space decreases only the spatial, not the temporal discretization dimension. The latter is fixed to the original number of time steps K that is necessary to obtain satisfactorily truth solutions. Moreover, the computation of a periodic reduced solution $u_{N_{FP}}(\mu)$ necessitates a full fixed-point procedure, resulting in an online computational effort of $\mathcal{O}(IKN_{FP}^3)$. In contrast, a space-time RB space of dimension N_{ST} leads to an online effort of $\mathcal{O}(N_{ST}^3)$, as only one dense linear equation system has to be solved.*

4.2.1. *A posteriori estimates.* Unlike the space-time setting, the time-stepping framework allows the derivation of a-posteriori error estimates only in a time-discrete norm that may be considered as an approximation of $\|\cdot\|_{\mathcal{Y}}$. More specifically, we have the following error bound.

Proposition 4.4. *Denote by $e_N^k(\mu) := u_h(t_k,\mu) - u_N^k(\mu)$ the error in the k-th time step and by $r_N^k(\cdot;\mu) : V \to \mathbb{R}$ the corresponding residual , i.e.*

$$r_N^k(w;\mu) := \Delta t\, \langle f(t_k;\mu), w\rangle - \langle u_N^k(\mu) - u_N^{k-1}(\mu), w\rangle - \Delta t\, a(t_k, u_N^k(\mu), w; \mu).$$

The error $u_h(\mu) - u_N(\mu)$ is then bounded in a discrete spatio-temporal norm as

$$\left(\Delta t \sum_{k=1}^{K} \|e_N^k(\mu)\|_V^2\right)^{\frac{1}{2}} \le \left(\frac{\Delta t}{\alpha_{LB}^2(\mu)} \sum_{k=1}^{K} \|r_N^k(\cdot;\mu)\|_{V'}^2\right)^{\frac{1}{2}} =: \Delta_N^{FP}(\mu). \tag{4.2}$$

Proof. For $k = 1, \dots, K$, the error e_N fulfills the recursion

$$\langle e_N^k(\mu), w_N\rangle + \Delta t\, a(t_k, e_N^k(\mu), w_N; \mu) = \langle e_N^{k-1}(\mu), w_N\rangle + \Delta t\, r_N^k(w_N;\mu)\ \forall\, w_N \in V_N.$$

Testing with $w_N := e_N^k(\mu)$ and using coercivity (2.5) as well as Young's inequality, we obtain

$$\begin{aligned}
\langle e_N^k(\mu), e_N^k(\mu)\rangle + \Delta t\, \alpha_{LB}(\mu)\|e_N^k(\mu)\|_V^2 &\le \langle e_N^k(\mu), e_N^k(\mu)\rangle + \Delta t\, a(t_k, e_N^k(\mu), e_N^k(\mu)) \\
&= \langle e_N^{k-1}(\mu), e_N^k(\mu)\rangle + \Delta t\, r_N^k(e_N^k(\mu);\mu) \\
&\le \sqrt{\langle e_N^{k-1}(\mu), e_N^{k-1}(\mu)\rangle}\sqrt{\langle e_N^k(\mu), e_N^k(\mu)\rangle} + \Delta t\, \|r_N^k(\cdot;\mu)\|_{V'}\|e_N^k(\mu)\|_V \\
&\le \frac{1}{2}\Big(\langle e_N^{k-1}(\mu), e_N^{k-1}(\mu)\rangle + \langle e_N^k(\mu), e_N^k(\mu)\rangle \\
&\qquad + \Delta t \frac{\|r_N^k(\cdot;\mu)\|_{V'}^2}{\alpha_{LB}(\mu)} + \Delta t\, \alpha_{LB}(\mu)\|e_N^k(\mu)\|_V^2\Big),
\end{aligned}$$

so that

$$\langle e_N^k(\mu), e_N^k(\mu)\rangle - \langle e_N^{k-1}(\mu), e_N^{k-1}(\mu)\rangle + \Delta t\, \alpha_{LB}(\mu)\|e_N^k(\mu)\|_V^2 \le \Delta t \frac{\|r_N^k(\cdot;\mu)\|_{V'}^2}{\alpha_{LB}(\mu)}.$$

