Fast Solvers with Block-Diagonal Preconditioners for Linear FEM-BEM Coupling

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FAST SOLVERS WITH BLOCK-DIAGONAL PRECONDITIONERS FOR LINEAR FEM-BEM COUPLING

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ABSTRACT. The purpose of this paper is to present optimal preconditioned iterative methods to solve indefinite linear systems of equations arising from symmetric coupling of finite elements and boundary elements [14]. This is a block-diagonal preconditioner together with a conjugate residual method (PCR) and a preconditioned inner-outer iteration (PIO). We prove the efficiency of these methods by showing that the number of iterations to preserve a given accuracy is bounded independently of the number of unknowns. Numerical examples underline the efficiency of these methods.

1. INTRODUCTION

This paper deals with the problem of efficiently solving systems of linear equations $\mathcal{A}\boldsymbol{x} = \boldsymbol{b}$ where $\mathcal{A}$ has a 2×2 block structure such that the diagonal matrices are positive semidefinite and negative definite. Particularly, we consider matrices arising from the symmetric coupling of finite element method (FEM) and boundary element method (BEM) when dealing with elliptic transmission problems.

In the case of an indefinite and symmetric FEM-BEM coupling matrix, we may write

$$\mathcal{A} := \begin{pmatrix} A+H & B^\top \\ B & -C \end{pmatrix} \tag{1.1}$$

where $A, H \in \mathbb{R}^{m\times m}$ are both positive semidefinite $(A, H \geq 0)$ and $C \in \mathbb{R}^{n\times n}$ is positive definite $(C > 0)$. Let a block-diagonal preconditioner

$$\mathcal{P} := \begin{pmatrix} P_A & 0 \\ 0 & P_C \end{pmatrix} \tag{1.2}$$

with symmetric submatrices P_A and P_C be given, which are spectrally equivalent to the Schur complement $A + H + B^\top C^{-1}B$ and C. This idea of using matrices as preconditioners which are spectrally equivalent to diagonal submatrices of $\mathcal{A}$ in the context of solvers for linear FEM-BEM equations was also used in [23, 24, 27, 32]. In the latter works, the matrix $\mathcal{A}$ is substructured into a 3×3 system. The theoretical and numerical results [32] indicate, that the convergence rate of a preconditioned conjugate residual method depends on the discretization. We show in this paper, that the block-diagonal preconditioned conjugate residual method (PCR) which we use, leads to convergence rates, which are independent of the mesh size h. The used theoretical tool gives also results for the case, such that the preconditioner $\mathcal{P}$ is a 2×2 block diagonal matrix. Bramble and Pasciak [7] introduced for problems like (1.1) a special inner product which then gives a symmetric and positive definite system. But the system is based on the assumption that there exists a matrix P_C and positive constants α_0, α_1 such that $\alpha_0 C \leq P_C \leq \alpha_1 C$ where $\alpha_1 < 1$ is desired. This can always be satisfied by scaling but it affects the rate of convergence of the applied iteration scheme e.g. conjugate gradient method. There will be no parameter to choose in the here presented block diagonal preconditioner resulting in an optimal rate of convergence.

Sylvester's law of inertia gives together with the following congruent transform of $\mathcal{A}$

$$\mathcal{A} = \begin{pmatrix} I & -B^\top C^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A+H+B^\top C^{-1}B & 0 \\ 0 & -C \end{pmatrix} \begin{pmatrix} I & 0 \\ -C^{-1}B & I \end{pmatrix} \tag{1.3}$$

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shows that the matrix $\mathcal{A}$ has m positive and n negative eigenvalues, where I denotes a generic identity matrix. In Section 2 we give bounds of intervals which contain the spectrum of $\mathcal{A}$ in a general setting, i.e., $\Lambda(\mathcal{A}) \subset [-a, -b] \cup [c, d]$ $(a, b, c, d > 0)$. In the subsequent section we introduce the interface model problem and rewrite it with boundary integral operators into an equivalent weak formulation. In Section 4 we introduce discrete basis functions and discretize the integral operators appearing in the symmetric coupling problem. In the following Section 5 we deal firstly with the fact that the convergence rate of the PCR is bounded by a term depending on a, b, c, d. Secondly, we present a block-diagonal preconditioner for which we prove the existence of constants θ, Θ, and Δ independently of the mesh size h. Using these constants Theorem 2 gives some h independent bounds of $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$ for the preconditioned Matrix $\overline{\mathcal{A}} := \mathcal{P}^{-1/2}\mathcal{A}\mathcal{P}^{-1/2}$. Hence, we get an upper bound for the convergence rate of the block-diagonal preconditioned PCR-algorithm, which is independent of the discretization. In Section 7 we analyse the inner-outer iteration [3], applied to the FEM-BEM coupling matrix. First, we present a convergence analysis of the inner-outer iteration for a general saddlepoint problem and apply finally this procedure to the FEM-BEM problem that we want to solve. Here, we also get a convergence rate, which is independent of the discretization (Theorem 4).

Numerical experiments in Sections 5 and 7 will give approximations for the constants θ, Θ, and Δ and underline that they are independent of the mesh size h. They also show optimal convergence for the preconditioned PCR method and the preconditioned inner-outer iteration, in the sense, that the number of iterations to reach a given exactness is bounded from above.

2. Block-Diagonal Preconditioner in a general setting

In the following we will use bold letters $\boldsymbol{x}, \boldsymbol{y}, \dots$ for column vectors and $\mathbb{R}^{k\times k}_{sym}$ will denote real-valued symmetric $k \times k$-matrices. The next theorem provides a basic conditioning estimate for the operator $\mathcal{P}^{-1/2}\mathcal{A}\mathcal{P}^{-1/2}$.

Theorem 1. *Let $H \in \mathbb{R}^{m\times m}_{sym}$ be a positive semidefinite matrix, $C \in \mathbb{R}^{n\times n}_{sym}$ be positive definite, and $B \in R^{n\times m}$. Further, let $A \in \mathbb{R}^{m\times m}_{sym}$ s.t. $A + H + B^\top C^{-1}B$ is positive definite, and $T \in \mathbb{R}^{m\times m}_{sym}$ s.t. $A + T$ is positive definite. Let the positive constants Δ, Θ, θ be given such that the following inequalities hold for all $\boldsymbol{x} \in \mathbb{R}^m \setminus \{0\}$*

$$\frac{\boldsymbol{x}^\top (A + H)\,\boldsymbol{x}}{\boldsymbol{x}^\top (A + T)\,\boldsymbol{x}} \le \Delta\,, \quad \frac{\boldsymbol{x}^\top B^\top C^{-1}B\,\boldsymbol{x}}{\boldsymbol{x}^\top (A + T)\,\boldsymbol{x}} \le \Theta^2\,, \tag{2.1}$$

and

$$\theta^2 \le \frac{\boldsymbol{x}^\top (A + H + B^\top C^{-1}B)\,\boldsymbol{x}}{\boldsymbol{x}^\top (A + T)\,\boldsymbol{x}}\,. \tag{2.2}$$

