Fast Solvers with Block-Diagonal Preconditioners for Linear FEM-BEM Coupling

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Preprint Series: 2007-08

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FAST SOLVERS WITH BLOCK-DIAGONAL PRECONDITIONERS FOR LINEAR FEM-BEM COUPLING

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Abstract. The purpose of this paper is to present optimal preconditioned iterative methods to solve indefinite linear systems of equations arising from symmetric coupling of finite elements and boundary elements [14]. This is a block-diagonal preconditioner together with a conjugate residual method (PCR) and a preconditioned inner-outer iteration (PIO). We prove the efficiency of these methods by showing that the number of iterations to preserve a given accuracy is bounded independently of the number of unknowns. Numerical examples underline the efficiency of these methods.

1. Introduction

This paper deals with the problem of efficiently solving systems of linear equations $Ax = b$ where $A$ has a $2 \times 2$ block structure such that the diagonal matrices are positive semidefinite and negative definite. Particularly, we consider matrices arising from the symmetric coupling of finite element method (FEM) and boundary element method (BEM) when dealing with elliptic transmission problems.

In the case of an indefinite and symmetric FEM-BEM coupling matrix, we may write

$$A := \begin{pmatrix} A + H & B^\top \\ B & -C \end{pmatrix}$$

where $A, H \in \mathbb{R}^{m \times m}$ are both positive semidefinite ($A, H \geq 0$) and $C \in \mathbb{R}^{n \times n}$ is positive definite ($C > 0$). Let a block-diagonal preconditioner

$$P := \begin{pmatrix} P_A & 0 \\ 0 & P_C \end{pmatrix}$$

with symmetric submatrices $P_A$ and $P_C$ be given, which are spectrally equivalent to the Schur complement $A + H + B^\top C^{-1} B$ and $C$. This idea of using matrices as preconditioners which are spectrally equivalent to diagonal submatrices of $A$ in the context of solvers for linear FEM-BEM equations was also used in [23, 24, 27, 32]. In the latter works, the matrix $A$ is substructured into a $3 \times 3$ system. The theoretical and numerical results [32] indicate, that the convergence rate of a preconditioned conjugate residual method depends on the discretization. We show in this paper, that the block-diagonal preconditioned conjugate residual method (PCR) which we use, leads to convergence rates, which are independent of the mesh size $h$. The used theoretical tool gives also results for the case, such that the preconditioner $P$ is a $2 \times 2$ block diagonal matrix. Bramble and Pasciak [7] introduced for problems like (1.1) a special inner product which then gives a symmetric and positive definite system. But the system is based on the assumption that there exists a matrix $P_C$ and positive constants $\alpha_0, \alpha_1$ such that $\alpha_0 C \leq P_C \leq \alpha_1 C$ where $\alpha_1 < 1$ is desired. This can always be satisfied by scaling but it affects the rate of convergence of the applied iteration scheme e.g. conjugate gradient method. There will be no parameter to choose in the here presented block diagonal preconditioner resulting in an optimal rate of convergence.

Sylvester’s law of inertia gives together with the following congruent transform of $A$

$$A = \begin{pmatrix} I & -B^\top C^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A + H + B^\top C^{-1} B & 0 \\ 0 & -C^{-1} B \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

1991 Mathematics Subject Classification. 65 F 10, 65 F 15, 65 N 22, 65 N 30, 65 N 38.

Key words and phrases. coupling of finite elements and boundary elements, Krylov-methods block-diagonal preconditioner, fast solver.
shows that the matrix $A$ has $m$ positive and $n$ negative eigenvalues, where $I$ denotes a generic identity matrix. In Section 2 we give bounds of intervals which contain the spectrum of $A$ in a general setting, i.e., $\Lambda(A) \subset [-a, -b] \cup [c, d]$ $(a, b, c, d > 0)$. In the subsequent section we introduce the interface model problem and rewrite it with boundary integral operators into an equivalent weak formulation. In Section 4 we introduce discrete basis functions and discretize the integral operators appearing in the symmetric coupling problem. In the following Section 5 we deal firstly with the fact that the convergence rate of the PCR is bounded by a term depending on $a, b, c, d$. Secondly, we present a block-diagonal preconditioner for which we prove the existence of constants $\theta, \Theta, \Delta$ independently of the mesh size $h$. Using these constants Theorem 2 gives some $h$ independent bounds of $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ for the preconditioned Matrix $A := A^{-1/2}A^{-1/2}$. Hence, we get an upper bound for the convergence rate of the block-diagonal preconditioned PCR-algorithm, which is independent of the discretization. In Section 7 we analyse the inner-outer iteration [3], applied to the FEM-BEM coupling matrix. First, we present a convergence analysis of the inner-outer iteration for a general saddlepoint problem and apply finally this procedure to the FEM-BEM problem that we want to solve. Here, we also get a convergence rate, which is independent of the discretization (Theorem 4).

Numerical experiments in Sections 5 and 7 will give approximations for the constants $\theta, \Theta, \Delta$ and underline that they are independent of the mesh size $h$. They also show optimal convergence for the preconditioned PCR method and the preconditioned inner-outer iteration, in the sense, that the number of iterations to reach a given exactness is bounded from above.

2. Block-Diagonal Preconditioner in a general setting

In the following we will use bold letters $x, y, \ldots$ for column vectors and $\mathbb{R}^{k \times k}_{\text{sym}}$ will denote real-valued symmetric $k \times k$-matrices. The next theorem provides a basic conditioning estimate for the operator $P^{-1/2}A^{-1/2}$.

**Theorem 1.** Let $H \in \mathbb{R}^{m \times m}_{\text{sym}}$ be a positive semidefinite matrix, $C \in \mathbb{R}^{n \times n}$ be positive definite, and $B \in \mathbb{R}^{m \times m}$. Further, let $A \in \mathbb{R}^{m \times m}_{\text{sym}}$ s.t. $A + H + B^T C^{-1} B$ is positive definite, and $T \in \mathbb{R}^{m \times m}_{\text{sym}}$ s.t. $A + T$ is positive definite. Let the positive constants $\Delta, \Theta, \theta$ be given such that the following inequalities hold for all $x \in \mathbb{R}^m \setminus \{0\}$

$$
(2.1) \quad \frac{x^T (A + H) x}{x^T (A + T) x} \leq \Delta, \quad \frac{x^T B^T C^{-1} B x}{x^T (A + T) x} \leq \Theta^2,
$$

and

$$
(2.2) \quad \theta^2 \leq \frac{x^T (A + H + B^T C^{-1} B) x}{x^T (A + T) x}.
$$

We define

$$
A := \begin{pmatrix} A & B^T \\ B & -C \end{pmatrix} \quad \text{and} \quad P := \begin{pmatrix} P_A & 0 \\ 0 & P_C \end{pmatrix}
$$

where $P_A \in \mathbb{R}^{m \times m}_{\text{sym}}$ and $P_C \in \mathbb{R}^{n \times n}_{\text{sym}}$ are positive definite matrices. The eigenvalues of $P_A^{-1} (A + T)$ are denoted by

$$
0 < \eta_1 \leq \eta_2 \leq \ldots \leq \eta_m
$$

and eigenvalues of $P_C^{-1} C$ by

$$
0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n.
$$

Then, the eigenvalues $\mu_n \leq \ldots \leq \mu_m$ of $P^{-1/2}A^{-1/2}$ lie in the union of intervals

$$
\left[ \frac{1}{2} (\lambda_n + \sqrt{\lambda_n^2 + 4 \eta_m \lambda_n \Theta^2}), -\lambda_1 \right] \\
\cup \left[ \frac{1}{2} \left(-\lambda_1 + \sqrt{\lambda_1^2 + 4 \eta_1 \lambda_1 \theta^2}\right) , \frac{1}{2} \left(-\lambda_1 - \Delta \eta_m \right) + \sqrt{(\lambda_1 + \Delta \eta_m)^2 + 4 \eta_m \lambda_n \Theta^2} \right]
$$
Proof. The proof uses techniques similar to that used in [30] and is given here for completeness. Here, additional matrices $H$ and $T$ occur and constants $\Delta$, $\theta$ and $\Theta$ are defined in differently. First, we derive an upper bound for the singular values of $P^{-1/2}_C B P^{-1/2}_A$, continue with estimates for the negative eigenvalues of $P^{-1/2}_A P^{-1/2}_T$, and conclude with bounds for the positive eigenvalues.

We define matrix $\bar{A}$ and its submatrices by

$$P^{-\frac{1}{2}}_A P^{-\frac{1}{2}}_T = \begin{pmatrix} A \frac{1}{2} (A + H) P^{-\frac{1}{2}}_A & P^{-\frac{1}{2}}_A B P^{-\frac{1}{2}}_C \\ P^{-\frac{1}{2}}_C B P^{-\frac{1}{2}}_A & -P^{-\frac{1}{2}}_C C P^{-\frac{1}{2}}_C \end{pmatrix} = \bar{A}$$

and order the singular values of $\bar{B}$ as $0 \leq \sigma_1 \leq \sigma_2 \ldots \leq \sigma_n$. First, we give an upper bound for the singular values of $\bar{B}$. For all $x \in \mathbb{R}^n$ we get with $\lambda_n$, $\eta_m$, and $\Theta$ defined by (2.1)

$$x^\top \bar{B}^\top \bar{B} x = x^\top P^{-\frac{1}{2}}_A B^\top P^{-\frac{1}{2}}_A B P^{-\frac{1}{2}}_C x \leq \lambda_n x^\top P^{-\frac{1}{2}}_A B P^{-\frac{1}{2}}_C x \leq \lambda_n \Theta^2 x^\top x.$$

This gives the upper bound

$$(2.3) \quad \sigma_n \leq \Theta \sqrt{\lambda_n \eta_m}.$$ 

Let $\mu$ be an eigenvalue of $\bar{A}$. Then, there is a vector $(x, y) \in \mathbb{R}^{m+n} \setminus \{0\}$ with

$$(2.4) \quad (\bar{A} + \bar{H}) x + \bar{B}^\top y = \mu x, \quad \bar{B} x - \bar{C} y = \mu y,$$

which follows from the following consideration. If $\mu > 0$, then $x \neq 0$, since otherwise (2.5) implies $\mu = 0$, as $\bar{C}$ is positive definite. If $\mu < 0$, then $y \neq 0$, since otherwise (2.4) implies $x = 0$, as $\bar{A} + \bar{H}$ is positive semidefinite.

For $\mu < 0$, $I - \mu^{-1}(\bar{A} + \bar{H})$ is invertible. We take the scalar product of (2.5) with $\mu y$ and substitute $x$ from (2.4), i.e., $x = \mu^{-1}(I - \frac{1}{\mu}(\bar{A} + \bar{H}))^{-1}\bar{B}^\top y$. This gives

$$\mu^2 y^\top y = y^\top \bar{B} (I - \mu^{-1}(\bar{A} + \bar{H}))^{-1} \bar{B}^\top y - \mu y^\top \bar{C} y \leq y^\top \bar{B} \bar{B}^\top y - \mu y^\top \bar{C} y,$$

since all eigenvalues of $(I - \frac{1}{\mu}(\bar{A} + \bar{H}))^{-1}$ are less or equal than 1 for $\mu < 0$. We obtain

$$0 \geq \mu^2 y^\top y + \mu y^\top \bar{C} y - y^\top \bar{B} \bar{B}^\top y \geq \mu^2 y^\top y + \mu\lambda_n y^\top y - \sigma_n^2 y^\top y = \left\{\mu + \lambda_n/2\right\} y^\top y.$$

Hence, we have

$$-\frac{1}{2} \left(\lambda_n + \sqrt{\lambda_n^2 + 4\sigma_n^2}\right) \leq \mu \leq -\frac{1}{2} \left(\lambda_n - \sqrt{\lambda_n^2 + 4\sigma_n^2}\right).$$

Together with (2.3) we get a lower bound for negative eigenvalues of $P^{-1} A$.

$$\mu_{-n} \geq -\frac{1}{2} \left(\lambda_n + \sqrt{\lambda_n^2 + 4\sigma_n^2}\right) \geq -\frac{1}{2} \left(\lambda_n + \sqrt{\lambda_n^2 + 4\lambda_n \eta_n \Theta^2}\right).$$

Let $\mu < 0$. Taking the scalar product of (2.4) with $x$ and the scalar product of (2.5) with $y$ and subtracting gives

$$x^\top (\bar{A} + \bar{H}) x + y^\top \bar{C} y = \mu x^\top x - \mu y^\top y \geq \lambda_1 y^\top y.$$ 

Since $\mu$ was assumed to be negative, as mentioned above $y^\top y$ is positive and we deduce from the last inequality an upper bound for the negative eigenvalues

$$0 \geq \mu x^\top x \geq (\lambda_1 + \mu) y^\top y \Rightarrow \mu_{-1} \leq -\lambda_1.$$ 