Summing over all time steps yields with the periodicity of the error

$$\begin{aligned}
\langle e_N^K(\mu), e_N^K(\mu)\rangle - \langle e_N^0(\mu), e_N^0(\mu)\rangle + \Delta t\, \alpha_{LB}(\mu)\sum_{k=1}^K \|e_N^k(\mu)\|_V^2 \\
= \Delta t\, \alpha_{LB}(\mu)\sum_{k=1}^K \|e_N^k(\mu)\|_V^2 \le \frac{\Delta t}{\alpha_{LB}(\mu)}\sum_{k=1}^K \|r_N^k(\cdot;\mu)\|_{V'}^2.
\end{aligned}$$

□

Remark 4.5. *Due to the periodic structure of both truth and RB solution, $\|e_N^0(\mu) - e_N^K(\mu)\| \le tol$, so that the norm in (4.2) is a trapezoidal approximation of $\|\cdot\|_{\mathcal{Y}}$. Note that the quadrature quality is restricted a-priori by the choice of time discretization.*

4.2.2. *Time-independent reduced basis.* The construction of the spatial reduced space $V_N \subset V_h$ is usually done with a so-called *POD-Greedy* procedure [7] that roughly consists of the following two steps:

(1) Based on the error estimator (4.2), choose greedily the next snapshot parameter μ out of some training set and compute the corresponding snapshot, i.e. the trajectory $\{u_h^k(\mu), k = 1, \dots, K\}$.
(2) Project the trajectory onto the existing reduced basis, subject the projection error to a POD with respect to time and add the first mode as (time-independent) basis function.

As in the space-time approach, the computation of the error bound involves lower bounds of the coercivity constant $\alpha_a(\mu)$ as well as dual norms of the residuals, here for each time step. Offline-online decomposition can be exploited in the calculation of these quantities in an analogous way if additionally to the assumption of affine structure (4.1) in the parameter, time and space can also be separated, i.e. if the involved linear and bilinear forms have the structure

$$a(t, w, v; \mu) = \sum_{q=1}^Q \theta_q(t,\mu)\, a_q(w, v), \qquad w, v \in V,$$

with $\theta_q : [0,T] \times \mathcal{D} \to \mathbb{R}$ and parameter-independent forms $a_q : V \times V \to \mathbb{R}$, $1 \le q \le Q$. If this assumption is not met, either spatial reduced bases V_N^k have to be constructed for each time step separately [4] or a temporal EIM [5] has to be employed to obtain the required structure.

The calculation of $\alpha_{LB}(\mu)$ then involves spatial generalized eigenvalue problems, while the computation of $\|r_N^k(\cdot;\mu)\|_{V'}$ requires again $Q_f + NQ_a$ Riesz representors, here in V.

Remark 4.6 (Separation of offline and online solution method)**.** *In principle, the solution method for the truth solutions in the offline phase and that for the reduced solutions in the online phase can be chosen independently. Space-time snapshots can be subjected to a POD in order to form a spatial reduced basis; and similarly the trajectories $\{u_h^k(\mu), k = 1,\dots,K\}$ can be reinterpreted as space-time basis functions for an appropriate time basis.*

However, both methods cannot be separated completely. As the a-posteriori error bounds involve reduced solutions, their type is implied by the choice of RB solution method, even during training. Moreover, recall that in the offline phase we calculate not only the basis functions but all $\mathcal{N}$-dependent, parameter-independent quantities necessary for RB solutions and error bounds, like the evaluation of the bilinear forms at the basis functions, the Riesz representor inner products etc. Obviously, this has to be done in accordance with the desired online method. In order to construct space-time reduced bases, for example, space-time Riesz representors have to be determined, even if a time-stepping method is used to compute the snapshots themselves.

In the following, we use always the same method for both offline and online solutions.