We define

$$\mathcal{A} := \begin{pmatrix} A + H & B^\top \\ B & -C \end{pmatrix} \quad \text{and} \quad \mathcal{P} := \begin{pmatrix} P_A & 0 \\ 0 & P_C \end{pmatrix}$$

where $P_A \in \mathbb{R}^{m\times m}_{sym}$ and $P_C \in \mathbb{R}^{n\times n}_{sym}$ are positive definite matrices. The eigenvalues of $P_A^{-1}(A + T)$ are denoted by

$$0 < \eta_1 \le \eta_2 \le \ldots \le \eta_m$$

and eigenvalues of $P_C^{-1}C$ by

$$0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_n\,.$$

Then, the eigenvalues $\mu_{-n} \le \ldots \le \mu_m$ of $\mathcal{P}^{-1/2}\mathcal{A}\mathcal{P}^{-1/2}$ lie in the union of intervals

$$\left[-\frac{1}{2}\left(\lambda_n + \sqrt{\lambda_n^2 + 4\eta_m\lambda_n\Theta^2}\right), -\lambda_1\right]$$
$$\cup \left[\frac{1}{2}\left(-\lambda_1 + \sqrt{\lambda_1^2 + 4\eta_1\lambda_1\theta^2}\right), \frac{1}{2}\left(-(\lambda_1 - \Delta\eta_m) + \sqrt{(\lambda_1 + \Delta\eta_m)^2 + 4\eta_m\lambda_n\Theta^2}\right)\right]$$

Proof. The proof uses techniques similar to that used in [30] and is given here for completeness. Here, additional matrices H and T occur and constants Δ, θ and Θ are defined in differently. First, we derive an upper bound for the singular values of $P_C^{-1/2}BP_A^{-1/2}$, continue with estimates for the negative eigenvalues of $\mathcal{P}^{-1/2}\mathcal{A}\mathcal{P}^{-1/2}$, and conclude with bounds for the positive eigenvalues.

We define matrix $\overline{\mathcal{A}}$ and its submatrices by

$$\mathcal{P}^{-\frac{1}{2}}\mathcal{A}\mathcal{P}^{-\frac{1}{2}} = \begin{pmatrix} P_A^{-\frac{1}{2}}(A+H)P_A^{-\frac{1}{2}} & P_A^{-\frac{1}{2}}B^\top P_C^{-\frac{1}{2}} \\ P_C^{-\frac{1}{2}}BP_A^{-\frac{1}{2}} & -P_C^{-\frac{1}{2}}CP_C^{-\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} \overline{A}+\overline{H} & \overline{B}^\top \\ \overline{B} & -\overline{C} \end{pmatrix} = \overline{\mathcal{A}}$$

and order the singular values of $\overline{B}$ as $0 \le \sigma_1 \le \sigma_2 \ldots \le \sigma_n$. First we give an upper bound for the singular values of $\overline{B}$. For all $\boldsymbol{x} \in \mathbb{R}^m$ we get with λ_n, η_m, and Θ defined by (2.1)

$$\begin{aligned} \boldsymbol{x}^\top \overline{B}^\top \overline{B}\boldsymbol{x} &= \boldsymbol{x}^\top P_A^{-\frac{1}{2}} B^\top P_C^{-1} B P_A^{-\frac{1}{2}} \boldsymbol{x} \le \lambda_n\, \boldsymbol{x}^\top P_A^{-\frac{1}{2}} B^\top C^{-1} B P_A^{-\frac{1}{2}} \boldsymbol{x} \\ &\le \lambda_n \Theta^2\, \boldsymbol{x}^\top P_A^{-\frac{1}{2}}(A+T)P_A^{-\frac{1}{2}}\boldsymbol{x} \le \lambda_n \eta_m \Theta^2\, \boldsymbol{x}^\top \boldsymbol{x} \,. \end{aligned}$$

This gives the upper bound

$$\sigma_n \le \Theta\sqrt{\lambda_n \eta_m}\,. \tag{2.3}$$

Let μ be an eigenvalue of $\overline{\mathcal{A}}$. Then, there is a vector $(\boldsymbol{x},\boldsymbol{y})^\top \in \mathbb{R}^{m+n} \setminus \{0\}$ with

$$(\overline{A}+\overline{H})\,\boldsymbol{x} + \overline{B}^\top \boldsymbol{y} = \mu\,\boldsymbol{x}\,, \tag{2.4}$$

$$\overline{B}\boldsymbol{x} - \overline{C}\,\boldsymbol{y} = \mu\,\boldsymbol{y}\,, \tag{2.5}$$

which follows from the following consideration. If $\mu > 0$, then $\boldsymbol{x} \neq 0$, since otherwise (2.5) implies $\mu = 0$, as $\overline{C}$ is positive definite. If $\mu < 0$, then $\boldsymbol{y} \neq 0$, since otherwise (2.4) implies $\boldsymbol{x} = 0$, as $\overline{A}+\overline{H}$ is positive semidefinite.

For $\mu < 0$, $I - \mu^{-1}(\overline{A}+\overline{H})$ is invertible. We take the scalar product of (2.5) with $\mu\,\boldsymbol{y}$ and substitute $\boldsymbol{x}$ from (2.4), i.e., $\boldsymbol{x} = \mu^{-1}(I - \frac{1}{\mu}(\overline{A}+\overline{H}))^{-1}\overline{B}^\top \boldsymbol{y}$. This gives

$$\mu^2\,\boldsymbol{y}^\top \boldsymbol{y} = \boldsymbol{y}^\top\, \overline{B}\left(I - \mu^{-1}(\overline{A}+\overline{H})\right)^{-1} \overline{B}^\top \boldsymbol{y} - \mu \boldsymbol{y}^\top\, \overline{C}\,\boldsymbol{y} \le \boldsymbol{y}^\top\, \overline{B}\,\overline{B}^\top \boldsymbol{y} - \mu \boldsymbol{y}^\top \overline{C}\,\boldsymbol{y}\,,$$

since all eigenvalues of $\left(I - \frac{1}{\mu}(\overline{A}+\overline{H})\right)^{-1}$ are less or equal than 1 for $\mu < 0$. We obtain

$$\begin{aligned} 0 &\ge \mu^2 \boldsymbol{y}^\top \boldsymbol{y} + \mu\,\boldsymbol{y}^\top \overline{C}\boldsymbol{y} - \boldsymbol{y}^\top \overline{B}\,\overline{B}^\top \boldsymbol{y} \\ &\ge \mu^2 \boldsymbol{y}^\top \boldsymbol{y} + \mu\lambda_n\,\boldsymbol{y}^\top \boldsymbol{y} - \sigma_n^2\,\boldsymbol{y}^\top \boldsymbol{y} = \left\{(\mu + \lambda_n/2)^2 - (\lambda_n^2 + 4\sigma_n^2)/4\right\}\boldsymbol{y}^\top \boldsymbol{y}\,. \end{aligned}$$

Hence, we have

$$-\frac{1}{2}\left(\lambda_n + \sqrt{\lambda_n^2 + 4\sigma_n^2}\right) \le \mu \le -\frac{1}{2}\left(\lambda_n - \sqrt{\lambda_n^2 + 4\sigma_n^2}\right)\,.$$