Next we prove a lower bound for the positive eigenvalues of $\bar{A}$. For $\mu > 0$, $\bar{C} + \mu I$ is invertible. Substituting $y$ from (2.5), i.e., $y = (\bar{C} + \mu I)^{-1} \bar{B} x$, into the scalar product of (2.4) and $x$, gives

$$\mu x^\top x = x^\top (\bar{A} + \bar{H}) x + x^\top \bar{B} (\bar{C} + \mu I)^{-1} \bar{B} x$$

$$= x^\top (\bar{A} + \bar{H}) x + x^\top \bar{B} (\bar{C} + \mu I)^{-1} \bar{B} x + x^\top \bar{B} \bar{B}^\top (\bar{C} + \mu I)^{-1} \bar{B} x.$$
Since for the eigenvalues of \((I + \mu C^{-1})^{-1}\) there holds
\[
0 < \left(1 + \frac{\mu}{\lambda_1}\right)^{-1} \leq \left(1 + \frac{\mu}{\lambda_2}\right)^{-1} \leq \cdots \leq \left(1 + \frac{\mu}{\lambda_n}\right)^{-1} \leq 1
\]
we have
\[
\mu \mathbf{x}^\top \mathbf{x} \geq \mathbf{x}^\top (\mathbf{A} + H) \mathbf{x} + (1 + \mu/\lambda_1)^{-1} \mathbf{x}^\top B^\top C^{-1} B \mathbf{x} \\
\geq \left(1 + \frac{\mu}{\lambda_1}\right)^{-1} \left(\mathbf{x}^\top (\mathbf{A} + H) \mathbf{x} + \mathbf{x}^\top B^\top C^{-1} B \mathbf{x}\right).
\]
Rewriting this inequality in terms of the blocks of the original unpreconditioned matrix \(\mathbf{A}\) gives
\[
(1 + \mu/\lambda_1)^{-1} \mathbf{x}^\top P_A^{-\frac{1}{2}} \left(\mathbf{A} + H + B^\top C^{-1} B\right) P_A^{-\frac{1}{2}} \mathbf{x} \leq \mu \mathbf{x}^\top \mathbf{x}.
\]
We employ the definition (2.2) of \(\theta\) to obtain
\[
\theta^2 (1 + \mu/\lambda_1)^{-1} \mathbf{x}^\top P_A^{-\frac{1}{2}} (\mathbf{A} + \mathbf{T}) P_A^{-\frac{1}{2}} \mathbf{x} \leq \mu \mathbf{x}^\top \mathbf{x}.
\]
This gives with the definition of \(\eta_1\)
\[
\eta_1 \theta^2 (1 + \mu/\lambda_1)^{-1} \mathbf{x}^\top \mathbf{x} \leq \mu \mathbf{x}^\top \mathbf{x}.
\]
We assumed \(\mu > 0\). Hence, we have \(\mathbf{x} \neq 0\) and obtain
\[
0 \leq \mu^2 + \lambda_1 \mu - \lambda_1 \eta_1 \theta^2 = \left(\mu + \frac{\lambda_1}{2}\right)^2 - \left(\frac{\lambda_1^2 + 4\lambda_1 \eta_1 \theta^2}{4}\right)
\]
which yields \(\mu \geq \frac{1}{2}(-\lambda_1 + \sqrt{\lambda_1^2 + 4\lambda_1 \eta_1 \theta^2})\). Therefore, the result for the lower bound of the positive eigenvalues is proven.

We take now the scalar product of \(\mathbf{x}\) and (2.4), substitute \(\mathbf{y}\) from (2.5), and use \(\Delta\) defined by (2.1), to deduce
\[
\mu \mathbf{x}^\top \mathbf{x} = \mathbf{x}^\top (\mathbf{A} + H) \mathbf{x} + \mathbf{x}^\top B^\top (\mathbf{C} + \mu \mathbf{I})^{-1} B \mathbf{x} \leq \Delta \eta_m \mathbf{x}^\top \mathbf{x} + \mathbf{x}^\top B^\top (\mathbf{C} + \mu \mathbf{I})^{-1} B \mathbf{x}
\]
\[
\leq \Delta \eta_m \mathbf{x}^\top \mathbf{x} + (\lambda_1 + \mu)^{-1} \mathbf{x}^\top B^\top B \mathbf{x} \leq (\Delta \eta_m + (\lambda_1 + \mu)^{-1} \sigma_n^2) \mathbf{x}^\top \mathbf{x}
\]
from which we obtain
\[
0 \geq \mu^2 + (\lambda_1 - \Delta \eta_m) \mu - \lambda_1 \Delta \eta_m - \sigma_n^2 = \left(\mu + \frac{\lambda_1 - \Delta \eta_m}{2}\right)^2 - \left(\frac{\lambda_1 + \Delta \eta_m)^2 + 4\sigma_n^2}{4}\right)
\]
for \(\mathbf{x} \neq 0\) in this case. From inequality (2.3) with considering different cases we obtain an upper bound for the positive eigenvalues. \(\square\)

3. Model problem

In this section we present the interface problem and we rewrite it equivalently, using boundary integral operators. We discretize the resulting system by FEM/BEM coupling, which leads to linear systems that will be solved by several efficient methods as described below.

Let \(\Omega\) be a bounded Lipschitz-domain and \(\Omega_c = \mathbb{R}^2 \setminus \overline{\Omega}\) be its complement. The partial differential equation to be considered in \(\Omega\) will involve \(D = (d_{ij})_{i,j=1,2}\). Let the coefficients \(d_{ij} = d_{ji} \in L^\infty(\Omega)\) be uniformly bounded in \(\Omega\), i.e.,
\[
(3.1) \quad \exists d_0 > 0: \quad \sum_{i,j=1}^2 d_{ij}(\mathbf{x}) \xi_i \xi_j \leq d_0 \sum_{i=1}^2 \xi_i^2 \quad (\mathbf{x} \in \mathbb{R}^2, \ \xi_1, \xi_2 \in \mathbb{R})
\]
and positive definite, i.e.,
\[
(3.2) \quad \exists d_1 > 0: \quad \sum_{i,j=1}^2 d_{ij}(\mathbf{x}) \xi_i \xi_j \geq d_1 \sum_{i=1}^2 \xi_i^2 \quad (\mathbf{x} \in \mathbb{R}^2, \ \xi_1, \xi_2 \in \mathbb{R}).
\]
We consider the following problem involving the prescribed jumps \(u_0, t_0\) across the interface \(\Gamma := \partial \Omega:\)

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Interface Problem: Given \( f \in H^{-1}(\Omega) \), \( u_0 \in H^{1/2}(\Omega) \) and \( t_0 \in H^{1/2}(\Gamma) \). Find \( u_1 \in H^1(\Omega) \), \( u_2 \in H^1_{loc}(\Omega_c) \), such that
\[
\text{div} (D \cdot \text{grad} u_1) + f = 0 \quad \text{in} \ \Omega,
\]
\[
\text{div} (\text{grad} u_2) = 0 \quad \text{in} \ \Omega_c, \quad \text{and}
\]
\[
u_1 = u_2 + u_0, \quad n \cdot (D \cdot \text{grad} u_1) = \frac{\partial u_2}{\partial n} + t_0 \quad \text{on} \ \Gamma,
\]
where \( n \) is the normal on \( \Gamma \) pointing from \( \Omega \) into \( \Omega_c \) and the regularity condition on \( u_2 \) at infinity,
\[
u_2(x) = a + b \log |x| + o(1) \quad \text{for} \ |x| \to \infty,
\]
with \( a, b \in \mathbb{R} \).

Let \( H^s(\Omega) \) denote the usual Sobolev spaces \([26]\) with the trace spaces \( H^{s-1/2}(\Gamma) \) \((s \in \mathbb{R})\) for a bounded Lipschitz domain \( \Omega \) with boundary \( \Gamma \). Let \( \| \cdot \|_{H^k(\omega)} \) and \( | \cdot |_{H^k(\omega)} \) denote the norm and semi-norm in \( H^k(\omega) \) for \( \omega \subseteq \Omega \) and an integer \( k \). Recall that \( \langle \cdot , \cdot \rangle \) denotes the \( L^2(\Omega) \)-scalar product while \( \langle \cdot , \cdot \rangle \) denotes duality between \( H^s(\Gamma) \) and \( H^{-s}(\Gamma) \) (defined by extending the scalar product in \( L^2(\Gamma) \)). Given \( v \in H^{1/2}(\Gamma) \) and \( \phi \in H^{-1/2}(\Gamma) \), the boundary integral operators which we will use in the following are defined, for \( z \in \Gamma \), by
\[
(V\phi)(z) := \frac{1}{\pi} \int_{\Gamma} \phi(\zeta) \log |z - \zeta| \, ds_{\zeta},
\]
\[
(Kv)(z) := \frac{1}{\pi} \int_{\Gamma} v(\zeta) \frac{\partial}{\partial n_{\zeta}} \log |z - \zeta| \, ds_{\zeta},
\]
\[
(K^*\phi)(z) := \frac{1}{\pi} \int_{\Gamma} \phi(\zeta) \frac{\partial}{\partial n_{\zeta}} \log |z - \zeta| \, ds_{\zeta},
\]
\[
(Wv)(z) := \frac{1}{\pi} \frac{\partial}{\partial n_z} \int_{\Gamma} v(\zeta) \frac{\partial}{\partial n_{\zeta}} \log |z - \zeta| \, ds_{\zeta}.
\]

The linear and boundary integral operators are continuous when mapping between the following Sobolev–spaces
\[
V: H^{s-1/2}(\Gamma) \to H^{s+1/2}(\Gamma), \quad K: H^{s+1/2}(\Gamma) \to H^{s+1/2}(\Gamma),
\]
\[
K^*: H^{s-1/2}(\Gamma) \to H^{s+1/2}(\Gamma), \quad W: H^{s+1/2}(\Gamma) \to H^{s+1/2}(\Gamma),
\]
where \( s \in [-1/2, 1/2] \) \([13]\). The single layer potential \( V \) and the hyper singular operator \( W \) are symmetric, the double layer potential \( K \) has the dual \( K^* \). Both, \( V \) and \( W \) are strongly elliptic in the sense that they satisfy a Gårding inequality (in the above spaces with \( s = 0 \)) \([13]\).

There are various of formulae which characterise the Cauchy data \((u_2, \partial u_2/\partial n)|_{\Gamma}\) of a function \( u_2 \) with \((3.4), (3.6)\) and we quote only one from the literature.

**Lemma 1** \(([15])\). Let \( u_2 \in H^1_{loc}(\Omega_c) \) satisfy \((3.4)\) and \((3.6)\), then \((\xi, \phi) := (u_2, \partial u_2/\partial n)|_{\Gamma} \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)\) satisfies
\[
2 \begin{pmatrix} \xi \\ \phi \end{pmatrix} = \begin{pmatrix} I + K & -V \\ -W & I - K^* \end{pmatrix} \begin{pmatrix} \xi \\ \phi \end{pmatrix} + \begin{pmatrix} 2a \\ 0 \end{pmatrix}.
\]

Conversely, for each \((\xi, \phi) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)\) there exists a function \( u_2 \in H^1_{loc}(\Omega_c) \) with \((3.4), (3.6)\) if and only if \((3.7)\) holds. The function \( u_2 \) is given by the representation formula, for \( x \in \Omega_c\),
\[
u_2(x) = \frac{1}{2\pi} \int_{\Gamma} \phi(z) \log |x - z| \, ds_z - \frac{1}{2\pi} \int_{\Gamma} \xi(z) \frac{\partial}{\partial n_z} \log |x - z| \, ds_z + a.
\]
Notice that \( W 1 = 0 = (K + 1) 1 \) (proved by \((3.7)\) for \((\xi, \phi) = (1, 0)\) and \( a = 1 \)).

The interface problem \((3.3)-(3.6)\) is equivalent to the following weak formulation which is known as symmetric coupling method \([13]\).

**Weak Formulation:** Find \( (u, \phi) \in H^1(\Omega) \times H^{-1/2}(\Gamma) \), such that
\[
B((u, \phi), (v, \psi)) = L(v) \quad \left( (v, \psi) \in H^1(\Omega) \times H^{-1/2}(\Gamma) \right)
\]
with

\[ B((\phi, \psi), (\phi', \psi')) = 2 \cdot \int_{\Omega} (\nabla u)^\top \cdot D \cdot \nabla v dx + \langle Wu + (K^* - 1)\phi, v \rangle + \langle \psi, (K - 1)u - V\phi \rangle \]

and

\[ L((\phi, \psi)) = 2 \cdot \int_{\Omega} f v dx + (Wu_0 + (K^* + 1)t_0, v) + \langle \psi, (K - 1)u_0 - Vt_0 \rangle. \]

The problem (3.3)-(3.6) has a unique solution and so the equivalent problem (3.9) has a unique solution as well.

Notice that the variable \( u_2 \) is determined by (3.9) up to the additive constant \( a \) defined by (3.6), e.g. \( u_1 = u + a, u_2|_\Gamma = u|_\Gamma - u_0 + a \) with \((u_1, u_2)\) satisfying (3.3)-(3.6) and \((u, \phi)\) satisfying (3.9).