5. Numerical Results

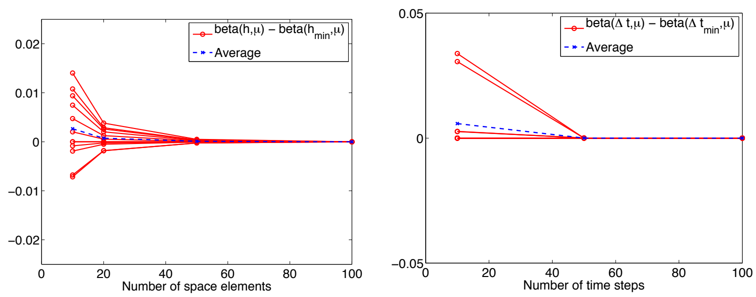
We consider the following parameterized periodic convection-diffusion-reaction problem on $[0,T]\times\Omega$ with $T=1$, $\Omega=(0,1)$:

$$
\begin{aligned}
&u_t - u_{xx} + \mu_1(\tfrac{1}{2} - x)u_x + \mu_2 u = \cos(2\pi t) \qquad \text{on } \Omega,\\
&u(t,0) = u(t,1) = 0, \quad u(0,x) = u(T,x),
\end{aligned}
\tag{5.1}
$$

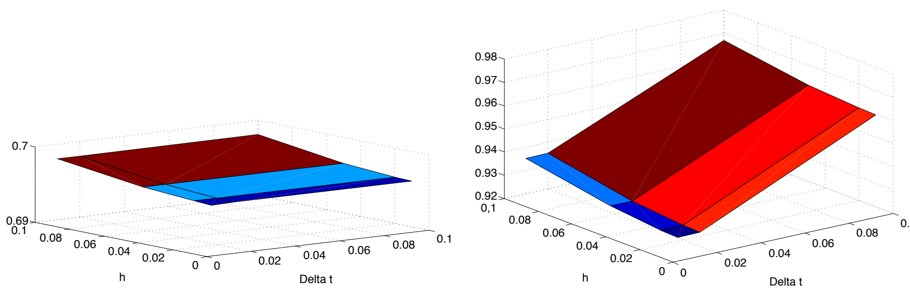
with parameter domain $\mathcal{D} = [0,30]\times[-9,15]$.

For the numerical experiments, we use for the space-time approach a Matlab implementation of the discretization described in Section 3.1, while the fixed point calculations are based on the `rbOOmit` framework [11], a plugin to the finite element library `libmesh` [9].

Inf-sup constant. Non-coercive initial value problems can be treated with time-stepping methods using a specific energy bound stability factor (respectively its lower bound) at each step t_k, $k=1,\dots,K$ [10]. This bound, however, decreases exponentially in T, rendering long-term integration impossible. It has been observed for an initial value problem similar to (5.1) that the space-time inf-sup constant $\beta(\mu)$, in contrast, decreases only linearly in T if there is no reaction ($\mu_2 = 0$), while $\beta((0,0)) \equiv 1$ and $\beta((0,\mu_2)) \sim e^{-\mu_2 T}$ [16] . In order to analyze the behaviour of $\beta(\mu) = \beta(\Delta t, h;\mu)$ in the periodic context, we compute the corresponding generalized eigenvalue problem for different discretizations $\Delta t \in \{10,50,100\}$, $h \in \{10,50,70,100\}$ at chosen parameter values $\mu \in \{(0,-10),(0,-7),(0,-3),(0,0),$ $(0,10),(10,-12),$ $(10,-7),(10,0),(10,10),(20,-15),(20,-5),(20,5)\}$. The results are presented in Figure 1, where we display in Figures 1(a) and 1(b) the behaviour of $\beta(\mu)$ for the considered parameter values at $\Delta t_{\min} = 0.01$ for decreasing h and at $h_{\min} = 0.01$ for decreasing Δt, respectively; both normalized by considering the absolute error with respect to $\beta(\Delta t_{\min}, h_{\min}; \mu)$. It is apparent that the inf-sup constant depends little on the discretization. The full dependence of both time and space discretization for two parameters, depicted in Figures 1(c) and 1(d), underlines this observation. Note that the energy approach of [10] is not even applicable in the periodic setting (mainly because the initial error e_0 cannot be bounded here).