Together with (2.3) we get a lower bound for negative eigenvalues of $\mathcal{P}^{-1}\mathcal{A}$,

$$\mu_{-n} \ge -\frac{1}{2}\left(\lambda_n + \sqrt{\lambda_n^2 + 4\sigma_n^2}\right) \ge -\frac{1}{2}\left(\lambda_n + \sqrt{\lambda_n^2 + 4\lambda_n\eta_m\Theta^2}\right)\,.$$

Let $\mu < 0$. Taking the scalar product of (2.4) with $\boldsymbol{x}$ and the scalar product of (2.5) with $\boldsymbol{y}$ and subtracting gives

$$\boldsymbol{x}^\top(\overline{A}+\overline{H})\boldsymbol{x} + \boldsymbol{y}^\top \overline{C}\boldsymbol{y} = \mu\,\boldsymbol{x}^\top \boldsymbol{x} - \mu\,\boldsymbol{y}^\top \boldsymbol{y} \ge \lambda_1\,\boldsymbol{y}^\top \boldsymbol{y}\,.$$

Since μ was assumed to be negative, as mentioned above $\boldsymbol{y}\top\boldsymbol{y}$ is positive and we deduce from the last inequality an upper bound for the negative eigenvalues

$$0 \ge \mu\,\boldsymbol{x}^\top \boldsymbol{x} \ge (\lambda_1 + \mu)\,\boldsymbol{y}^\top \boldsymbol{y} \quad \Rightarrow \quad \mu_{-1} \le -\lambda_1\,.$$

Next we prove a lower bound for the positive eigenvalues of $\overline{\mathcal{A}}$. For $\mu > 0$, $\overline{C} + \mu I$ is invertible. Substituting $\boldsymbol{y}$ from (2.5), i.e., $\boldsymbol{y} = (\overline{C} + \mu I)^{-1}\overline{B}\boldsymbol{x}$, into the scalar product of (2.4) and $\boldsymbol{x}$, gives

$$\begin{aligned} \mu \boldsymbol{x}^\top \boldsymbol{x} &= \boldsymbol{x}^\top(\overline{A}+\overline{H})\boldsymbol{x} + \boldsymbol{x}^\top \overline{B}^\top (\overline{C} + \mu I)^{-1}\overline{B}\boldsymbol{x} \\ &= \boldsymbol{x}^\top(\overline{A}+\overline{H})\boldsymbol{x} + \boldsymbol{x}^\top \overline{B}\,\overline{C}^{-\frac{1}{2}}(I + \mu\overline{C}^{-1})^{-1}\overline{C}^{-\frac{1}{2}}\overline{B}^\top \boldsymbol{x}\,. \end{aligned}$$

Since for the eigenvalues of $(I + \mu\overline{C}^{-1})^{-1}$ there holds

$$0 < \left(1 + \frac{\mu}{\lambda_1}\right)^{-1} \leq \left(1 + \frac{\mu}{\lambda_2}\right)^{-1} \cdots \leq \left(1 + \frac{\mu}{\lambda_n}\right)^{-1} \leq 1$$

we have

$$\begin{aligned} \mu \boldsymbol{x}^\top \boldsymbol{x} &\geq \boldsymbol{x}^\top (\overline{A} + \overline{H})\boldsymbol{x} + (1 + \mu/\lambda_1)^{-1} \boldsymbol{x}^\top \overline{B}^\top \overline{C}^{-1} \overline{B}\boldsymbol{x} \\ &\geq (1 + \mu/\lambda_1)^{-1} \left(\boldsymbol{x}^\top (\overline{A} + \overline{H})\boldsymbol{x} + \boldsymbol{x}^\top \overline{B}^\top \overline{C}^{-1} \overline{B}\boldsymbol{x}\right) . \end{aligned}$$

Rewriting this inequality in terms of the blocks of the original unpreconditioned matrix $\mathcal{A}$ gives

$$(1 + \mu/\lambda_1)^{-1} \boldsymbol{x}^\top P_A^{-\frac{1}{2}} \left(A + H + B^\top C^{-1} B\right) P_A^{-\frac{1}{2}} \boldsymbol{x} \leq \mu \boldsymbol{x}^\top \boldsymbol{x} .$$

We employ the definition (2.2) of θ to obtain

$$\theta^2 (1 + \mu/\lambda_1)^{-1} \boldsymbol{x}^\top P_A^{-\frac{1}{2}} (A + T) P_A^{-\frac{1}{2}} \boldsymbol{x} \leq \mu \boldsymbol{x}^\top \boldsymbol{x} .$$

This gives with the definition of η_1

$$\eta_1 \theta^2 (1 + \mu/\lambda_1)^{-1} \boldsymbol{x}^\top \boldsymbol{x} \leq \mu \boldsymbol{x}^\top \boldsymbol{x} .$$

We assumed $\mu > 0$. Hence, we have $\boldsymbol{x} \neq 0$ and obtain

$$0 \leq \mu^2 + \lambda_1 \mu - \lambda_1 \eta_1 \theta^2 = \left(\mu + \frac{\lambda_1}{2}\right)^2 - \frac{\lambda_1^2 + 4\lambda_1 \eta_1 \theta^2}{4} \tag{2.6}$$

which yields $\mu \geq \frac{1}{2}(-\lambda_1 + \sqrt{\lambda_1^2 + 4\lambda_1 \eta_1 \theta^2})$. Therefore, the result for the lower bound of the positive eigenvalues is proven.

We take now the scalar product of $\boldsymbol{x}$ and (2.4), substitute $\boldsymbol{y}$ from (2.5), and use Δ defined by (2.1), to deduce

$$\begin{aligned} \mu \boldsymbol{x}^\top \boldsymbol{x} &= \boldsymbol{x}^\top (\overline{A} + \overline{H})\boldsymbol{x} + \boldsymbol{x}^\top \overline{B}^\top \left(\overline{C} + \mu I\right)^{-1} \overline{B}\boldsymbol{x} \leq \Delta \eta_m \, \boldsymbol{x}^\top \boldsymbol{x} + \boldsymbol{x}^\top \overline{B}^\top (\overline{C} + \mu I)^{-1} \overline{B}\boldsymbol{x} \\ &\leq \Delta \eta_m \, \boldsymbol{x}^\top \boldsymbol{x} + (\lambda_1 + \mu)^{-1} \boldsymbol{x}^\top \overline{B}^\top \overline{B}\boldsymbol{x} \leq (\Delta \eta_m + (\lambda_1 + \mu)^{-1} \sigma_n^2) \, \boldsymbol{x}^\top \boldsymbol{x} \end{aligned}$$

from which we obtain

$$0 \geq \mu^2 + (\lambda_1 - \Delta\eta_m)\mu - \lambda_1 \Delta \eta_m - \sigma_n^2 = \left(\mu + \frac{\lambda_1 - \Delta\eta_m}{2}\right)^2 - \frac{(\lambda_1 + \Delta\eta_m)^2 + 4\sigma_n^2}{4}$$

for $\boldsymbol{x} \neq 0$ in this case. From inequality (2.3) with considering different cases we obtain an upper bound for the positive eigenvalues. □

3. Model problem

In this section we present the interface problem and we rewrite it equivalently, using boundary integral operators. We discretize the resulting system by FEM/BEM coupling, which leads to linear systems that will be solved by several efficient methods as described below.