4. The Discrete Problem

In this section we consider the discretization of problem (3.9). Therefore we define finite dimensional vector spaces \( H_h \subset H^1(\Omega), H_h^{-1/2} \subset H^{-1/2}(\Gamma) \). Let \( H_h \) be the space of continuous piecewise linear functions of a quasi-uniform and regular triangulation of \( \Omega \) and \( H_h^{-1/2} \) be the space of piecewise constant functions on the discretization of the boundary \( \Gamma \) induced by \( H_h \). Let \( \eta_i, i = 1, \ldots, m \) be basis functions of \( H_h \), i.e., span \( \{\eta_i\} = H_h \) and \( t_i, i = 1, \ldots, n \) be basis functions of \( H_h^{-1/2} \), i.e., span \( \{t_i\} = H_h^{-1/2} \). Furthermore, let \( u = (u_1, \ldots, u_m)^\top \in \mathbb{R}^m, \phi = (\phi_1, \ldots, \phi_n)^\top \in \mathbb{R}^n \) be the vectors with \( u = \sum_{i=1}^m u_i \eta_i \in H_h \) and \( \phi = \sum_{i=1}^n \phi_i t_i \in H_h^{-1/2} \). We define the discretization of the operators \( W, K, V \), etc.

\[
A_h := \{2 \int_{\Omega} (\nabla \eta_i)^\top D \nabla \eta_j d\Omega\}_{i,j=1}^{m,n}, \quad W_h := \{(W \eta_i|_\Gamma, \eta_j|_\Gamma)\}_{i,j=1}^{m,n}, \quad V_h := \{(V t_i, t_j)\}_{i,j=1}^{n,n}.
\]

Hence, we have (with \( \approx \) denoting equivalence of norms)

\[
2 \int_{\Omega} (\nabla u)^\top D \nabla v d\Omega = u^\top A_h v, \quad 2 \| D \frac{1}{2} \nabla u \|^2_{L^2(\Omega)} \approx u^\top A_h u \quad (u, v \in H_h),
\]

\[
\langle W u, v \rangle = u^\top W_h v, \quad \| u \|^2_{H^1(\Gamma) \setminus \mathbb{R}} \approx u^\top W_h u \quad (u, v \in H_h),
\]

\[
\langle \phi, V \psi \rangle = \phi^\top V_h \psi, \quad \| \phi \|^2_{H^{-1/2} \setminus \Gamma} \approx \phi^\top V_h \phi \quad (\phi, \psi \in H_h^{-1/2}).
\]

5. Block–Diagonal Preconditioner

In the following we consider linear systems of equations with symmetric matrices of the form

\[
A_h = \begin{pmatrix}
A_h & W_h \\
K_h & I_h & V_h
\end{pmatrix}.
\]

Given the preconditioner

\[
P_h = \begin{pmatrix}
P_{A_h} & 0 \\
0 & P_{V_h}
\end{pmatrix},
\]

where \( P_{A_h} \) and \( P_{V_h} \) are both symmetric and positive definite. Consequent, \( P_h \) is a symmetric and positive definite matrix, too. In [12, Thm. 3.2] it was proven that if the eigenvalues \( \mu_i \) of a preconditioned matrix \( P_h^{-1/2} A_h P_h^{-1/2} \) lie in intervals of the form

\[
[-\hat{a}, -\hat{b}] \cup [\hat{c}, \hat{d}]
\]

\[6\]
with \( \hat{a} - \hat{b} = \hat{d} - \hat{c} > 0 \), where \( \hat{a}, \hat{b}, \hat{c} \) and \( \hat{d} \) are positive constants, then the PCR convergence rate is

\[
(5.4) \quad \left( \frac{\| P_h^{-1/2} (b - A_h x_k) \|_2}{\| P_h^{-1/2} (b - A_h x_0) \|_2} \right)^2 \leq 2 \left( 1 - \frac{\sqrt{b\hat{c}/\hat{a}\hat{d}}}{1 + \sqrt{b\hat{c}/\hat{a}\hat{d}}} \right)^k
\]

where \( \| \cdot \|_2 \) denotes the Euclidean norm and \( x_k \) is the \( k^{th} \) iterate.

In the following we will use Theorem 1 to estimate the extreme eigenvalues of the preconditioned matrix \( \bar{A}_h \)

\[
(5.5) \quad \bar{A}_h := P_h^{-1/2} A_h P_h^{-1/2} = \begin{pmatrix}
P^{-1/2}_{A_h} (A_h + W_h) P^{-1/2}_{A_h} & P^{-1/2}_{A_h} (K_h - I_h) P^{-1/2}_{V_h} \\
-P^{-1/2}_{V_h} V_h^{-1} P^{-1/2}_{A_h} & P^{-1/2}_{V_h} \end{pmatrix},
\]

in order to estimate the convergence of the PCR-method.

We obtain from Sylvester’s law of inertia together with the congruent transform (1.3) of \( A_h \) that the matrix \( A_h \) has \( m \) positive and \( n \) negative eigenvalues. We get the same result for \( \bar{A}_h \), if we apply this law again to the transform (5.5). We denote the eigenvalues of \( \bar{A}_h \) by

\[
\mu_{-n} \leq \mu_{-n+1} \leq \ldots \leq \mu_{1} < 0 < \mu_{2} \leq \ldots \leq \mu_{m}.
\]

We define bounds of the spectrum of \( P^{-1/2}_{V_h} V_h^{-1/2} \) which may depend on the mesh size by

\[
(5.6) \quad 0 < \lambda_{\min}(h) \leq \frac{x^\top V_h x}{x^\top P_{A_h} x} \leq \lambda_{\max}(h) \quad (x \in \mathbb{R}^n \setminus \{0\}).
\]

Since \( A_h \) is positive semi-definite we add a symmetric matrix \( T_h \), s.t. \( A_h + T_h \) is positive definite. In our model problem we can choose for example \( T_h = W_h + \gamma D_h^T D_h^{-1} I_d \) where \( \gamma > 0 \) which \( D_h := \{t_i, t_j\}_{i,j=1,...,n} \) denotes a diagonal matrix, or let \( T_h = M_h := \{ \int_{\Omega} \eta \eta_j d\Omega \}_{i,j=m} \) be the mass matrix. Since \( P_{A_h} \) and \( A_h \) are both symmetric and positive definite, we can define constants depending on the mesh size \( h = h(n) \) to bound the extreme eigenvalues

\[
(5.7) \quad 0 < \eta_{\min}(h) \leq \frac{x^\top (A_h + T_h)x}{x^\top P_{A_h} x} \leq \eta_{\max}(h) \quad (x \in \mathbb{R}^m \setminus \{0\}).
\]

For our assertion in the following Theorem 2 we have to show, that there exist some constants \( \theta, \Theta, \) and \( \Delta \), which are independent of the mesh size \( h \), satisfying for all \( x \in \mathbb{R}^n \setminus \{0\} \)

\[
(5.8) \quad \frac{x^\top (A_h + W_h)x}{x^\top (A_h + T_h)x} \leq \Delta, \quad \frac{x^\top \{ (K_h^T - I_h^T) V_h^{-1} (K_h - I_h) \} x}{x^\top (A_h + T_h)x} \leq \Theta^2 , \quad \text{and}
\]

\[
(5.9) \quad \theta^2 \leq \frac{x^\top \{ A_h + W_h + (K_h^T - I_h^T) V_h^{-1} (K_h - I_h) \} x}{x^\top (A_h + T_h)x}.
\]

In the following we motivate and prove the existence of these constants for both before mentioned choices of \( T_h \), i.e. \( T_h = M_h \) and \( T_h = W_h + \gamma D_h^T D_h^{-1} I_d \).

5.1. Existence of \( \theta, \Theta, \) and \( \Delta \) for choice \( T_h = M_h \).

In the sequel \( c_i \) (\( i = 1, 2, \ldots \)) will denote positive constants which are independent of mesh size \( h \). First we show the existence a positive constant \( \Delta \) satisfying (5.8) independent of the mesh size \( h \). Such an estimate can be obtained directly by estimating the numerator and denominator of (5.8) as follows: For all \( u \in H_h \) there holds with \( d_0 \) from (3.1)

\[
u^\top A_h u = \frac{1}{2} \| D^{1/2} \nabla u \|_{L^2(\Omega)}^2 \leq \frac{d_0}{2} \| D^{1/2} \nabla u \|_{L^2(\Omega)}^2 \quad \text{and} \quad \nu^\top W_h u \approx \| u \|_{H^{1/2}(\Omega)/\mathbb{R}}^2.
\]

This yields with a trace mapping that there exists a positive constant \( c_1 \) independent of mesh size \( h \) s.t.

\[
u^\top (A_h + W_h) u \leq c_1 \| u \|_{H^1(\Omega)}^2.
\]
Furthermore with the choice $T_h = M_h$ there exists a positive constant $c_2$ independent of $h$ satisfying 
(5.10) \[ u^\top (A_h + M_h) u \geq c_2 \| u \|^2_{H^1(\Omega)}. \]
Altogether this shows $\Delta \leq c_1/c_2$. Assuming that the capacity of the boundary $\Gamma$ is smaller than one, the single layer potential is positive definite on $H^{-1/2}(\Gamma)$ (see, e.g. [13, 15]). Next we employ the facts, that $V$ is bijective from $H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ under the assumption that $\text{cap}(\Gamma) < 1$ and that $(K - I)$ is a continuous mapping from $H^{1/2}(\Gamma)$ onto itself and the trace operator from $H^1(\Omega) \to H^{1/2}(\Gamma)$ is continuous. Furthermore, for $u \in H_h$ there holds with positive and mesh independent constant $c_3$, $c_4$, and $c_5$
(5.11) \[ u^\top (K_h^\top - I_h^\top) V_h^{-1}(K_h - I_h) u \leq c_3 \| (K - I) u \|^2_{H^{1/2}(\Gamma)} \leq c_4 \| u \|^2_{H^{1/2}(\Gamma)} \leq c_5 \| u \|^2_{H^1(\Omega)}. \]
This yields together with (5.10), that there exists a constant $\Theta \in \mathbb{R}$ independent of $h$ with 
(5.12) \[ \frac{x^\top \{(K_h^\top - I_h^\top) V_h^{-1}(K_h - I_h)\} x}{x^\top (A_h + M_h) x} \leq \Theta^2 \quad (x \in \mathbb{R}^n \setminus \{0\}). \]
In [13] it is proven, that we can define an extension operator $T$, with the following properties. Given $v \in H^{1/2}(\Gamma)$, then there exists a $w := Tv$ with $w = v$ on $\Gamma$ and 
(5.13) \[ \| w \|_{H^1(\Omega)} \leq c_6 \| v \|_{H^{1/2}(\Gamma)}. \]
Now we have for all $u \in H^1(\Omega)$, if we use the first Friedrich’s inequality and (5.13) 
(5.14) \[ \| u \|_{H^1(\Omega)} \leq \| u - Tu|_\Gamma \|^2_{H^{1/2}(\Gamma)} + \| Tu|_\Gamma \|^2_{H^{1/2}(\Omega)} \leq c_7 \| u - Tu|_\Gamma \|^2_{H^{1/2}(\Gamma)} + \| Tu|_\Gamma \|^2_{H^{1/2}(\Omega)} \]
\[ \leq c_7 \| u^2_{H^1(\Omega)} + 3 \| Tu|_\Gamma \|^2_{H^{1/2}(\Omega)} \leq c_8 \left( \| u^2_{H^1(\Omega)} + \| u \|^2_{H^{1/2}(\Gamma)} \right), \]
with $c_8 = \max\{c_7, 3 c_4^2\}$. It is known, that the operator $W + (K^\top - 1)V^{-1}(K - 1) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is elliptic on $H^{1/2}(\Gamma)$ [11]. Hence we have together with (5.14) for all $u \in H_h$ the inequality 
(5.15) \[ \| u \|^2_{H^1(\Omega)} \leq c_8 \left( \| u^2_{H^1(\Omega)} + \| u \|^2_{H^{1/2}(\Gamma)} \right) \leq c_9 u^\top \{ A_h + W_h + (K_h^\top - I_h^\top) V_h^{-1}(K_h - I_h)\} u. \]
From the equivalence of $\| u \|^2_{H^1(\Omega)}$ and $u^\top (A_h + M_h) u$ for $u \in H_h$ we conclude the existence of $\theta \in \mathbb{R}$ with 
(5.16) \[ \theta^2 \leq \frac{x^\top \{ A_h + W_h + (K_h^\top - 1)V_h^{-1}(K_h - 1)\} x}{x^\top (A_h + M_h) x} \quad (x \in \mathbb{R}^n \setminus \{0\}) \]
which is independent of $h$.