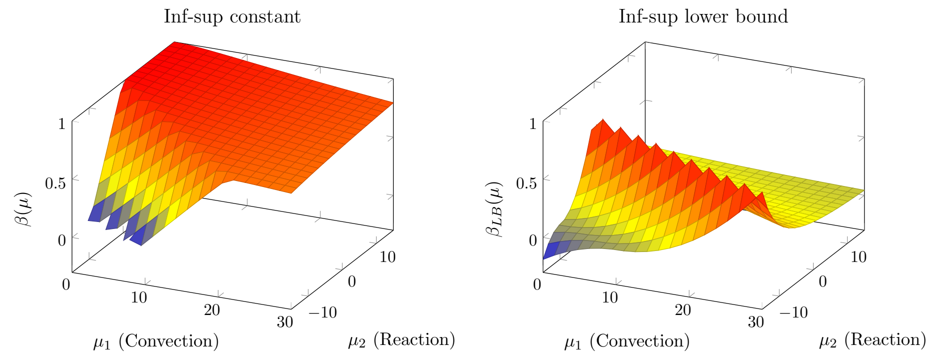
(a) Absolute error of $\beta(\Delta t_{\min}, h; \mu)$ (b) Absolute error of $\beta(\Delta t, h_{\min}; \mu)$



(c) $\beta(\Delta t, h; \mu)$ for $\mu = (0, -3)$ (d) $\beta(\Delta t, h; \mu)$ for $\mu = (10, 10)$

FIGURE 1. Stability of inf-sup constant

Moreover, to investigate the quality of the analytical bound in (2.6), we plot in Figure 2 both the inf-sup constant β_μ as well as the lower bound $\beta_{LB}(\mu)$ over the extended parameter domain $\widetilde{\mathcal{D}} = [0, 30] \times [-15, 15]$. Note that we resort to eigenvalue problems in space in order to determine continuity and coercivity constants of the (symmetric part) of the bilinear form $a(\cdot, \cdot; \mu)$ in (2.6). This still reduces the computational effort significantly in comparison with the space-time eigenvalue problem for $\beta(\mu)$. As expected, the inf-sup constant (Fig. 2(a)) degrades with growing convection as well as with decreasingly negative reaction parameters and reliably indicates the parameter region of non-stability (note that the values with $\beta^2(\mu) < 0$ for parameters near (0,-15) are not plotted). The bound (Fig. 2(b)) mirrors this behaviour in the region of low and no stability, but proves to provide poor estimates for large parameter values. This is mainly due to the influence of the continuity constant M_a that grows in both parameters and enters as M_a^{-2} into the lower bound. For this example, one thus would rather resort to a SCM to obtain efficient and good online estimates for the inf-sup constant.