Let Ω be a bounded Lipschitz-domain and $\Omega_c = \mathbb{R}^2 \backslash \overline{\Omega}$ be its complement. The partial differential equation to be considered in Ω will involve $D = (d_{ij})_{i,j=1,2}$. Let the coefficients $d_{ij} = d_{ji} \in L^\infty(\Omega)$ be uniformly bounded in Ω, i.e.,

$$\exists\, d_0 > 0 : \quad \sum_{i,j=1}^{2} d_{ij}(\boldsymbol{x})\, \xi_i \xi_j \leq d_0 \sum_{i=1}^{2} \xi_i^2 \qquad (\boldsymbol{x} \in \mathbb{R}^2 ,\ \xi_1, \xi_2 \in \mathbb{R}) \tag{3.1}$$

and positive definite, i.e.,

$$\exists\, d_1 > 0 : \quad \sum_{i,j=1}^{2} d_{ij}(\boldsymbol{x})\, \xi_i \xi_j \geq d_1 \sum_{i=1}^{2} \xi_i^2 \qquad (\boldsymbol{x} \in \mathbb{R}^2 ,\ \xi_1, \xi_2 \in \mathbb{R}) . \tag{3.2}$$

We consider the following problem involving the prescribed jumps u_0, t_0 across the interface $\Gamma := \partial\Omega$:

Interface Poblem: *Given $f \in H^{-1}(\Omega)$, $u_0 \in H^{1/2}(\Gamma)$ and $t_0 \in H^{-1/2}(\Gamma)$. Find $u_1 \in H^1(\Omega)$, $u_2 \in H^1_{\mathrm{loc}}(\Omega_c)$, such that*

$$\begin{aligned} \operatorname{div}(D \cdot \operatorname{grad} u_1) + f &= 0 \text{ in } \Omega\,, \quad (3.3)\\ \operatorname{div}(\operatorname{grad} u_2) &= 0 \text{ in } \Omega_c, \quad \textit{and} \quad (3.4)\\ u_1 &= u_2 + u_0, \qquad n \cdot (D \cdot \operatorname{grad} u_1) = \frac{\partial u_2}{\partial n} + t_0 \quad \textit{on}\ \Gamma\,, \quad (3.5)\end{aligned}$$

where n is the normal on Γ pointing from Ω into Ω_c and the regularity condition on u_2 at infinity,

$$u_2(x) = a + b \log|x| + o(1) \qquad \textit{for}\ |x| \to \infty\,, \tag{3.6}$$

with $a, b \in \mathbb{R}$.

Let $H^s(\Omega)$ denote the usual Sobolev spaces [26] with the trace spaces $H^{s-1/2}(\Gamma)$ ($s \in \mathbb{R}$) for a bounded Lipschitz domain Ω with boundary Γ. Let $\|\cdot\|_{H^k(\omega)}$ and $|\cdot|_{H^k(\omega)}$ denote the norm and semi-norm in $H^k(\omega)$ for $\omega \subseteq \Omega$ and an integer k. Recall that $(\cdot,\cdot)$ denotes the $L^2(\Omega)$-scalar product while $\langle\cdot,\cdot\rangle$ denotes duality between $H^s(\Gamma)$ and $H^{-s}(\Gamma)$ (defined by extending the scalar product in $L^2(\Gamma)$). Given $v \in H^{1/2}(\Gamma)$ and $\phi \in H^{-1/2}(\Gamma)$, the boundary integral operators which we will use in the following are defined, for $z \in \Gamma$, by

$$\begin{aligned} (V\phi)(z) &:= -\frac{1}{\pi}\int_\Gamma \phi(\zeta)\log|z-\zeta|\,ds_\zeta,\\ (Kv)(z) &:= -\frac{1}{\pi}\int_\Gamma v(\zeta)\frac{\partial}{\partial n_\zeta}\log|z-\zeta|\,ds_\zeta,\\ (K^*\phi)(z) &:= -\frac{1}{\pi}\int_\Gamma \phi(\zeta)\frac{\partial}{\partial n_z}\log|z-\zeta|\,ds_\zeta,\\ (Wv)(z) &:= \frac{1}{\pi}\frac{\partial}{\partial n_z}\int_\Gamma v(\zeta)\frac{\partial}{\partial n_\zeta}\log|z-\zeta|\,ds_\zeta. \end{aligned}$$

The linear and boundary integral operators are continuous when mapping between the following Sobolev–spaces

$$\begin{aligned} V : H^{s-1/2}(\Gamma) &\to H^{s+1/2}(\Gamma), & K : H^{s+1/2}(\Gamma) &\to H^{s+1/2}(\Gamma),\\ K^* : H^{s-1/2}(\Gamma) &\to H^{s-1/2}(\Gamma), & W : H^{s+1/2}(\Gamma) &\to H^{s-1/2}(\Gamma), \end{aligned}$$

where $s \in [-1/2, 1/2]$ [13]. The single layer potential V and the hyper singular operator W are symmetric, the double layer potential K has the dual K^*. Both, V and W are strongly elliptic in the sense that they satisfy a G**a**rding inequality (in the above spaces with $s = 0$) [13].

There are various of formulae which characterise the Cauchy data $(u_2, \partial u_2/\partial n)|_\Gamma$ of a function u_2 with (3.4), (3.6) and we quote only one from the literature.

Lemma 1 ([15])**.** *Let $u_2 \in H^1_{loc}(\Omega_c)$ satisfy* (3.4) *and* (3.6)*, then $(\xi, \phi) := (u_2, \partial u_2/\partial n)|_\Gamma \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ satisfies*

$$2\begin{pmatrix} \xi \\ \phi \end{pmatrix} = \begin{pmatrix} I+K & -V \\ -W & I-K^* \end{pmatrix}\begin{pmatrix} \xi \\ \phi \end{pmatrix} + \begin{pmatrix} 2a \\ 0 \end{pmatrix}. \tag{3.7}$$

Conversely, for each $(\xi, \phi) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ there exists a function $u_2 \in H^1_{loc}(\Omega_c)$ with (3.4)*,* (3.6) *if and only if* (3.7) *holds. The function u_2 is given by the representation formula, for $x \in \Omega_c$,*

$$u_2(x) = \frac{1}{2\pi}\int_\Gamma \phi(z)\,\log|x-z|\,ds_z - \frac{1}{2\pi}\int_\Gamma \xi(z)\,\frac{\partial}{\partial n_z}\log|x-z|\,ds_z + a\,. \tag{3.8}$$

Notice that $W\,1 = 0 = (K+1)\,1$ (proved by (3.7) for $(\xi, \phi) = (1, 0)$ and $a = 1$).

The interface problem (3.3)-(3.6) is equivalent to the following weak formulation which is known as symmetric coupling method [13].