5.2. Alternative choice $T_h = W_h + \gamma Id_h^\top D_h^{-1} Id_h$. Let $\gamma > 0$ and $D_h = \{< t_i, t_j \rangle \}_{i,j=1,...,n}$ denote a diagonal matrix. Before we show the existence of constants $\Delta$, $\theta_1$, and $\Theta$ defined by (2.1) and (2.2) for $T_h = W_h + \gamma Id_h^\top D_h^{-1} Id_h$, we make a few preparations. Let $\Gamma_i \subset \Gamma$ denote the support of $t_i$ ($i = 1, \ldots, n$), $h_i$ its diameter, and $h_{\text{max}} := \max_{i=1,...,n} \{h_i\}$. For the integral mean $u_m := < u, 1 > / |\Gamma|$ of $u \in L^2(\Gamma)$ we get by straightforward calculation 
(5.17) \[ \| u \|^2_{L^2(\Gamma)} = \| u - u_m \|^2_{L^2(\Gamma)} + < u, 1 >^2 / |\Gamma|. \]
Using Hölder’s-inequality and the definitions of $I_h$ and $D_h$ we deduce 
(5.18) \[ < u, 1 >^2 = \sum_{i=1}^n h_i^{1/2} h_i^{-1/2} \left( \int_{\Gamma_i} u \, dx \right)^2 \leq |\Gamma| \sum_{i=1}^n h_i^{-1} \left( \int_{\Gamma_i} u \, dx \right)^2 \leq |\Gamma| \, u^\top I_h^\top D_h^{-1} I_h u. \]
By using the two last results (5.17) and (5.18) we get for $u_m \in H_h \setminus \{0\} \subset H^{1/2}(\Gamma)$ and using a standard approximation property $\| u - u_m \|^2_{L^2(\Gamma)} \leq c_9 h_{\text{max}}^2 \| u \|^2_{H^{1/2}(\Gamma)} \leq c_{10} h_{\text{max}}^2 u^\top W_h u$
(5.19) \[ \| u \|^2_{L^2(\Gamma)} = \| u - u_m \|^2_{L^2(\Gamma)} + < u, 1 >^2 / |\Gamma| \leq c_{10} h_{\text{max}} u^\top W_h u + u^\top I_h^\top D_h^{-1} I_h u. \]
Using this result and the compactness property of $K$ as in (5.11) we obtain
\begin{equation}
(5.20) \quad \mathbf{u}^\top (K_h^\top - I)V_h^{-1}(K_h - I)\mathbf{u} \leq c_{11}\|(K - I)\mathbf{u}\|^2_{H^{1/2}(\Gamma)} \leq c_{12}\|\mathbf{u}\|^2_{H^{1/2}(\Gamma)} \\
\leq c_{13}\|\mathbf{u}\|^2_{H^{1/2}(\Gamma)} + \|\mathbf{u}\|^2_{\mathbb{R}^m} \leq c_{14}(1 + c_{10}h_{\text{max}})\mathbf{u}^\top W_h\mathbf{u} + c_{13}\mathbf{u}^\top I_h^T D_h^{-1}I_h\mathbf{u}.
\end{equation}
That proves the existence of a positive constant $\Delta < \max\{c_{13}, c_{14}(1 + c_{10} |\Gamma|)\}$ which is bounded independent of $h$. Since $\gamma I_h^T D_h^{-1}I_h$ is positive semidefinite, we obtain
\[ x^\top (A_h + W_h)x \leq x^\top (A_h + W_h + \gamma I_h^T D_h^{-1}I_h)x \quad (x \in \mathbb{R}^m). \]

Hence, we can choose $\Delta = 1$ in (5.8) independently of the mesh size $h$. Taking into account that $I_h^T D_h^{-1}I_h$ can be rewritten as sum of local quantities as in (5.18), we get by using Hölder’s inequality
\[ \mathbf{u}^\top (I_h^T D_h^{-1}I_h)\mathbf{u} = \sum_{i=1}^n h_i^{-1} \left( \int_{\Gamma_i} u dx \right)^2 \leq \sum_{i=1}^n \int_{\Gamma_i} u^2 dx = \mathbf{u}^\top M_h \mathbf{u} \]
Hence, we obtain for $u \in H_h$
\[ \mathbf{u}^\top (A_h + W_h + \gamma I_h^T D_h^{-1}I_h)\mathbf{u} \leq \mathbf{u}^\top (A_h + W_h + \gamma M_h)\mathbf{u} \leq c_{15} \max\{1, \gamma\}\|\mathbf{u}\|_{H^1(\Omega)}. \]
Together with (5.15) this proves, that there exists a mesh size independent constant $\theta^2 \geq c_9 c_{15} \max\{1, \gamma\} > 0$.

What will be a good choice of $\gamma$? Let $1$ denote a generic 1-vector, i.e. $1 = (1, \ldots, 1)^\top$. Since $\text{kern}\{A_h + W_h\} = \{1\}$, we expect
\[ \gamma \approx \frac{1^\top (K_h^\top - I_h)V_h^{-1}(K_h - I_h)1}{1^\top I_h^T D_h^{-1}I_h 1} \]
to be a good choice. Note, from the representation formula (3.7) we get $K1 = -1$ and numerical observations show $(1^\top I_h V_h^{-1}I_h 1)/(1^\top I_h D_h^{-1}I_h 1) \approx (1^\top D_h 1)/(1^\top V_h 1) = |\Gamma|/(1^\top V_h 1)$ (see Section 7 below for numerical experiments). Therefore, we get
\[ 1^\top (K_h^\top - I_h)V_h^{-1}(K_h - I_h)1 = 4 \cdot 1^\top I_h V_h^{-1}I_h 1 \approx \frac{4|\Gamma|}{1^\top V_h 1} 1^\top I_h D_h^{-1}I_h 1 \]
Hence, we end up with
\begin{equation}
(5.21) \quad \gamma = \frac{4|\Gamma|}{1^\top V_h 1} = \frac{-4\pi |\Gamma|}{\int_\Gamma \int_\Gamma \log |x - y| ds_x ds_y}.
\end{equation}
Note, that $1^\top V_h 1$ and also $|\Gamma|$ are independent of the discretisation.

From Subsections 5.1 and 5.2 we deduce the following lemma.

**Lemma 2.** Let $T_h$ be $M_h$ or $W_h + \gamma I_h D_h^{-1}I_h$ ($\gamma > 0$). Then, there exist positive constants $\Delta$, $\theta$, and $\Theta$ satisfying (5.8) and (5.9) independent of the mesh size $h$ and the discretisation of $\Gamma$.

If $\Delta$, $\theta$, and $\Theta$ are independent from the mesh size $h$, Theorem 1 gives bounds for the spectrum of $P_h^{-1/2}A_h P_h^{-1/2}$ depending only on the extreme eigenvalues of $P_h^{-1/2}A_h T_h$ (5.7) and $P_h^{-1/2}V_h$ (5.6). Using (5.4), also the rate of convergence of the PCR-method can be bounded by the extreme eigenvalues (5.6) and (5.7). With Lemma 2, we get from Theorem 1 the following result.

**Theorem 2.** Let $P_h$, $V_h$ be spectral equivalent to $A_h + T_h$ and $V_h$, respectively. Then, there exist independently of the mesh size $h$ positive constants $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, and $\tilde{d}$ such that the extreme eigenvalues of the preconditioned matrix $P_h^{-1/2}A_h P_h^{-1/2}$ lie in intervals of the form (5.3). Hence, the convergence rate of the PCR-method applied to $A_h$ with preconditioner $P_h$ is bounded independently of the mesh size $h$. 

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Remark 1. i.) In the situation that there holds only $\eta_{\text{max}} = 1$ and $\eta_{\text{min}}(h) \to 0$ as $h \to 0$, and $P_{V_h}$ is spectrally equivalent to $V_h$ the eigenvalues of the preconditioned matrix lie in the union of the intervals
\[
\left[ \frac{-1}{2} \left( \lambda_{\text{max}} + \sqrt{\lambda_{\text{max}}^2 + 4\lambda_{\text{max}}\Theta^2} \right), -\lambda_{\text{min}} \right]
\cup \left[ O(\sqrt{\eta_{\text{min}}(h)}), \frac{1}{2} \left( -\left( \lambda_{\text{min}} - \Delta \right) + \sqrt{(\lambda_{\text{min}} + \Delta)^2 + 4\lambda_{\text{max}}\Theta^2} \right) \right].
\]

This leads to a asymptotic convergence rate of the PCR method
\[
\lim_{k \to \infty} \frac{\| P_h^{-1/2}(b - Ax^k) \|_2}{\| P_h^{-1/2}(b - Ax^0) \|_2} = 1 - O(\sqrt{\lambda_{\text{min}}})
\]
where we used Lebedev’s results [25].

ii.) Using the assumptions on the preconditioners as mentioned in i. If we choose $P_{A_h}$ as the diagonal of $A_h$ we get $\eta_{\text{min}}(h) = h^2$. Hence $O(h^{-1})$ preconditioned PCR iterates are required for convergence.

iii. Numerical results ([17]) indicate that also the QMR, bi-CGstab, and GMRES methods [4] when applied the diagonal preconditioned matrix $P_h^{-1/2} A_h P_h^{-1/2}$ bounded number of iterations (independent of the meshsize $h$) to reduce the error to a given tolerance.

6. MULTIGRID ALGORITHM FOR SINGLE LAYER POTENTIAL $V$.

We assumed that there exists symmetric matrices $P_A$ and $P_C$, which are spectrally equivalent to the Schur complement $A + H + B^T C^{-1} B$ and $C$. In the following numerical examples we used multigrid algorithms. The $h$-independence of these methods is well known (e.g. see [6, 8, 9, 10, 22, 33]). There exists also alternative approaches as well for the finite element part as for the boundary element part. (See e.g. [5, 16, 19, 28, 31, 29] and the references therein.)

For completeness we present a multigrid $V$-cycle algorithm for single layer potential $V$ which will be used in the following numerical examples. Since $V$ is an pseudo-differential operator of order $-1$ the smoothing step has to be modified in comparison to standard multigrid algorithms. For the analysis we refer to [6] where the three dimensional case is considered.

We assume that a coarse triangulation $T_1$ of $\Omega$ is given and develop a sequence of nested triangulations of $\Omega$ in the usual way. Successively finer triangulations $T_k$ for $k > 1$ are defined by subdividing each triangle into four by connecting the midpoints of the edges. Each triangulation $T_k$ induces a triangulation $G_k$ of the boundary $\Gamma$ in an obvious way.

We let the number of levels in the multigrid algorithm be determined by $J \geq 1$ and define $\mathcal{M}_k$ ($k = 1, \ldots, J$) to be the space of functions which are piecewise constant with respect to the triangulation $G_k$. Since the triangulations $T_k$ are nested and consequently $G_k$, it follows that
\[
\mathcal{M}_1 \subset \mathcal{M}_2 \subset \ldots \subset \mathcal{M}_J.
\]

For $k = 1, \ldots, J$, let $t_i^k$, $i = 1, \ldots, n_k$ be basis functions of $\mathcal{M}_k$, $h_i^k = \text{diam}(\text{supp}(t_i^k))$, $h_{\text{max}}^k = \max_i h_i^k$, and the coarsening matrix
\[
P_k = \{p_{ij}^{k+1}\}_{i=1}^{n_{k+1}}, \quad p_{ij}^{k+1} \text{ be defined by } t_i^k = \sum_{j=1}^{n_{k+1}} p_{ij}^{k+1} t_j^{k+1}.
\]

The space $S_1^k$ is defined to be the space of functions which are affine between two successive midpoints with respect to the triangulation $G_k$ and continuous on $\Gamma$. Also, we define $S_2^k$ to be the space of splines of order three. A plot of the B-spline basis functions $\phi_i^{k,\mu}$ ($i = 1, \ldots, n_k$) for $S_2^k$ ($\mu = 1, 2$) is given in Fig. 1. Note, that the basis functions $\phi_i^{k,\mu}$ form a decomposition of unity.
In the following we neglect the upper index to keep the notation simple. For \( k = 1, \ldots, J \) we define
\[
L_k := \text{diag}\{h_1, \ldots, h_{n_k}\}, \quad D_{k,\mu} := \{(1 + \frac{\partial^2}{\partial s^2})\phi_{i,\mu}^k, \phi_{j,\mu}^k\}_{i,j = 1,n_k}
\]
\[
M_{k,\mu} := \{(t_{i,1}^k, \phi_{i,\mu}^k)\}_{i,j = 1,n_k}, \quad V_k := \{(Vt_{i,1}^k, t_{i,1}^k)\}_{i,j = 1,n_k}.
\]
Further, let \( \chi_k^* \) be an upper bound of
\[
\chi_k = \sup_{x \in \mathbb{R}^{n_k}} \frac{x^\top V_k x}{x^\top M_{k,\mu} D_{k,\mu} M_{k,\mu}^\top M_{k,\mu} x}.
\]

\[\text{Algorithm \( \text{Mg}^V_k \):} \] Matrix form of multigrid V-cycle algorithm for single layer potential with \( \nu \) pre and post smoothing steps.

Set \( \text{Mg}_1^V = V_1^{-1} \). Assume \( \text{Mg}_{k-1}^V \) has been defined and define \( \text{Mg}_k^V y \in \mathbb{R}^{n_k} \) as follows:

i) Set \( x^0 = 0 \) and define \( x^\ell \) for \( \ell = 1, \ldots, \nu \) by
\[
x^\ell = x^{\ell-1} + \frac{1}{\chi_k} M_{k,\mu}^{-\top} D_{k,\mu} M_{k,\mu}^{-1} (y - V_k x^{\ell-1}).
\]

ii) Define \( x^{\nu+1} = x^{\nu} + P_k^{-\top} q \) where
\[
q = \text{Mg}_{k-1}^V P_{k-1} (y - V_k x^\nu).
\]

iii) Finally, set \( \text{Mg}_k^V y = x^{2\nu+1} \) where \( x^\ell \) for \( \ell = \nu + 2, \ldots, 2\nu + 1 \) are defined by
\[
x^\ell = x^{\ell-1} + \frac{1}{\chi_k} M_{k,\mu}^{-\top} D_{k,\mu} M_{k,\mu}^{-1} (y - V_k x^{\ell-1}).
\]

**Remark 2.** i) The smoother \( \frac{1}{\chi_k} M_{k,\mu}^{-\top} D_{k,\mu} M_{k,\mu}^{-1} \) is symmetric and hence \( \text{Mg}^V \) is a symmetric operator.

ii) Note, that \( M_{k,\mu} \) is sparse and diagonal dominant, so an approximate inverse can be computed by the Jacobi iteration. Numerical examples in Section 7 indicate that only a few iterations will be needed to keep the contraction number of the multigrid method bounded. See Tab. 9 and Tab. 10 below where only \( 3 \) Jacobi iterations were used.

iii) The following numerical examples show, that we get also very good results if we choose \( \mu = 1 \) and substitute \( M_{k,\mu} \) just by the diagonal matrix \( L_k \).