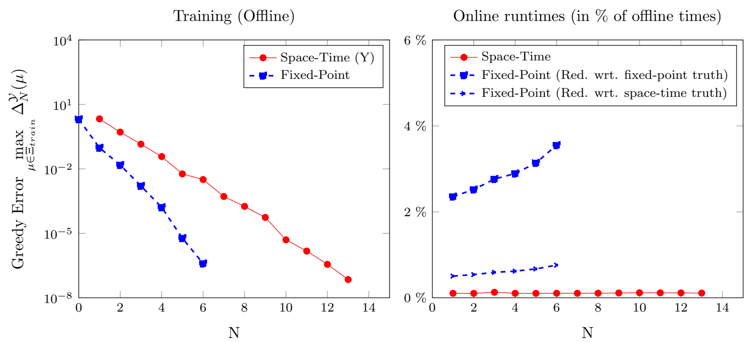


(a) Inf-sup constant (b) Inf-sup lower bound

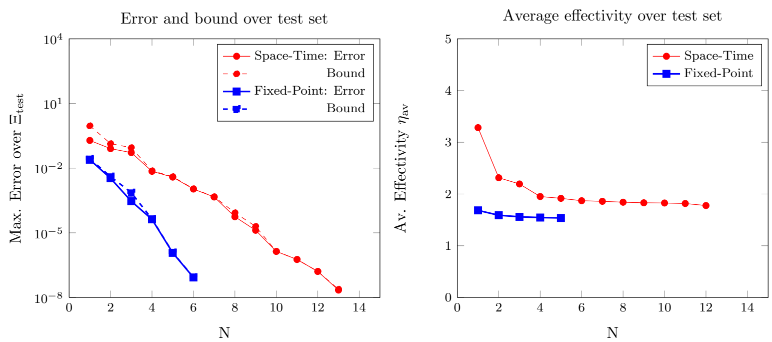
FIGURE 2. Inf-sup constant and bound

Reduced Basis construction. The offline training phase is controlled by the *Greedy error* $\max_{\mu\in\Xi_{\text{train}}} \Delta_N^{\mathcal{Y}}(\mu)$, which serves as approximation quality indicator of the reduced basis. Its exponential decrease with growing number of basis functions is depicted in Figure 3(a). As training set, we use for both space-time and fixed point method a uniform set $\Xi_{\text{train}} \subset \mathcal{D}$ of parameters with $n_{\text{train}} = |\Xi_{\text{train}}| = 20 \times 20 = 400$. Space and time discretization are $\Delta t = 0.02$, $h = \frac{1}{32}$ for both methods.
The Greedy error behaviour determines the size of the reduced basis that is used in the online phase to calculate the reduced solutions $u_N(\mu)$. We see in Figure 3(a) that the space-time approach yields approximately twice the number of basis functions (denoted by N_{ST}) compared to the fixed point method (N_{FP} basis functions) for the same approximation quality. Recall that this implies an online effort of $\mathcal{O}(N_{\text{ST}}^3) = \mathcal{O}(8N_{\text{FP}}^3)$, which is a substantial improvement over $\mathcal{O}(I \cdot 50 \cdot N_{\text{FP}}^3)$, the effort of an online fixed point solution. In our example, the number of fixed point iterations I is not prohibitively large: $I \in \{1, \ldots, 11\}$, depending on parameter and RB size ($I \in \{2, \ldots, 5\}$ for most parameters). Note that the fixed point convergence can sometimes be accelerated by systematic choices of initial values, yet a certain number of iterations as well as of time steps K is necessary to maintain online solution quality.

The difference in online computational effort is also visualized in Figure 3(b), where we compare the runtime reduction of the reduced bases obtained for each method. We measure this reduction by the ratio between the time needed to compute an online solution $u_N(\mu)$ and that to calculate the corresponding truth solution $u_{\mathcal{N}}(\mu)$, i.e. $\frac{\text{time in } s \text{ to obtain } u_N(\mu)}{\text{time in } s \text{ to obtain } u_{\mathcal{N}}(\mu)}$, in % for $N = 1, \ldots, N_{\text{FP}}$ and $N = 1, \ldots, N_{\text{ST}}$, respectively. However, due to the different computational complexities of both the truth and the reduced solutions this comparison may be somewhat misleading. To provide some more insight, we additionally plot the runtime gain of the fixed-point RB with respect to the (more expensive) space-time truth solution. Note that this



(a) Training (b) Runtime reduction



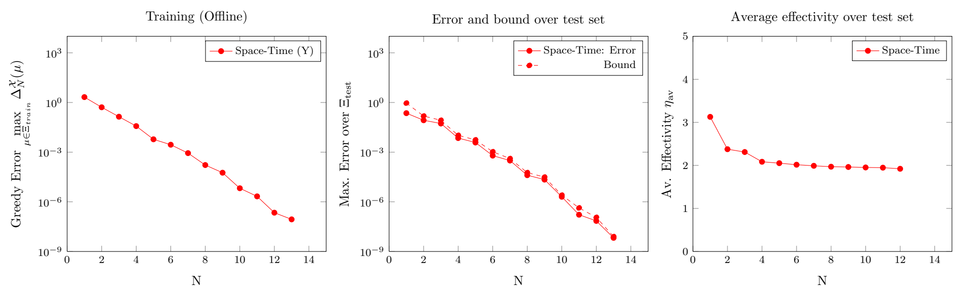
(c) Test (d) Effectivity of error bounds

FIGURE 3. Comparison of space-time and fixed-point approach for the convection-diffusion-reaction example.

corresponds to a direct comparison of online runtimes. It is clearly visible that the space-time reduced solution provides a greater reduction for any approximation quality (even for the maximal basis size N_{ST}) and can be obtained faster than any fixed-point RB solution. (Recall, however, that both methods are implemented in different frameworks.)