Weak Formulation: *Find $(\boldsymbol{u}, \boldsymbol{\phi}) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$, such that*

$$\mathcal{B}\left(\left(\begin{smallmatrix} u \\ \phi \end{smallmatrix}\right), \left(\begin{smallmatrix} v \\ \psi \end{smallmatrix}\right)\right) = \mathcal{L}\left(\begin{smallmatrix} v \\ \psi \end{smallmatrix}\right) \qquad \left(\left(\begin{smallmatrix} v \\ \psi \end{smallmatrix}\right) \in H^1(\Omega) \times H^{-1/2}(\Gamma)\right) \tag{3.9}$$

with

$$\mathcal{B}\left(\left(\begin{smallmatrix}u\\ \phi\end{smallmatrix}\right), \left(\begin{smallmatrix}v\\ \psi\end{smallmatrix}\right)\right) = 2\cdot\int_\Omega (\nabla u)^\top \cdot D\cdot \nabla v\,dx + \langle Wu + (K^*-1)\phi, v\rangle + \langle \psi, (K-1)u - V\phi\rangle$$

and

$$\mathcal{L}\left(\begin{smallmatrix}v\\ \psi\end{smallmatrix}\right) = 2\cdot\int_\Omega f\,v\,dx + \langle Wu_0 + (K^*+1)t_0, v\rangle + \langle \psi, (K-1)u_0 - Vt_0\rangle\,.$$

The problem (3.3)-(3.6) has a unique solution and so the equivalent problem (3.9) has a unique solution as well.

Notice that the variable u_2 is determined by (3.9) up to the additive constant a defined by (3.6), e.g. $u_1 = u + a$, $u_2|_\Gamma = u|_\Gamma - u_0 + a$ with (u_1, u_2) satisfying (3.3)-(3.6) and (u, ϕ) satisfying (3.9).

4. The Discrete Problem

In this section we consider the discretization of problem (3.9). Therefore we define finite dimensional vector spaces $H_h \subset H^1(\Omega)$, $H_h^{-1/2} \subset H^{-1/2}(\Gamma)$. Let H_h be the space of continuous piecewise linear functions of a quasi-uniform and regular triangulation of Ω and $H_h^{-1/2}$ be the space of piecewise constant functions on the discretization of the boundary Γ induced by H_h. Let η_i, $i = 1, \ldots, m$ be basis functions of H_h, i.e., $\mathrm{span}_{i=1,m}\{\eta_i\} = H_h$ and t_i, $i = 1, \ldots, n$ be basis functions of $H_h^{-1/2}$, i.e., $\mathrm{span}_{i=1,n}\{t_i\} = H_h^{-1/2}$. Furthermore, let $\boldsymbol{u} = (u_1, \ldots, u_m)^\top \in \mathbb{R}^m$, $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_n)^\top \in \mathbb{R}^n$ be the vectors with $u = \sum_{i=1}^m u_i\,\eta_i \in H_h$ and $\phi = \sum_{i=1}^n \phi_i\,t_i \in H_h^{-1/2}$. We define the discretization of the operators W, K, V, etc.

$$\begin{array}{ll} A_h := \left\{2\int_\Omega (\nabla\eta_i)^\top D\nabla\eta_j\,d\Omega\right\}_{i=1,m}^{j=1,m}, & W_h := \{\langle W\,\eta_i|_\Gamma, \eta_j|_\Gamma\rangle\}_{i=1,m}^{j=1,m}, \\ K_h := \{\langle t_i, K\,\eta_j|_\Gamma\rangle\}_{i=1,n}^{j=1,m}, & V_h := \{\langle V\,t_i, t_j\rangle\}_{i=1,n}^{j=1,n} \\ I_h := \{\langle t_i, \eta_j|_\Gamma\rangle\}_{i=1,n}^{j=1,m}. & \end{array}$$

Hence, we have (with $\approx$ denoting equivalence of norms)

$$\begin{array}{rclrcll} 2\int_\Omega (\nabla u)^\top D\nabla v\,d\Omega &=& \boldsymbol{u}^\top A_h \boldsymbol{v}, & 2\|\,D^{\frac12}\,\nabla u\,\|^2_{L^2(\Omega)} &=& \boldsymbol{u}^\top A_h \boldsymbol{u} & (u, v\in H_h), \\ \langle W\,u, v\rangle &=& \boldsymbol{u}^\top W_h \boldsymbol{v}, & \|\,u\,\|^2_{H^{\frac12}(\Gamma)\backslash\mathbb{R}} &\approx& \boldsymbol{u}^\top W_h \boldsymbol{u} & (u, v\in H_h), \\ \langle \phi, V\psi\rangle &=& \boldsymbol{\phi}^\top V_h \boldsymbol{\psi}, & \|\,\phi\,\|^2_{H^{-\frac12}(\Gamma)} &\approx& \boldsymbol{\phi}^\top V_h \boldsymbol{\phi} & (\phi, \psi\in H_h^{-\frac12}), \\ \langle \phi, (K-1)\,u\rangle &=& \boldsymbol{\phi}^\top (K_h - I_h)\,\boldsymbol{u} & (u\in H_h\,,\ \phi\in H_h^{-\frac12}). & & & \end{array}$$

5. Block–Diagonal Preconditioner

In the following we consider linear systems of equations with symmetric matrices of the form

$$\mathcal{A}_h = \begin{pmatrix} A_h + W_h & K_h^\top - I_h^\top \\ K_h - I_h & -V_h \end{pmatrix}. \tag{5.1}$$

Given the preconditioner

$$\mathcal{P}_h = \begin{pmatrix} P_{A_h} & 0 \\ 0 & P_{V_h} \end{pmatrix}, \tag{5.2}$$

where P_{A_h} and P_{V_h} are both symmetric and positive definite. Consequential, $\mathcal{P}_h$ is a symmetric and positive definite matrix, too. In [12, Thm. 3.2] it was proven that if the eigenvalues μ_i of a preconditioned matrix $\mathcal{P}_h^{-1/2}\mathcal{A}_h\mathcal{P}_h^{-1/2}$ lie in intervals of the form

$$[-\hat{a}, -\hat{b}] \cup [\hat{c}, \hat{d}] \tag{5.3}$$

with $\hat{a}-\hat{b}=\hat{d}-\hat{c}>0$, where $\hat{a}$, $\hat{b}$, $\hat{c}$ and $\hat{d}$ are positive constants, then the PCR convergence rate is

$$\left(\frac{\| \mathcal{P}_h^{-1/2}(\boldsymbol{b}-\mathcal{A}_h\boldsymbol{x}_k) \|_2}{\| \mathcal{P}_h^{-1/2}(\boldsymbol{b}-\mathcal{A}_h\boldsymbol{x}_0) \|_2}\right)^2 \leq 2\left(\frac{1-\sqrt{\hat{b}\hat{c}/\hat{a}\hat{d}}}{1+\sqrt{\hat{b}\hat{c}/\hat{a}\hat{d}}}\right)^k \tag{5.4}$$

where $\|\cdot\|_2$ denotes the Euclidean norm and $\boldsymbol{x}_k$ is the kth iterate.