7. Numerical Example

The following numerical experiments illustrates the sharpness of the bounds given by Theorem 2 in case of a FEM-BEM coupling problem (3.9), underlines the choices of \( \beta \) in \( T_h = \beta M_h \), resp. \( \gamma \) (5.21) in \( T_h = W_h + \gamma I_d h D_{h}^{-1} I_d h \), the smoother in \( \text{Mg}^V \), and highlights the quasi-optimal convergence rate of the PCR-method, independent of the mesh size \( h \) and mesh-structure.
Example 1: (L-shaped domain, uniform mesh).
Let $\Omega$ be the L-shaped domain with vertices $(0,0)$, $(s,0)$, $(s,s)$, $(-s,s)$, $(-s,-s)$, $(0,-s)$ with $s = 0.25$. Therefore the single layer potential operator $V$ is positive definite. On a shape-regular mesh with triangular elements we use piecewise linear functions in $\Omega$ and piecewise constant functions on $\square$ for discretization. The coarsest triangulation consists of six have squares with edge length $h_1 = 0.25$ as depicted in Fig. 2 (left). The implementation is realized in Matlab in the way like [1, 2, 18].

In Tab. 1 we give the extreme eigenvalues of the unpreconditioned matrix $A_h$ (5.1) and the resulting behavior of the bounds in terms of the mesh size $h$.

<table>
<thead>
<tr>
<th>$h/h_1$</th>
<th>$-ve$ ev’s min / max</th>
<th>$+ve$ ev’s min / max</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/1</td>
<td>-0.6194194 -0.014306039</td>
<td>0.4035585 9.280412</td>
</tr>
<tr>
<td>1/2</td>
<td>-0.2813471 -0.003824051</td>
<td>0.1689828 13.20866</td>
</tr>
<tr>
<td>1/4</td>
<td>-0.1216139 -0.000998319</td>
<td>0.0630067 15.92962</td>
</tr>
<tr>
<td>1/8</td>
<td>-0.0515328 -0.000255008</td>
<td>0.0214023 15.98179</td>
</tr>
<tr>
<td>1/16</td>
<td>-0.0219900 -0.000064265</td>
<td>0.0067335 15.98179</td>
</tr>
<tr>
<td>1/32</td>
<td>-0.0096163 -0.000016083</td>
<td>0.0019840 15.98179</td>
</tr>
</tbody>
</table>

As $P^{-1}_\lambda$ we choose a symmetric multigrid V-cycle with Gauss-Seidel smoother applied to the matrix $A_h + T_h$ and as $P^{-1}_\lambda$ we take the modified multigrid V-cycle algorithm $Mg^V$ as explained in Section 6 with $\chi^*_k = 2.5/2^k$ $(k = 1, \ldots, J)$ and $\frac{1}{L_k}D_{k,1}L_k^{-1}$ as smoother. In both cases we use one pre and post smoothing step on each level.

By solving generalized eigenvalue problems using the QZ algorithm [20] we approximated numerically the constants $\theta$, $\Theta$, $\Delta$, $\lambda_1$, $\eta_1$, and $\eta_m$. The values are given in Tab. 2 for $T_h = 50M_h$ and in Tab. 3 for $T_h = W_h + \gamma I_d_1^{-1} I_d_h$ (For $\gamma$ see (5.21)). All results underline for both choices of $T_h$ the $h$-independence of the constants $\theta$, $\Theta$, $\Delta$, $\lambda_1$, $\eta_1$, and $\eta_m$.

<table>
<thead>
<tr>
<th>$h/h_1$</th>
<th>$\lambda_1$</th>
<th>$\lambda_m$</th>
<th>$\eta_1$</th>
<th>$\eta_m$</th>
<th>$\Delta$</th>
<th>$\theta^2$</th>
<th>$\Theta^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/1</td>
<td>1.000000</td>
<td>1.000000</td>
<td>1.000000</td>
<td>1.000000</td>
<td>1.000000</td>
<td>1.000000</td>
<td>1.000000</td>
</tr>
<tr>
<td>1/2</td>
<td>0.94793</td>
<td>0.73407</td>
<td>1.04108</td>
<td>1.31705</td>
<td>2.21626</td>
<td>3.13444</td>
<td>3.8436</td>
</tr>
<tr>
<td>1/4</td>
<td>0.96172</td>
<td>0.75646</td>
<td>1.54046</td>
<td>1.59630</td>
<td>0.87648</td>
<td>4.2884</td>
<td>4.2579</td>
</tr>
<tr>
<td>1/8</td>
<td>0.96266</td>
<td>0.75627</td>
<td>1.63252</td>
<td>0.87492</td>
<td>2.47739</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/16</td>
<td>0.96275</td>
<td>0.75631</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/32</td>
<td>0.96276</td>
<td>0.75629</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In Tab. 4, resp. 5 we give the extreme eigenvalues of the preconditioned matrix $P^{-1}_h \cdot A_h$ and in brackets its estimates by Theorem 2 using values from Tab. 2, resp. 3. The bounds for the
Table 3. Constants $\theta$, $\Theta$, $\Delta$, $\eta_1$ and $\eta_m$ ($T_h = W_h + \gamma D_h \Gamma^{-1} D_h$)

<table>
<thead>
<tr>
<th>$h/h_0$</th>
<th>$\eta_{\min}$</th>
<th>$\eta_{\max}$</th>
<th>$\Delta$</th>
<th>$\theta^2$</th>
<th>$\Theta^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/1$</td>
<td>1.00000</td>
<td>1</td>
<td>1</td>
<td>0.84419</td>
<td>1.03581</td>
</tr>
<tr>
<td>$1/2$</td>
<td>0.80521</td>
<td>1</td>
<td>1</td>
<td>0.82932</td>
<td>1.05373</td>
</tr>
<tr>
<td>$1/4$</td>
<td>0.75072</td>
<td>1</td>
<td>1</td>
<td>0.82016</td>
<td>1.07151</td>
</tr>
<tr>
<td>$1/8$</td>
<td>0.71708</td>
<td>1</td>
<td>1</td>
<td>0.81680</td>
<td>1.08549</td>
</tr>
<tr>
<td>$1/16$</td>
<td>0.69343</td>
<td>1</td>
<td>1</td>
<td>0.81564</td>
<td>1.09695</td>
</tr>
<tr>
<td>$1/32$</td>
<td>0.68075</td>
<td>1</td>
<td>1</td>
<td>0.81526</td>
<td>1.11419</td>
</tr>
</tbody>
</table>

Negative eigenvalues are very close to the exact values (relative error between estimated and exact value < 5%).

Table 4. Extreme eigenvalues of the preconditioned matrix $P_h^{-1} \cdot A_h$ ($T_h = A_h + 50M_h$)

<table>
<thead>
<tr>
<th>$h/h_1$</th>
<th>-ve ev’s min / max</th>
<th>+ve ev’s min / max</th>
<th>bounds by Theorem 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/1$</td>
<td>-2.04598 (-2.07043)</td>
<td>-1.00000 (-1.00000)</td>
<td>0.88180 (0.63625)</td>
</tr>
<tr>
<td>$1/2$</td>
<td>-2.0471 (-2.10107)</td>
<td>-0.96621 (-0.96172)</td>
<td>0.79333 (0.52267)</td>
</tr>
<tr>
<td>$1/4$</td>
<td>-2.04718 (-2.13671)</td>
<td>-0.97127 (-0.96266)</td>
<td>0.70534 (0.42116)</td>
</tr>
<tr>
<td>$1/8$</td>
<td>-2.04754 (-2.14559)</td>
<td>-0.97275 (-0.96275)</td>
<td>0.68945 (0.39916)</td>
</tr>
<tr>
<td>$1/16$</td>
<td>-2.04777 (-2.15148)</td>
<td>-0.97286 (-0.96276)</td>
<td>0.67847 (0.39057)</td>
</tr>
</tbody>
</table>

Table 5. Extreme eigenvalues of the preconditioned matrix $P_h^{-1} \cdot A_h$ ($T_h = W_h + \gamma D_h \Gamma^{-1} D_h$)

<table>
<thead>
<tr>
<th>$h/h_1$</th>
<th>-ve ev’s min / max</th>
<th>+ve ev’s min / max</th>
<th>bounds by Theorem 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/1$</td>
<td>-1.63094 (-1.63393)</td>
<td>-1.00000 (-1.00000)</td>
<td>0.61307 (0.54603)</td>
</tr>
<tr>
<td>$1/2$</td>
<td>-1.60470 (-1.64180)</td>
<td>-0.95174 (-0.94793)</td>
<td>0.58339 (0.45212)</td>
</tr>
<tr>
<td>$1/4$</td>
<td>-1.59517 (-1.64956)</td>
<td>-0.96604 (-0.96172)</td>
<td>0.57321 (0.42653)</td>
</tr>
<tr>
<td>$1/8$</td>
<td>-1.59086 (-1.65563)</td>
<td>-0.97099 (-0.96266)</td>
<td>0.56680 (0.41058)</td>
</tr>
<tr>
<td>$1/16$</td>
<td>-1.58904 (-1.66058)</td>
<td>-0.97233 (-0.96275)</td>
<td>0.56343 (0.39967)</td>
</tr>
<tr>
<td>$1/32$</td>
<td>-1.58828 (-1.66798)</td>
<td>-0.97243 (-0.96276)</td>
<td>0.56176 (0.39386)</td>
</tr>
</tbody>
</table>

The rate of convergence of the PCR-method can be bounded by a function of $((\hat{b} \cdot \hat{c})/(\hat{a} \cdot \hat{d}))^{1/2}$ (see (5.4)). In Fig. 3 we show the dependence of $((\hat{b} \cdot \hat{c})/(\hat{a} \cdot \hat{d}))^{1/2}$ on $\beta$ in (5.4) for $P_h^{-1} \cdot A_h$ and $T_h = A_h + \beta M_h$ and the rate of convergence of the PCR method. We get from the figure, that $\beta \approx 50$ minimizes the expression $((\hat{b} \cdot \hat{c})/(\hat{a} \cdot \hat{d}))^{1/2}$ for the computed meshesizes $h_1/h = 1, 2, 4, 8, 16$.

![Figure 3](image-url)
Example 2: (Z-shaped domain, quasi-uniform mesh).
In our next example we consider a domain with a stronger singularity at the reentrant corner and a non equally sized mesh as depicted in Fig. 2 (right). The vertices are (0, 0), (0.5, 0), (0.15, 0.22), (0.6, 0.2), (0, 0.4). We choose $P_{A_h}^{-1}$ and $F_{V_h}$ to be the same multigrid V-cycle algorithms as in Example 1 and use $\nu = 1, 3, 5$ pre and post smoothing steps. For all computations we get $\lambda_n = \eta_m = 1$. In Tab. 6 and 7 we give the results for the Z-shaped domain.

Table 6. Lower bounds $\lambda_1$ and $\eta_1$ of the applied multigrid methods for different number of pre and postsmoothing steps ($T_h = 50 M_h$ and $T_h = W_h + \gamma I d_h^T D_h^{-1} I d_h$)

<table>
<thead>
<tr>
<th>$h/h_1$</th>
<th>$\lambda_1$</th>
<th>$\eta_1 (T_h = 50 M_h)$</th>
<th>$\eta_1 (T_h = W_h + \gamma I d_h^T D_h^{-1} I d_h)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu = 1$</td>
<td>$\nu = 3$</td>
<td>$\nu = 5$</td>
<td>$\nu = 1$</td>
</tr>
<tr>
<td>$1/2$</td>
<td>0.66961</td>
<td>0.46635</td>
<td>0.21464</td>
</tr>
<tr>
<td>$1/4$</td>
<td>0.64043</td>
<td>0.22201</td>
<td>0.21609</td>
</tr>
<tr>
<td>$1/8$</td>
<td>0.63512</td>
<td>0.22201</td>
<td>0.21609</td>
</tr>
<tr>
<td>$1/16$</td>
<td>0.63420</td>
<td>0.21609</td>
<td>0.21609</td>
</tr>
<tr>
<td>$1/32$</td>
<td>0.63405</td>
<td>0.21609</td>
<td>0.21609</td>
</tr>
</tbody>
</table>

The numerical experiments indicate that the contraction number of both V-cycles improves uniformly as the number of smoothing steps is increased and is bounded independently from the meshsize $h$. (For a prove see [10] for second order elliptic pde's.)

Table 7. Constants $\Delta$, $\theta$, and $\Theta$, ($T_h = 50 M_h$ and $T_h = W_h + \gamma I d_h^T D_h^{-1} I d_h$)

<table>
<thead>
<tr>
<th>$h/h_1$</th>
<th>$T_h = 50 M_h$</th>
<th>$T_h = W_h + \gamma I d_h^T D_h^{-1} I d_h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h/h_1$</td>
<td>$\Delta$</td>
<td>$\theta^2$</td>
</tr>
<tr>
<td>$1/1$</td>
<td>1.62331</td>
<td>0.97108</td>
</tr>
<tr>
<td>$1/2$</td>
<td>1.74237</td>
<td>0.94476</td>
</tr>
<tr>
<td>$1/4$</td>
<td>1.81163</td>
<td>0.93656</td>
</tr>
<tr>
<td>$1/8$</td>
<td>1.85523</td>
<td>0.93422</td>
</tr>
<tr>
<td>$1/16$</td>
<td>1.88295</td>
<td>0.93359</td>
</tr>
<tr>
<td>$1/32$</td>
<td>1.89515</td>
<td>0.93396</td>
</tr>
</tbody>
</table>

All results underline for both choices of $T_h$ the $h$-independence of the constants $\theta$, $\Theta$, $\Delta$, $\lambda_1$, $\lambda_n$, $\eta_1$, and $\eta_m$.