It is also of interest whether the error bounds are adequate surrogates for the true approximation error. We have proven in Sections 4.1.1 and 4.2.1 that the respective estimates are indeed upper bounds, so that the error is never underestimated. However, large overestimation may lead to unnecessarily large reduced bases. In Figure 3(c), the maximal true error over a test set $\Xi_{\text{test}} \subset \mathcal{D}$, uniformly spaced over $[0.5, 29.5] \times [-8.5, 14.5]$ with $n_{\text{test}} = |\Xi_{\text{test}}| = 15 \times 15 = 225$, and the corresponding error bounds for both methods are shown. It is obvious that both error bounds reliably reflect the behaviour of the true errors. To further quantify this property, we present in Figure 3(d) the *average effectivities* of the bounds over the test set, i.e. $\eta_{\text{av}} := \frac{1}{n_{\text{test}}} \sum_{\mu\in\Xi_{\text{test}}} \frac{\Delta_N(\mu)}{\|u_{\mathcal{N}}(\mu)-u_N(\mu)\|}$. This value is slightly larger for the space-time error bound but sufficiently small in both methods: for $N \geq 4$ the true error is never overestimated by a factor larger than 2. Note that for the maximum basis size both error bounds and errors are in the order of (the square root of) machine precision, effectivity computations are hence little reliable and not plotted.

Finally, note that all presented results for the space-time approach up to this point are obtained with respect to the norm in $\mathcal{Y}$ to ensure comparability with the fixed-point method, as the time-stepping approach does not allow for another measure. In Figure 4, we show without further comment the training and test behaviour of the space-time method with respect to norm and error bound in $\mathcal{X}$. It is obvious that all above observations for $\|\cdot\|_{\mathcal{Y}}$ can be transferred without modification.



(a) Training (b) Test (c) Effectivity of error bounds

FIGURE 4. Space-time approach using $\|\cdot\|_{\mathcal{X}}$ and $\Delta_N^{\mathcal{X}}(\mu)$.

6. CONCLUSION

We have considered space-time methods for time-periodic problems and established conditions for well-posedness as well as lower bounds for the inf-sup constant.

In order to apply reduced basis methods in the space-time setting, we derived error bounds for the state variable in both $\mathcal{X}$- and $\mathcal{Y}$-norm as well as for linear time-averaged outputs and discussed basis construction by a Greedy method and the computation of the involved residual dual norms and stability constants. This was contrasted with both POD-Greedy basis construction and the discrete error bounds obtainable in a time-stepping setting using fixed-point methods.

The numerical results for a convection-diffusion-reaction equation show that the space-time inf-sup constant correctly reflects the stability of the problem and is stable with respect to the underlying discretization. Unfortunately, is has also been observed that in this example the effectivity of the analytical lower bound degrades for large parameters as the continuity constant grows in μ. As thus the true stability of the problem is not adequately represented, other efficient inf-sup estimators have to be used in online calculations.

RB training for both space-time and fixed-point method revealed that the basis constructed by the space-time approach is approximately twice as large as the fixed-point basis for the same approximation quality. However, as the online fixed-point problem is avoided, significantly lower online computation times can be observed for the space-time approach. Moreover, we demonstrated that all error bounds are sufficiently efficient to provide good surrogates for the true error.

In future, we would like to address the problem of the computationally intensive offline space-time truth solutions. We have seen in this paper that space-time reduced bases may for some problems significantly reduce online computation costs while offline costs are rather large. Other problems, especially those involving time-variant operators, may however not be computable directly with time-stepping methods but require additional treatment, e.g. approximations of the operators by EIMs in time. In such cases, space-time approaches may be preferable.

Moreover, we will investigate the application of adaptive space-time solutions methods, e.g. with wavelet algorithms, which reduce the increase in computational cost that is due to the additional (time) dimension. First experiments in that direction indicate the need for an accurate monitoring of the tolerable approximation error, particularly in the computations of the Riesz representors.

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