In the following we will use Theorem 1 to estimate the extreme eigenvalues of the preconditioned matrix $\overline{\mathcal{A}}_h$

$$\overline{\mathcal{A}}_h := \mathcal{P}_h^{-1/2}\mathcal{A}_h\mathcal{P}_h^{-1/2} = \begin{pmatrix} P_{A_h}^{-1/2}(A_h+W_h)P_{A_h}^{-1/2} & P_{A_h}^{-1/2}(K_h-I_h)P_{V_h}^{-1/2} \\ P_{V_h}^{-1/2}(K_h^\top-I_h^\top)P_{A_h}^{-1/2} & -P_{V_h}^{-1/2}V_hP_{V_h}^{-1/2} \end{pmatrix}, \tag{5.5}$$

in order to estimate the convergence of the PCR-method.

We obtain from Sylvester's law of inertia together with the congruent transform (1.3) of $\mathcal{A}_h$ that the matrix $\mathcal{A}_h$ has m positive and n negative eigenvalues. We get the same result for $\overline{\mathcal{A}}_h$, if we apply this law again to the transform (5.5). We denote the eigenvalues of $\overline{\mathcal{A}}_h$ by

$$\mu_{-n}\leq\mu_{-n+1}\leq\ldots\leq\mu_{-1}<0<\mu_1\leq\mu_2\leq\ldots\leq\mu_m\,.$$

We define bounds of the spectrum of $P_{V_h}^{-1/2}V_hP_{V_h}^{-1/2}$ which may depend on the mesh size by

$$0<\lambda_{\min}(h)\leq\frac{\boldsymbol{x}^\top V_h\boldsymbol{x}}{\boldsymbol{x}^\top P_{V_h}\boldsymbol{x}}\leq\lambda_{\max}(h) \qquad (\boldsymbol{x}\in\mathbb{R}^n\setminus\{0\})\,. \tag{5.6}$$

Since A_h is positive semi-definite we add a symmetric matrix T_h, s.t. A_h+T_h is positive definite. In our model problem we can choose for example $T_h=W_h+\gamma Id_h^\top D_h^{-1}Id_h$ $(\gamma>0)$ where $D_h:=\{t_i,t_j\}_{i,j=1,\ldots,n}$ denotes a diagonal matrix, or let $T_h=M_h:=\{\int_\Omega \eta_i\eta_j\,d\Omega\}_{i,j=m}$ be the mass matrix. Since P_{A_h} and A_h are both symmetric and positive definite, we can define constants depending on the mesh size $h=h(n)$ to bound the extreme eigenvalues

$$0<\eta_{\min}(h)\leq\frac{\boldsymbol{x}^\top(A_h+T_h)\boldsymbol{x}}{\boldsymbol{x}^\top P_{A_h}\boldsymbol{x}}\leq\eta_{\max}(h) \qquad (\boldsymbol{x}\in\mathbb{R}^m\setminus\{0\})\,. \tag{5.7}$$

For our assertion in the following Theorem 2 we have to show, that there exist some constants θ, Θ, and Δ, which are independent of the mesh size h, satisfying for all $\boldsymbol{x}\in\mathbb{R}^m\setminus\{0\}$

$$\frac{\boldsymbol{x}^\top(A_h+W_h)\boldsymbol{x}}{\boldsymbol{x}^\top(A_h+T_h)\boldsymbol{x}}\leq\Delta\,, \qquad \frac{\boldsymbol{x}^\top\{(K_h^\top-I_h^\top)V_h^{-1}(K_h-I_h)\}\boldsymbol{x}}{\boldsymbol{x}^\top(A_h+T_h)\boldsymbol{x}}\leq\Theta^2 \quad \text{, and} \tag{5.8}$$

$$\theta^2\leq\frac{\boldsymbol{x}^\top\{A_h+W_h+(K_h^\top-I_h^\top)V_h^{-1}(K_h-I_h)\}\boldsymbol{x}}{\boldsymbol{x}^\top(A_h+T_h)\boldsymbol{x}} \tag{5.9}$$

In the following we motivate and prove the existence of these constants for both before mentioned choices of T_h, i.e. $T_h=M_h$ and $T_h=W_h+\gamma Id_h^\top D_h^{-1}Id_h$.

5.1. **Existence of θ, Θ, and Δ for choice $T_h=M_h$.**

In the sequel c_i $(i=1,2,\ldots)$ will denote positive constants which are independent of mesh size h. First we show the existence a positive constant Δ satisfying (5.8) independent of the mesh size h. Such an estimate can be obtained directly by estimating the numerator and denominator of (5.8) as follows: For all $u\in H_h$ there holds with d_0 from (3.1)

$$\boldsymbol{u}^\top A_h\boldsymbol{u}=\frac{1}{2}\| D^{1/2}\nabla u\|^2_{L^2(\Omega)}\leq\frac{d_0}{2}\| D^{1/2}\nabla u\|^2_{L^2(\Omega)} \quad\text{and}\quad \boldsymbol{u}^\top W_h\boldsymbol{u}\approx\| u\|^2_{H^{1/2}(\Gamma)/\mathbb{R}}$$

This yields with a trace mapping that there exists a positive constant c_1 independent of mesh size h s.t.

$$\boldsymbol{u}^\top(A_h+W_h)\boldsymbol{u}\leq c_1\| u\|^2_{H^1(\Omega)}\,.$$

Furthermore with the choice $T_h = M_h$ there exists a positive constant c_2 independent of h satisfying

$$\boldsymbol{u}^\top (A_h + M_h)\boldsymbol{u} \geq c_2 \| u \|^2_{H^1(\Omega)} . \tag{5.10}$$

Altogether this shows $\Delta \leq c_1/c_2$. Assuming that the capacity of the boundary Γ is smaller than one, the single layer potential is positive definite on $H^{-1/2}(\Gamma)$ (see, e.g. [13, 15]). Next we employ the facts, that V is bijective from $H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ under the assumption that $\text{cap}(\Gamma) < 1$ and that $(K - I)$ is a continuous mapping from $H^{1/2}(\Gamma)$ onto itself and the trace operator from $H^1(\Omega) \to H^{1/2}(\Gamma)$ is continuous. Furthermore, for $u \in H_h$ there holds with positive and mesh independent constant c_3, c_4, and c_5

$$\boldsymbol{u}^\top (K_h^\top - I_h^\top) V_h^{-1} (K_h - I_h)\, \boldsymbol{u} \leq c_3 \| (K-1)u \|^2_{H^{\frac{1}{2}}(\Gamma)} \leq c_4 \| u \|^2_{H^{\frac{1}{2}}(\Gamma)} \leq c_5 \| u \|^2_{H^1(\Omega)} . \tag{5.11}$$

This yields together with (5.10), that there exists a constant $\Theta \in \mathbb{R}$ independent of h with

$$\frac{\boldsymbol{x}^\top \{(K_h^\top - I_h^\top) V_h^{-1} (K_h - I_h)\} \boldsymbol{x}}{\boldsymbol{x}^\top (A_h + M_h) \boldsymbol{x}} \leq \Theta^2 \qquad (\boldsymbol{x} \in \mathbb{R}^n \setminus \{0\}) . \tag{5.12}$$