![Figure 4](image-url) Dependence of the rate of convergence of the PCR method on $\gamma$ using $T_h = W_h + \gamma I d_h^T D_h^{-1} I d_h$ (left) resp. on $\beta$ using $T_h = \beta M_h$ (right).

In Subsection 5.2 we stated the property

$$c_{19} \frac{1^T D_h 1}{1^T V_h 1} \leq \frac{1^T I_h V_h^{-1} I_h 1}{1^T I_h D_h^{-1} I_h 1} \leq c_{20} \frac{1^T D_h 1}{1^T V_h 1} = c_{20} |\Gamma|/(1^T V_h 1)$$
For both examples and different meshesizes $h$ the ratio between $(1^\top I_h V^{-1}_h I_h 1)/(1^\top I_h V^{-1}_h I_h 1)$ and $(1^\top D_h 1)/(1^\top V_h 1)$ is given in Tab. 8. Further numerical experiments on different domains show $c_{19} \leq 0.9$ and $c_{20} \geq 1$.

**Table 8.** Ratio between $(1^\top I_h V^{-1}_h I_h 1)/(1^\top I_h V^{-1}_h I_h 1)$ and $(1^\top D_h 1)/(1^\top V_h 1)$

<table>
<thead>
<tr>
<th>$h/h_1$</th>
<th>$\frac{1^\top I_h V^{-1}_h I_h 1}{1^\top D_h 1}$</th>
<th>$\frac{1^\top I_h V^{-1}_h I_h 1}{1^\top V_h 1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Example 1</td>
<td>Example 1</td>
<td>Example 2</td>
</tr>
<tr>
<td>1/1</td>
<td>0.98215</td>
<td>0.95306</td>
</tr>
<tr>
<td>1/2</td>
<td>0.97590</td>
<td>0.94109</td>
</tr>
<tr>
<td>1/4</td>
<td>0.97256</td>
<td>0.93609</td>
</tr>
<tr>
<td>1/8</td>
<td>0.97127</td>
<td>0.93388</td>
</tr>
<tr>
<td>1/16</td>
<td>0.97075</td>
<td>0.93288</td>
</tr>
<tr>
<td>1/32</td>
<td>0.97055</td>
<td>0.93243</td>
</tr>
</tbody>
</table>

**Example 3:** (Different smoothers in $Mg^V$).

In our numerical example we compare our three proposed smoother in the multigrid algorithm $Mg^V$ with respect to the extreme eigenvalues $\lambda_1, \lambda_n$ of $Mg^V V_h$ for the geometries and meshes in Example 1 and 2. To execute the action $M^{-1}_{k,1}$, we use the Jacobi iteration; i.e. we replace $M^{-1}_{k,1}$ by $\tilde{M}^{-1}_{k,1}$, which is the matrix representation of 3 Jacobi iterations applied to $M_{k,1}$. We computed for all smoothers the damping parameter $\chi_k$ (6.3) and the resulting extreme eigenvalues $\lambda_1, \lambda_n$. Numerical experiments show, that the $\chi_k$ given by (6.3) is not optimal. By bisection we computed an optimal parameter $\tilde{\chi}_k$ (given in the last row), which minimizes the contraction number of the multigrid method $Mg^V$, i.e. maximizes $\lambda_1$ while $\lambda_n = 1$. The resulting minimal eigenvalue $\lambda_1$ is given in brackets.

In Tab. 9 we give the results for Example 1, resp. in Tab. 10 for Example 2. For all computations we get $\lambda_n = 1$.

**Table 9.** Constant $\lambda_1$ in Example 1 for different smoother.

<table>
<thead>
<tr>
<th>$k = # level$</th>
<th>$\frac{1}{\chi_k} L^{-1}<em>k D</em>{k,1} L^{-1}_k$</th>
<th>$\frac{1}{\chi_k} \tilde{M}^{-1}<em>{k,1} D</em>{k,1} \tilde{M}^{-1}_{k,1}$</th>
<th>$\frac{1}{\chi_k} M^{-1}<em>{k,2} D</em>{k,2} M^{-1}_{k,2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_k/2^k$</td>
<td>$\lambda_1$</td>
<td>$\chi_k/2^k$</td>
<td>$\lambda_1$</td>
</tr>
<tr>
<td>2</td>
<td>2.37308</td>
<td>0.96501 (0.96964)</td>
<td>1.32083</td>
</tr>
<tr>
<td>3</td>
<td>2.30758</td>
<td>0.97893 (0.98045)</td>
<td>1.36793</td>
</tr>
<tr>
<td>4</td>
<td>2.31010</td>
<td>0.98130 (0.98235)</td>
<td>1.38358</td>
</tr>
<tr>
<td>5</td>
<td>2.30250</td>
<td>0.98218 (0.98332)</td>
<td>1.38792</td>
</tr>
<tr>
<td>6</td>
<td>2.30178</td>
<td>0.98251 (0.98384)</td>
<td>1.38906</td>
</tr>
<tr>
<td>optimal</td>
<td>2.10867</td>
<td>1.17063</td>
<td>4.88401</td>
</tr>
</tbody>
</table>

All three smoothers used with $Mg^V$ provide constant bounds for the spectrum, which do not depend on the meshsize $h$. All three smoothers can be used to construct an efficient multigrid method for the single layer potential $V$; however the smoothers $M^{-1}_{k,\mu} D_{k,1} M^{-1}_{k,\mu}$ require twice the appilation of $M^{-1}_{k,\mu}$ resp. $M^{-1}_{k,\mu}$ compared with used a diagonal scaling when using $L^{-1}_k D_{k,1} L^{-1}_k$.

8. Inner–Outer Iteration

In this section we discuss the convergence of an inner-outer iteration, which can be written for a general system $Ax = b$ as follows:
Table 10. Constant \( \lambda_1 \) in Example 2 for different smoother.

<table>
<thead>
<tr>
<th># level</th>
<th>( \frac{1}{\chi}L^{-1}Dk,1L^{-1} )</th>
<th>( \frac{1}{\chi}M^{-1}k,1Dk,1M^{-1}k,1 )</th>
<th>( \frac{1}{\chi}M^{-1}k,2Dk,2M^{-1}k,2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \chi_k/2^k )</td>
<td>( \lambda_1 )</td>
<td>( \chi_k/2^k )</td>
</tr>
<tr>
<td>2</td>
<td>3.72814, 0.49728 (0.63956)</td>
<td>1.88333, 0.55408 (0.69623)</td>
<td>7.72710, 0.54991 (0.70662)</td>
</tr>
<tr>
<td>3</td>
<td>3.52397, 0.48804 (0.67501)</td>
<td>1.84899, 0.53167 (0.71827)</td>
<td>7.81520, 0.52436 (0.68449)</td>
</tr>
<tr>
<td>4</td>
<td>3.23460, 0.48541 (0.66984)</td>
<td>1.84878, 0.52720 (0.71923)</td>
<td>8.2547, 0.50133 (0.67973)</td>
</tr>
<tr>
<td>5</td>
<td>3.23460, 0.48493 (0.66894)</td>
<td>1.84876, 0.52634 (0.71955)</td>
<td>8.45619, 0.48847 (0.67886)</td>
</tr>
<tr>
<td>6</td>
<td>3.23460, 0.48485 (0.66879)</td>
<td>1.84876, 0.52620 (0.71956)</td>
<td>8.53971, 0.48307 (0.67875)</td>
</tr>
<tr>
<td>optimal</td>
<td>2.31715</td>
<td>1.20125</td>
<td>5.47552</td>
</tr>
</tbody>
</table>

Algorithm 1: Inner-Outer-Iteration applied to \( Ax = b \)

Let \( \tilde{A} \) be ‘close’ to \( A \).

Choose initial guess \( x^0 \).

\begin{algorithm}
for \( k = 0, 1, 2, \ldots \) until convergence do

\begin{enumerate}
\item compute residual \( r^k = b - Ax^k \)
\item compute approximation \( \tilde{d}^{k,s} \) to \( \tilde{A}^{-1}r^k \) by \( s \) iterations
\item update \( x^{k+1} = x^k + \tilde{d}^{k,s} \)
\end{enumerate}

end do
\end{algorithm}

The iteration was introduced by Axelsson and Vassilevski [3]. In the following we will analyze the convergence of this method for indefinite and symmetric matrices. First we give a general perturbation argument for perturbed linear systems of equations, which we will use in the sequel. We quote the convergence rate for the steepest descent method under consideration of perturbations. The steepest descent method will be used as inner iteration. After we have shown the convergence of the inner-outer iteration for indefinite and symmetric matrices in general, we prove the convergence of this method for discretized problems (3.9), independent of mesh size \( h \).

8.1. Preliminaries. In the sequel we consider an indefinite and symmetric system of equations

\[(8.1) \quad \begin{pmatrix} A & B^\top \\ B & -C \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \]

where \( 0 \leq A \in \mathbb{R}^{m \times m}_{\text{sym}} \) and \( 0 < C \in \mathbb{R}^{n \times n}, m - \text{rank}(A) \leq \text{rank}(C), B \in \mathbb{R}^{m \times n} \) be given, s.t. the Schur complement \( S := A + B^\top C^{-1}B \) is positive definite. Let \( 0 < P_C \in \mathbb{R}^{n \times n} \) and \( 0 < P_S \in \mathbb{R}^{m \times m} \) ‘close’ to \( C \), resp. \( \tilde{S} := A + B^\top P_C^{-1}B \) be given.

Now, let us motivate how to apply Algorithm 1 to system (8.1) with right hand side \( b = (b_1,b_2) \) and exact solution \( x = (x_1,x_2) \) which is approximated after \( k \) iterations by \( x^k = (x_1^k,x_2^k) \) \((k = 0, 1, 2, \ldots)\). In each outer iteration the residual \( r = (r_1^k, r_2^k) \) \((k = 0, 1, \ldots)\) given by

\[(8.2) \quad \begin{pmatrix} r_1^k \\ r_2^k \end{pmatrix} = \begin{pmatrix} b_1 - Ax_1^k - B^\top x_2^k \\ b_2 - Bx_1^k + Cx_2^k \end{pmatrix} \]

is calculated and an approximation to the exact defect \( (d_1^k, d_2^k) = (x_1 - x_1^k, x_2 - x_2^k) \) is calculated. Notice

\[(8.3) \quad \begin{pmatrix} r_1^k \\ r_2^k \end{pmatrix} = \begin{pmatrix} A(x_1 - x_1^k) + B^\top (x_2 - x_2^k) \\ B(x_1 - x_1^k) - C(x_2 - x_2^k) \end{pmatrix} = \begin{pmatrix} A & B^\top \\ B & -C \end{pmatrix} \begin{pmatrix} d_1^k \\ d_2^k \end{pmatrix} \]

Hence, \( x_i = d_i^k + x_i^k \) \((i = 1, 2)\) would give the exact solution. The inverse of \( \begin{pmatrix} A & B^\top \\ B & -C \end{pmatrix} \) can be written as

\[(8.4) \quad \begin{pmatrix} A & B^\top \\ B & -C \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1}(r_1^k + B^\top C^{-1}r_2^k) \\ C^{-1}((BS^{-1}(r_1^k + B^\top C^{-1}r_2^k)) - r_2^k) \end{pmatrix} \]
Our modification of the inner-outer iteration to compute an approximation to \((dddk_{1}, ddk_{2})\) resp. \((xxk_{1}, xxk_{2})\) is to use two different approximations. Firstly, using a matrix \(P_{C}\) which is "close" to \(C\). That gives an approximation \((\dddk_{1}, \dddk_{2})\) satisfying

\[
\begin{bmatrix}
A & B^{\top} \\
B & -P_{C}
\end{bmatrix}
\begin{bmatrix}
\dddk_{1} \\
\dddk_{2}
\end{bmatrix}
= \begin{bmatrix}
r_{1}^{k} \\
r_{2}^{k}
\end{bmatrix}
\text{ resp. }
\begin{bmatrix}
\dddk_{1} \\
\dddk_{2}
\end{bmatrix}
= \begin{bmatrix}
\tilde{S}^{-1}(r_{1}^{k} + B^{\top}P_{C}^{-1}r_{2}^{k}) \\
(P_{C}^{-1}(B\tilde{S}^{-1}(r_{1}^{k} + B^{\top}P_{C}^{-1}r_{2}^{k}))-r_{2}^{k})
\end{bmatrix}
\]

Secondly, an approximation \((\dddk_{1}, \dddk_{2})\) to \((\dddk_{1}, \dddk_{2})\) which will be calculated by \(s\) iterations of the steepest descent method, i.e. \(\dddk_{1}\) will be computed as approximation to \(\dddk_{1} = \tilde{S}^{-1}(r_{1}^{k} + B^{\top}P_{C}^{-1}r_{2}^{k})\) and \(\dddk_{2} = P_{C}^{-1}(B\dddk_{1} - r_{2}^{k})\). Notice, \(d_{i}^{k} = x_{i} - xx_{i}^{k}\) and \(xx_{i}^{k+1} = xx_{i}^{k} + \dddk_{i}^{k+1} (i = 1, 2)\). Hence, \(xx_{i} - xx_{i}^{k+1} = d_{i}^{k} - \dddk_{i}^{k}\) and from (8.4) and (8.5) we get

\[
SD_{1}^{k} = r_{1}^{k} + B^{\top}C^{-1}r_{2}^{k}, \quad d_{2}^{k} = C^{-1}(Bd_{1}^{k} - r_{2}^{k})
\]