In [13] it is proven, that we can define an extension operator T, with the following properties. Given $v \in H^{1/2}(\Gamma)$, then there exists a $w := Tv$ with $w = v$ on Γ and

$$\| w \|_{H^1(\Omega)} \leq c_6 \| v \|_{H^{\frac{1}{2}}(\Gamma)} . \tag{5.13}$$

Now we have for all $u \in H^1(\Omega)$, if we use the first Friedrich's inequality and (5.13)

$$\begin{aligned} \| u \|_{H^1(\Omega)} &\leq \| u - Tu|_\Gamma \|^2_{H^1(\Omega)} + \| Tu|_\Gamma \|^2_{H^1(\Omega)} \leq c_7 |u - Tu|_\Gamma|^2_{H^1(\Omega)} + \| Tu|_\Gamma \|^2_{H^1(\Omega)} \\ &\leq \; c_7 |u|^2_{H^1(\Omega)} + 3 \| Tu|_\Gamma \|^2_{H^1(\Omega)} \leq c_8 \left(|u|^2_{H^1(\Omega)} + \| u \|^2_{H^{\frac{1}{2}}(\Gamma)} \right) , \end{aligned} \tag{5.14}$$

with $c_8 = \max\{c_7, 3\, c_6^2\}$. It is known, that the operator $W + (K^\top - 1)V^{-1}(K - 1) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is elliptic on $H^{1/2}(\Gamma)$ [11]. Hence we have together with (5.14) for all $u \in H_h$ the inequality

$$\| u \|^2_{H^1(\Omega)} \leq c_8 \left(|u|^2_{H^1(\Omega)} + \| u \|^2_{H^{\frac{1}{2}}(\Gamma)} \right) \leq c_9\, \boldsymbol{u}^\top \{ A_h + W_h + (K_h^\top - I_h^\top) V_h^{-1} (K_h - I_h) \}\, \boldsymbol{u} . \tag{5.15}$$

From the equivalence of $\| u \|^2_{H^1(\Omega)}$ and $\boldsymbol{u}^\top (A_h + M_h)\boldsymbol{u}$ for $u \in H_h$ we conclude the existence of $\theta \in \mathbb{R}$ with

$$\theta^2 \leq \frac{\boldsymbol{x}^\top \{ A_h + W_h + (K_h^\top - 1) V_h^{-1} (K_h - 1) \} \boldsymbol{x}}{\boldsymbol{x}^\top (A_h + M_h) \boldsymbol{x}} \qquad (\boldsymbol{x} \in \mathbb{R}^n \setminus \{0\}) \tag{5.16}$$

which is independent of h.

5.2. **Alternative choice $T_h = W_h + \gamma Id_h^\top D_h^{-1} Id_h$.** Let $\gamma > 0$ and $D_h = \{< t_i, t_j >\}_{i,j=1,\ldots,n}$ denote a diagonal matrix. Before we show the existence of constants Δ, θ, and Θ defined by (2.1) and (2.2) for $T_h = W_h + \gamma Id_h^\top D_h^{-1} Id_h$, we make a few preparations. Let $\Gamma_i \subset \Gamma$ denote the support of t_i ($i = 1, \ldots, n$), h_i its diameter, and $h_{\max} := \max_{i=1,\ldots,n}\{h_i\}$. For the integral mean $u_m =< u, 1 > /|\Gamma|$ of $u \in L^2(\Gamma)$ we get by straight forward calculation

$$\|u\|^2_{L^2(\Gamma)} = \|u - u_m\|^2_{L^2(\Gamma)} + < u, 1 >^2 /|\Gamma| . \tag{5.17}$$

Using Hölder's-inequality and the definitions of I_h and D_h we deduce

$$< u, 1 >^2 = \left(\sum_{i=1}^n h_i^{1/2} h_i^{-1/2} \int_{\Gamma_i} u \, dx \right)^2 \leq |\Gamma| \sum_{i=1}^n h_i^{-1} \left(\int_{\Gamma_i} u \, dx \right)^2 \leq |\Gamma|\, \boldsymbol{u}^\top I_h^\top D_h^{-1} I_h \boldsymbol{u} . \tag{5.18}$$

By using the two last results (5.17) and (5.18) we get for $u_h \in H_h|_\Gamma \subset H^{1/2}(\Gamma)$ and using a standard approximation property $\|u - u_m\|^2_{L^2(\Gamma)} \leq c_9\, h_{\max} \|u\|^2_{H^{1/2}(\Gamma)\setminus\mathbb{R}} \leq c_{10}\, h_{\max}^{1/2} \boldsymbol{u}^\top W_h \boldsymbol{u}$

$$\|u\|^2_{L^2(\Gamma)} = \|u - u_m\|^2_{L^2(\Gamma)} + < u, 1 >^2 /|\Gamma| \leq c_{10}\; h_{\max}\, \boldsymbol{u}^\top W_h \boldsymbol{u} + \boldsymbol{u}^\top I_h^\top D_h^{-1} I_h \boldsymbol{u} . \tag{5.19}$$

Using this result and the compactness property of K as in (5.11) we obtain

$$\begin{aligned}(5.20)\quad \boldsymbol{u}^\top(K_h^\top - I)V_h^{-1}(K_h - I)\boldsymbol{u} &\le c_{11}\|(K-I)u\|^2_{H^{1/2}(\Gamma)} \le c_{12}\|u\|^2_{H^{1/2}(\Gamma)} \\ &\le c_{13}\|u\|^2_{H^{1/2}(\Gamma)\backslash\mathbb{R}} + \|u\|^2_{L^2(\Gamma)} \le c_{14}(1 + c_{10}\, h_{\max})\,\boldsymbol{u}^\top W_h \boldsymbol{u} + c_{13}\,\boldsymbol{u}^\top I_h^\top D_h^{-1} I_h \boldsymbol{u}\,.\end{aligned}$$

That proves the existence of a positive constant $\Delta < \max\{c_{13}, c_{14}(1 + c_{10}|\Gamma|)\}$ which is bounded independent of h. Since $\gamma I_h^\top D_h^{-1} I_h$ is positive semidefinite, we obtain

$$\boldsymbol{x}^\top(A_h + W_h)\boldsymbol{x} \le \boldsymbol{x}^\top(A_h + W_h + \gamma I_h^\top D_h^{-1} I_h)\boldsymbol{x} \quad (\boldsymbol{x} \in \mathbb{R}^m)\,.$$

Hence, we can choose $\Delta = 1$ in (5.8) independently of the mesh size h. Taking into account that $I_h^\top D_h^{-1} I_h$ can be rewritten as sum of local quantities as in (5.18), we get by using Hölder's inequality

$$\boldsymbol{u}^\top(I_h^\top D_h^{-1} I_h)\boldsymbol{u} = \sum_{i=1}^{n} h_i^{-1}\left(\int_{\Gamma_i} u\,dx\right)^2 \le \sum_{i=1}^{n}\int_{\Gamma_i} u^2\,dx = \boldsymbol{u}\top M_h \boldsymbol{u}$$

Hence, we obtain for $u \in H_h$

$$\boldsymbol{u}^\top(A_h + W_h + \gamma I_h^\top D_h^{-1} I_h)\boldsymbol{u} \le \boldsymbol{u}^\top(A_h + W_h + \gamma M_h)\boldsymbol{u} \le c_{15}\max\{1,\gamma\}\|u\|_{H^1(\Omega)}.$$

Together with (5.15) this proves, that there exists a mesh size independent constant $\theta^2 \ge c_9\, c_{15}\max\{1,\gamma\} > 0$.