Algorithm 2 applied to the system (8.1) with \(s\) inner iterations of the steepest descent method and the specific choice of approximations \(P_{C}\) and \(P_{S}\) can be written as pseudocode as follows

**Algorithm 2: Inner-Outer-Iteration applied to (8.1)**

Choose initial guess \(xx^{0} = (xx_{1}^{0}, xx_{2}^{0})\)

for \(k = 0, 1, 2, \ldots\) % begin outer iteration

\(r_{1}^{k} = b_{1} - Axx_{1}^{k} - B^{\top}xx_{2}^{k}\)
\(r_{2}^{k} = b_{2} - Bxx_{1}^{k} + Cxx_{2}^{k}\)

check convergence; continue if necessary

\(xx_{0}^{0} = 0\)
\(p_{i}^{0} = r_{i}^{k} + B^{\top}P_{C}^{-1}r_{2}^{k}\)

for \(l = 0, 1, 2, \ldots, s - 1\) do % begin inner iteration

\(q_{1}^{l} = P_{S}^{-1}p_{1}^{l}\)
\(a_{1}^{l} = (A + B^{\top}P_{C}^{-1}B)q_{1}^{l}\)
\(\beta_{l} = (p_{1}^{l}, q_{1}^{l})/(a_{1}^{l}, q_{1}^{l})\)
\(z_{1}^{l+1} = \beta_{l}q_{1}^{l}\)
\(p_{1}^{l+1} = p_{1}^{l} - \beta_{l}a_{1}^{l}\)

end do % end inner iteration

\(xx_{1}^{k+1} = xx_{1}^{k} + z_{1}^{l}\)
\(xx_{2}^{k+1} = xx_{2}^{k} + P_{C}^{-1}(Bxx_{1}^{k} - r_{2}^{k})\)

end do % end outer iteration

**Theorem 3.** Let the positive constants \(\xi_{1}, \xi_{m}, \lambda_{1}, \lambda_{n},\) and \(\tau\) be given such that the following inequalities hold for all \(xx \in \mathbb{R}^{m} \setminus \{0\}\)

\[
(8.7) \quad \xi_{1} \leq \frac{xx^{\top}\tilde{S}xx}{xx^{\top}P_{S}xx} \leq \xi_{m}
\]

resp. \(xx \in \mathbb{R}^{n} \setminus \{0\}\)

\[
(8.8) \quad \lambda_{1} \leq \frac{xx^{\top}Cxx}{xx^{\top}P_{C}xx} \leq \lambda_{n}, \quad \frac{xx^{\top}(BS^{-1}B)xx}{xx^{\top}Cxx} \leq \tau^{2}.
\]

Furthermore, let \([\lambda_{1}, \lambda_{n}] \subset [\alpha^{-1}, \alpha]\) with \(\alpha > 1\). Then, the iterates \((xx_{1}^{k}, xx_{2}^{k})\) \((k = 0, 1, 2, \ldots)\) of Algorithm 2 satisfy

\[
(8.9) \quad \|S^{1/2}(xx_{1} - xx_{1}^{k+1})\|_{2} + \|C^{1/2}(xx_{2} - xx_{2}^{k+1})\|_{2} \leq \rho\left(\|S^{1/2}(xx_{1} - xx_{1}^{k})\|_{2} + \|C^{1/2}(xx_{2} - xx_{2}^{k})\|_{2}\right)
\]
with rate of convergence

\[
\rho = 2\alpha (1 + \tau \alpha) \left( \alpha - 1 + \left( \frac{\xi_m - \xi_1}{\xi_m + \xi_1} \right) \right) + (\alpha - 1) \alpha \left( 1 + 2\tau + 2\tau \left( \frac{\xi_m - \xi_1}{\xi_m + \xi_1} \right) \right).
\]

**Remark 3.** i.) If \( s \) is large enough and \( \alpha \) sufficiently close to one we get \( 0 \leq \rho \leq c_{16} < 1 \).

ii.) Let \( \delta = \alpha - 1 > 0 \) and \( s \) such that \( ([\xi_m - \xi_1]/(\xi_m + \xi_1)]^s \leq \delta \). Then, we get from (8.10) \( \rho \leq \delta/(1 + \delta)(5 + 6\tau(1 + \delta)) \).

**Proof.** Before we give an estimate for the convergence rate of Algorithm 2 we show some technical estimates first. They will be used in (8.23) later.

The matrix \( \tilde{S} = A + B^T P_C^{-1} B \) is symmetric and positive definite, since we obtain from (8.8)

\[
\lambda_1 \leq \frac{x^T P_C^{-1} x}{x^T C^{-1} x} \leq \lambda_n \quad \text{and} \quad \min\{1, \lambda_1\} \leq \frac{x^T \tilde{S} x}{x^T S x} \leq \max\{1, \lambda_n\} \quad (x \in \mathbb{R}^n \setminus \{0\}).
\]

The second last inequality implies

\[
\max_{x \in \mathbb{R} \setminus \{0\}} \| (I - C^{1/2} P_C^{-1} C^{1/2}) x \|_2^2 \leq \max_{x \in \mathbb{R} \setminus \{0\}} \frac{x^T (I - C^{1/2} P_C^{-1} C^{1/2}) x}{x^T x} = \left( \max_{x \in \mathbb{R} \setminus \{0\}} \frac{x^T (I - C^{1/2} P_C^{-1} C^{1/2}) x}{x^T x} \right)^2 \leq \max\{(1 - \lambda_1)^2, (1 - \lambda_n)^2\} =: \kappa^2
\]

and similar

\[
\max_{x \in \mathbb{R} \setminus \{0\}} \| (I - P_C^{1/2} C^{-1} P_C^{1/2}) x \|_2^2 \leq \max\{(1 - \lambda_1)^2, (1 - \lambda_n)^2\} =: \sigma^2
\]

Since \( S \) and \( \tilde{S} \) are symmetric and positive definite, inequality (8.11) implies

\[
\max_{x \in \mathbb{R} \setminus \{0\}} \| (I - \tilde{S}^{1/2} S^{-1} \tilde{S}^{1/2}) x \|_2^2 = \left( \max_{x \in \mathbb{R} \setminus \{0\}} \frac{x^T (I - \tilde{S}^{1/2} S^{-1} \tilde{S}^{1/2}) x}{x^T x} \right)^2 \leq \max\{(1 - \min\{1, \lambda_1\})^2, (1 - \max\{1, \lambda_n\})^2\} = \max\{(1 - \lambda_1)^2, (\lambda_n - 1)^2\} = \kappa^2
\]

From inequalities (8.11), (8.8), and (8.12) we get for all \( x \in \mathbb{R}^n \setminus \{0\} \)

\[
\| \tilde{S}^{1/2} B^T (P_C^{-1} - C^{-1}) x \|_2 \leq \max\{1, \lambda_1^{-1/2}\} \| S^{-1/2} B^T C^{-1/2} (C^{1/2} P_C^{-1} C^{1/2} - I) C^{-1/2} x \|_2 \leq \max\{1, \lambda_1^{-1/2}\} \tau \kappa \| C^{-1/2} x \|_2 \]

From the last inequality we obtain

\[
\| \tilde{S}^{1/2} B^T (P_C^{-1} - C^{-1}) r_2^k \|_2 \leq \max\{1, \lambda_1^{-1/2}\} \tau \kappa \| C^{-1/2} r_2^k \|_2 \leq \max\{1, \lambda_1^{-1/2}\} \tau \kappa \| S^{-1/2}(x_1 - x_1^k) - C(x_2 - x_2^k) \|_2 \leq \max\{1, \lambda_1^{-1/2}\} \tau \kappa \| S^{-1/2}(x_1 - x_1^k) - C(x_2 - x_2^k) \|_2
\]

Using \( S(x_1 - x_1^k) = Sd_1^k = r_1^k + B^T C^{-1} r_2^k \) in (8.6) and inequality (8.11) we obtain

\[
\| \tilde{S}^{1/2}(r_1^k + B^T C^{-1} r_2^k) \|_2 = \| \tilde{S}^{1/2} S(x_1 - x_1^k) \|_2 \leq \max\{1, \lambda_1^{-1/2}\} \| S^{1/2}(x_1 - x_1^k) \|_2
\]

For the \( s \)-iterate \( \tilde{d}_s^k \) of the preconditioned steepest descent method applied to \( \tilde{S}d_1^k = r_1^k + B^T C^{-1} r_2^k \) with coefficient matrix \( \tilde{S} \) and preconditioner \( P_S \) and \( \xi_1, \xi_m \) from (8.7) there holds the following convergence result [21]

\[
\| \tilde{S}^{1/2}(\tilde{d}_1^k - \tilde{d}_1) \| \leq \left( \frac{\xi_m - \xi_1}{\xi_m + \xi_1} \right)^s \| \tilde{S}^{1/2} \tilde{d}_1^k \|
\]

By adding \( \pm \tilde{S}^{1/2} B^T C^{-1} r_2^k \), (8.6), and the triangle inequality we get

\[
\| \tilde{S}^{1/2} \tilde{d}_1^k \|_2 \leq \| \tilde{S}^{1/2}(r_1^k + B^T C^{-1} r_2^k) \|_2 + \| \tilde{S}^{1/2} B^T (P_C^{-1} - C^{-1}) r_2^k \|_2
\]
By the same technique (adding $\pm \sqrt{S}B^T C^{-1}r$) and inequality (8.14) we obtain

$$\| \sqrt{S}(d_1 - d_1) \|_2 \leq \| \sqrt{S}(S - \sqrt{S})(r_1 + B^T C^{-1}r_2) \|_2 + \| \sqrt{S}B^T(C^{-1} - P_C^{-1})r_2 \|_2$$

(8.20) \leq \| \sqrt{S}(S - \sqrt{S}I) \|_2 \leq \| \sqrt{S}B^T(C^{-1} - P_C^{-1})r_2 \|_2

$$\leq \| \sqrt{S}(r_1 + B^T C^{-1}r_2) \|_2 + \| \sqrt{S}B^T(C^{-1} - P_C^{-1})r_2 \|_2$$

From (8.18), (8.19) together with (8.16) and (8.17)

(8.21) \[ \| \sqrt{S}(d_1 - d_1) \|_2 \leq \| \sqrt{S}(d_1 - d_1) \|_2 + \| \sqrt{S}B^T(C^{-1} - P_C^{-1})r_2 \|_2 \]

\[ \leq (\kappa + \frac{\xi_m - \xi_1}{\xi_m + \xi_1}) \| \sqrt{S}(r_1 + B^T C^{-1}r_2) \|_2 + (1 + \frac{\xi_m - \xi_1}{\xi_m + \xi_1}) \| \sqrt{S}B^T(C^{-1} - P_C^{-1})r_2 \|_2 \]

\[ \leq (\kappa + \tau + (1 + \tau \kappa) \frac{\xi_m - \xi_1}{\xi_m + \xi_1}) \max\{1, \lambda_{-1/2}^{-}\} \| \sqrt{S}(x_1 - x_1) \|_2 \]

\[ + \tau (1 + \frac{\xi_m - \xi_1}{\xi_m + \xi_1}) \max\{1, \lambda_{-1/2}^{-}\} \| C^{1/2}(x_2 - x_2) \|_2 \]

Using $Bd_1 - r_2 = C(x_2 - x_1)$, $A$ is assumed to be positive semi-definite, and (8.6), (8.13), (8.11) we bound the error $x_2 - x_2$ as follows

(8.22) \[ \| P_C^{1/2}(x_2 - x_2)^{(k+1)} \|_2 \leq \| P_C^{1/2}(d_2 - d_2) \|_2 + \| P_C^{1/2}(d_2 - P_C^{-1}Bd_1 + P_C^{-1}Bd_1 - d_2) \|_2 \]

\[ \leq \| P_C^{1/2}(d_2 - d_2) \|_2 + \| (I - P_C^{-1}B)P_C^{1/2}Bd_1 - r_2 \|_2 \]

\[ \leq \| \sqrt{S}(d_2 - d_2) \|_2 + \| \sqrt{S}(d_1 - d_1) \|_2 + \| \sqrt{S}(a_{1/2} - d_1) \|_2 \]

We are now able to bound the error of the $(k + 1)$-th iterate by using (8.11), (8.8), (8.22), (8.21), identities $x_i - x_i^{(k+1)} = \tilde{d}_i^{(k+1)} - d_i^{(k+1)}$ and $\sigma$ defined by (8.13)

(8.23) \[ \| S^{1/2}(x_1 - x_1)^{(k+1)} \|_2 + \| C^{1/2}(x_2 - x_2)^{(k+1)} \|_2 \]

\[ \leq \max\{1, \lambda_{1/2}^{-}\} \| \sqrt{S}(d_1 - d_1) \|_2 + \max\{1, \lambda_{1/2}^{-}\} \| P_C^{1/2}(d_2 - d_2) \|_2 \]

\[ \leq (\kappa + \tau) + (1 + \tau \kappa) \frac{\xi_m - \xi_1}{\xi_m + \xi_1}) \max\{1, \lambda_{1/2}^{-}\} \| \sqrt{S}(x_1 - x_1) \|_2 \]

Let $[\lambda_1, \lambda_n] \subset [\alpha^{-1}, \alpha]$, with $\alpha > 1$. Hence, we get $\kappa, \sigma \leq \alpha - 1$ and

(8.24) \[ \| S^{1/2}(x_1 - x_1)^{(k+1)} \|_2 + \| C^{1/2}(x_2 - x_2)^{(k+1)} \|_2 \]

\[ \leq \left( (\alpha - 1)(1 + \tau) + (1 + \tau \alpha) \frac{\xi_m - \xi_1}{\xi_m + \xi_1})^\alpha \right) \| S^{1/2}(x_1 - x_1) \|_2 \]

\[ + \left( (\alpha - 1) + (\alpha - 1) \tau \right) \| \sqrt{S}(x_1 - x_1) \|_2 \]

\[ \leq \rho \| S^{1/2}(x_1 - x_1) \|_2 + \| C^{1/2}(x_2 - x_2) \|_2 \]

where

$$\rho := 2\alpha(1 + \tau \alpha) \left( \alpha - 1 + \left( \frac{\xi_m - \xi_1}{\xi_m + \xi_1} \right)^\alpha \right) + (\alpha - 1) \alpha \left( 2\tau + 2\tau \left( \frac{\xi_m - \xi_1}{\xi_m + \xi_1} \right)^\alpha \right)$$
8.2. The Inner-Outer Iteration Applied to the Coupling Matrix $A$. We consider now the inner-outer iteration Algorithm 2 applied to the linear system (5.1).

Since $P_{S_h}$ and $P_{V_h}$ are both assumed to be symmetric and positive definite, we can define positive constants depending on the mesh size $h = h(m)$ (resp. $h = h(n)$) to bound the extreme eigenvalues satisfying

\begin{equation}
0 < \eta_{\min}(h) \leq \frac{x^\top (A_h + T_h)x}{x^\top P_{S_h}x} \leq \eta_{\max}(h) \quad (x \in \mathbb{R}^m \setminus \{0\}).
\end{equation}

and

\begin{equation}
0 < \lambda_{\min}(h) \leq \frac{x^\top V_hx}{x^\top P_{V_h}x} \leq \lambda_{\max}(h) \quad (x \in \mathbb{R}^n \setminus \{0\}).
\end{equation}

Therefore by applying Section 5 we obtain the estimates

\begin{equation}
0 \leq \frac{x^\top \{A_h + W_h + (K_h^\top - I_h^\top)V_h^{-1}(K_h - I_h)\}x}{x^\top (A_h + T_h)x} \leq \Omega \quad (x \in \mathbb{R}^m \setminus \{0\})
\end{equation}

where $\omega, \Omega$ are positive constants independent of the mesh size $h$ (Let $\omega = \theta^2$ with $\theta$ from (5.8) and $\Omega = \max\{\Delta, \Theta^2\}$ with $\Delta$ from (5.8), $\Theta$ from (5.9), and notice Subsections 5.1 and 5.2 where the existence is proven.) and

\begin{equation}
\xi_{\min}(h) \leq \frac{x^\top \{A_h + W_h + (K_h^\top - I_h^\top)V_h^{-1}(K_h - I_h)\}x}{x^\top P_{S_h}x} \leq \xi_{\max}(h) \quad (x \in \mathbb{R}^m \setminus \{0\})
\end{equation}

which is analogous to (8.7) (Let $\xi_{\min}(h) = \omega \eta_{\min}(h)$ and $\xi_{\max}(h) = \Omega \eta_{\max}(h)$ with $\eta_{\min}(h), \eta_{\max}(h)$ from (8.25) and $\omega, \Omega$ from (8.27)). Then, with

$$\phi^\top (K_h - 1)(A_h + T_h)^{-1}(K_h^\top - 1)\phi \leq c_{17} \|(K_h^\top - 1)\phi\|_{H^{-\frac{1}{2}}(\Gamma)}^2 \leq c_{18} \phi^2 \|H^{-\frac{1}{2}}(\Gamma)\|
$$

and (8.27) we see that there is a constant $\tau \in \mathbb{R}$ with

\begin{equation}
\frac{x^\top (K_h - I_h)(A_h + W_h + (K_h^\top - I_h^\top)V_h^{-1}(K_h - I_h))^{-1}(K_h - I_h)^\top x}{x^\top V_hx} \leq \tau^2 \quad (x \in \mathbb{R}^n \setminus \{0\})
\end{equation}

which is analogous to (8.8).

If $\Delta, \theta$, and $\Theta$ are independent from the mesh size $h$, there exists also a constant $\tau$ (8.29) independent from the mesh size $h$, and Theorem 3 gives bounds for convergence rate of Algorithm 2 applied to (8.1) depending only on the extreme eigenvalues of $P_{S_h}^{-1}S_h$ (8.28) and $P_{V_h}^{-1}V_h$ (8.26). With Lemma 2, we get from Theorem 3 the following result.

**Theorem 4.** Let $P_{S_h}$, $P_{V_h}$ be spectral equivalent to $A_h + T_h$ and $V_h$, respectively. Then, for sufficiently large $s$ and sufficiently small spectrum of $P_{V_h}^{-1}V_h$ around one, Algorithm 2 applied to (8.1) converges and the convergence rate is bounded above independently of the mesh size $h$.

**Remark 4.** Notice that there holds

$$\|S^\frac{1}{2}x\|_2 \cong \|x\|_{H^1(\Omega_h)}, \quad \|V^\frac{1}{2}\phi\|_2 \cong \|\phi\|_{H^{-\frac{1}{2}}(\Gamma)}.$$

Therefore Theorem 4 implies convergence in the energy norm.

9. Numerical Example (Part 2)

We continue our numerical example of Section 7. We choose $P_{S_h}^{-1}$ and $P_{V_h}^{-1}$ to be the same multigrid V-cycle algorithms as in Example 1, resp. 2 and use $\nu = 1$ pre and post smoothing step in Example 1 and 2 to compute $P_{V_h}^{-1}$, and $\nu = 1$ pre and post smoothing step in Example 1 resp. $\nu = 1, 3, 5$ in Example 2 to compute $P_{S_h}^{-1}$. The computed eigenvalues and singular values were computed with Matlab-routines as explained in Section 7. In Tab. 11, there are given computed bounds for $\xi_1, \xi_m,$ and $\tau^2$ for Example 1 and 2. These results confirm in both examples our theory, namely that the lower resp. upper bounds for $\xi_1, \xi_m,$ and $\tau^2$ do not dependent on the meshsize $h$. 

Table 11. Constants $\xi_1$, $\xi_m$, and $\tau^2$

<table>
<thead>
<tr>
<th>$h/h_1$</th>
<th>$\xi_1$</th>
<th>$\xi_m$</th>
<th>$\tau^2$</th>
<th>$\xi_1$</th>
<th>$\xi_m$</th>
<th>$\tau^2$</th>
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<tr>
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<td>1</td>
<td>0.54705</td>
<td>1</td>
<td>1</td>
</tr>
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<td>1</td>
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<td>0.84660</td>
<td>0.95849</td>
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<td>1</td>
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<td>0.72817</td>
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<td>1</td>
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<td>1</td>
<td>0.22196</td>
<td>0.61774</td>
<td>0.80658</td>
</tr>
</tbody>
</table>

In Tab. 12 we present results for solving the linear system $Ax = b$ arising from the geometry and meshes in Example 1 for 20 randomly chosen right hand sides $b$ and $s = 1,3$ number of inner iterations. The average number of iterations and minimal/maximal number (in brackets) are given. The computations were done on a Laptop PC (3.2 GHz) using Matlab (Release 14). We used the stopping criterion

$$(9.1) \quad \|\tilde{S}_h^{1/2}(x_1 - x_k^h)\|_2 \leq tol\|\tilde{S}_h^{1/2}(x_1 - x_0^h)\|_2$$

with $tol = 10^{-8}$. Here, $\| \cdot \|_2$ is the Euclidean-norm, $(x_1, x_2)$ the exact solution of our discretized problem and $(x_1^0, x_2^0) = 0$ the starting vector.

Table 12. Number of iterations and CPU-time for preconditioned Inner-Outer-method (with $P_{S_h}^{-1} \approx V$-cycle multigrid method applied to $A_h + W_h + \gamma D_h^{-1} D_h^{-1} I_d$) and different number of inner iterations ($s = 1,3$)

<table>
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<th>$h_0/h$</th>
<th># unknowns</th>
<th># iterations</th>
<th>CPU-time [s]</th>
<th># iterations</th>
<th>CPU-time [s]</th>
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<tr>
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<td></td>
<td>$s = 1$</td>
<td></td>
<td>$s = 3$</td>
<td></td>
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<tr>
<td>1/1</td>
<td>16</td>
<td>13.6 (12/14)</td>
<td>0.0101</td>
<td>5.45 (5/6)</td>
<td>0.0094</td>
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<tr>
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<td>37</td>
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<td>0.0328</td>
<td>14.2 (13/15)</td>
<td>0.0516</td>
</tr>
<tr>
<td>1/4</td>
<td>97</td>
<td>15.3 (14/16)</td>
<td>0.0703</td>
<td>14.0 (14/14)</td>
<td>0.1126</td>
</tr>
<tr>
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<td>17.1 (16/18)</td>
<td>0.1875</td>
<td>13.8 (13/14)</td>
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</tr>
<tr>
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</tr>
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<td>2.5250</td>
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<td>13 (13/13)</td>
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</tr>
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<td>12.95 (12/13)</td>
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<td>199681</td>
<td>21 (21/21)</td>
<td>199.03</td>
<td>12.05 (12/13)</td>
<td>210.91</td>
</tr>
</tbody>
</table>

Both, the number of iterations and the CPU-time indicate the efficiency of the preconditioned inner-outer iteration.

For comparison we computed also the number of PCR-iterates $(x_1^k, x_2^k)$ where we used the stopping criterion

$$(9.2) \quad \|A_h^{1/2}(x_1 - x_1^k)\|_2 + \|V_h^{1/2}(x_2 - x_2^k)\|_2 \leq tol \left( \|A_h^{1/2}(x_1 - x_1^0)\|_2 + \|V_h^{1/2}(x_2 - x_2^0)\|_2 \right)$$

with $tol = 10^{-8}$.

All results presented here underline the efficiency of both preconditioned iterative methods to solve indefinite linear systems of equations arising from symmetric coupling of finite elements and boundary elements and confirm our theory. Both methods are optimal in the sense, that the number of iterations is bounded independently by the meshsize $h$. The preconditioned conjugate residual method is in our example twice as fast as the preconditioned inner-outer iteration.
Table 13. Number of iterations and CPU-time for PCR-method with block-diagonal preconditioning

<table>
<thead>
<tr>
<th>$h_0/h$</th>
<th># unknowns</th>
<th># iterations</th>
<th>CPU-time [s]</th>
<th># iterations</th>
<th>CPU-time [s]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>average (min/max)</td>
<td></td>
<td>average (min/max)</td>
<td></td>
</tr>
<tr>
<td>1/1</td>
<td>16</td>
<td>17 (17/17)</td>
<td>0.0078</td>
<td>16.35 (16/17)</td>
<td>0.0063</td>
</tr>
<tr>
<td>1/2</td>
<td>37</td>
<td>33.6 (32/35)</td>
<td>0.0273</td>
<td>23.7 (23/25)</td>
<td>0.0204</td>
</tr>
<tr>
<td>1/4</td>
<td>97</td>
<td>41.6 (41/43)</td>
<td>0.0648</td>
<td>26.85 (26/27)</td>
<td>0.0437</td>
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<tr>
<td>1/8</td>
<td>289</td>
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<td>0.1680</td>
<td>28 (28/28)</td>
<td>0.1156</td>
</tr>
<tr>
<td>1/16</td>
<td>961</td>
<td>50 (50/50)</td>
<td>0.5226</td>
<td>28.4 (28/30)</td>
<td>0.3725</td>
</tr>
<tr>
<td>1/32</td>
<td>3457</td>
<td>53 (53/53)</td>
<td>1.9141</td>
<td>29.5 (28/30)</td>
<td>1.4222</td>
</tr>
<tr>
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<td>55.2 (55/57)</td>
<td>7.7930</td>
<td>30 (30/30)</td>
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<tr>
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<td>50689</td>
<td>57.2 (57/59)</td>
<td>32.050</td>
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<tr>
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<td>58.05 (57/59)</td>
<td>133.32</td>
<td>30 (30/30)</td>
<td>101.31</td>
</tr>
</tbody>
</table>

Figure 5. CPU-time [s] for different solvers.

10. Conclusion

All the preconditioned iteration schemes presented here are optimal in the sense, that the number of iterations is bounded independently by the meshsize $h$. The best result with respect to cpu-time (and matrix-vector multiplication) was obtained by using the block-diagonal preconditioned conjugate residual method and $T_h = \beta M_h$. (Notice, one complete inner-outer iteration needs at least twice as much operations as one PCR-iteration.) All smoothers used in the multigrid-algorithm for the single layer potential provide constant bounds for the spectrum of $M \mu V^V h$, which do not depend on the meshsize $h$. In all examples is the damping parameter $\chi_k^*$ less than $\chi_k$ in (6.3) to get the optimal contraction rate of $M \mu V^V$.

References


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