What will be a good choice of γ? Let $\mathbf{1}$ denote a generic 1-vector, i.e. $\mathbf{1} = (1,\ldots,1)^\top$. Since $\operatorname{kern}\{A_h + W_h\} = \{\mathbf{1}\}$, we expect

$$\gamma \approx \frac{\mathbf{1}^\top(K_h^\top - I_h)V_h^{-1}(K_h - I_h)\mathbf{1}}{\mathbf{1}^\top I_h^\top D_h^{-1} I_h \mathbf{1}}$$

to be a good choice. Note, from the representation formula (3.7) we get $K1 = -1$ and numerical observations show $(\mathbf{1}^\top I_h V_h^{-1} I_h \mathbf{1})/(\mathbf{1}^\top I_h D_h^{-1} I_h \mathbf{1}) \approx (\mathbf{1}^\top D_h \mathbf{1})/(\mathbf{1}^\top V_h \mathbf{1}) = |\Gamma|/(\mathbf{1}^\top V_h \mathbf{1})$ (see Section 7 below for numerical experiments). Therefore, we get

$$\mathbf{1}^\top(K_h^\top - I_h)V_h^{-1}(K_h - I_h)\mathbf{1} = 4\cdot\mathbf{1}^\top I_h V_h^{-1} I_h \mathbf{1} \approx \frac{4|\Gamma|}{\mathbf{1}^\top V_h \mathbf{1}}\mathbf{1}^\top I_h D_h^{-1} I_h \mathbf{1}$$

Hence, we end up with

$$(5.21)\qquad \gamma = \frac{4|\Gamma|}{\mathbf{1}^\top V_h \mathbf{1}} = \frac{-\,4\,\pi\,|\Gamma|}{\int_\Gamma\int_\Gamma \log|x-y|\,ds_x ds_y}\,.$$

Note, that $\mathbf{1}^\top V_h \mathbf{1}$ and also $|\Gamma|$ are independent of the discretisation.

From Subsections 5.1 and 5.2 we deduce the following lemma.

Lemma 2. *Let T_h be M_h or $W_h + \gamma I_h D_h^{-1} I_h$ $(\gamma > 0)$. Then, there exist positive constants Δ, θ, and Θ satisfying* (5.8) *and* (5.9) *independent of the mesh size h and the discretisation of Γ.*

If Δ, θ, and Θ are independent from the mesh size h, Theorem 1 gives bounds for the spectrum of $\mathcal{P}_h^{-1/2}\mathcal{A}_h\mathcal{P}_h^{-1/2}$ depending only on the extreme eigenvalues of $P_{A_h}^{-1}(A_h + T_h)$ (5.7) and $P_{V_h}^{-1} V_h$ (5.6). Using (5.4), also the rate of convergence of the PCR-method can be bounded by the extreme eigenvalues (5.6) and (5.7). With Lemma 2, we get from Theorem 1 the following result.

Theorem 2. *Let P_{A_h}, P_{V_h} be spectral equivalent to $A_h + T_h$ and V_h, respectively. Then, there exist independently of the mesh size h positive constants $\hat{a}$, $\hat{b}$, $\hat{c}$, and $\hat{d}$ such that the extreme eigenvalues of the preconditioned matrix $\mathcal{P}_h^{-1/2}\mathcal{A}_h\mathcal{P}_h^{-1/2}$ lie in intervals of the form* (5.3)*. Hence, the convergence rate of the PCR-method applied to $\mathcal{A}_h$ with preconditioner $\mathcal{P}_h$ is bounded independently of the mesh size h.*

Remark 1. *i.) In the situation that there holds only $\eta_{\max} = 1$ and $\eta_{\min}(h) \to 0$ as $h \to 0$, and P_{V_h} is spectrally equivalent to V_h the eigenvalues of the preconditioned matrix lie in the union of the intervals*

$$\left[-\frac{1}{2}\left(\lambda_{\max} + \sqrt{\lambda_{\max}^2 + 4\lambda_{\max}\Theta^2}\right), -\lambda_{\min}\right]$$
$$\cup \left[\mathcal{O}(\sqrt{\eta_{\min}(h)}), \frac{1}{2}\left(-(\lambda_{\min} - \Delta) + \sqrt{(\lambda_{\min} + \Delta)^2 + 4\lambda_{\max}\Theta^2}\right).\right]$$

This leads to a asymptotic convergence rate of the PCR method

$$\lim_{k\to\infty} \frac{\| \mathcal{P}_h^{-1/2}(\boldsymbol{b} - \mathcal{A}\boldsymbol{x}^k) \|_2}{\| \mathcal{P}_h^{-1/2}(\boldsymbol{b} - \mathcal{A}\boldsymbol{x}^0) \|_2} = 1 - \mathcal{O}(\sqrt{\lambda_{\min}})$$

where we used Lebedev's results [25].
ii.) Using the assumptions on the preconditioners as mentioned in i. If we choose P_{A_h} as the diagonal of A_h we get $\eta_{\min}(h) = h^2$. Hence $\mathcal{O}(h^{-1})$ preconditioned PCR iterates are required for convergence.
iii. Numerical results ([17]*) indicate that also the QMR, bi-CGstab, and GMRES methods* [4] *when applied the diagonal preconditioned matrix $\mathcal{P}_h^{-1/2}\mathcal{A}_h\mathcal{P}_h^{-1/2}$ bounded number of iterations (independent of the meshsize h) to reduce the error to a given tolerance.*

6. Multigrid algorithm for single layer potential V.

We assumed that there exists symmetric matrices P_A and P_C, which are spectrally equivalent to the Schur complement $A + H + B^\top C^{-1}B$ and C. In the following numerical examples we used multigrid algorithms. The h-independence of these methods is well known (e.g. see [6, 8, 9, 10, 22, 33]). There exists also alternative approaches as well for the finite elment part as for the boundary element part. (See e.g. [5, 16, 19, 28, 31, 29] and the references therein.)

For completeness we present a multigrid V-cycle algorithm for single layer potential V which will be used in the following numerical examples. Since V is an pseudo-differential operator of order -1 the smoothing step has to be modified in comparison to standard multigrid algorithms. For the analysis we refer to [6] where the three dimensional case is considered.

We assume that a coarse triangulation $\mathcal{T}_1$ of Ω is given and develop a sequence of nested triangulations of Ω in the usual way. Successively finer triangulations $\mathcal{T}_k$ for $k > 1$ are defined by subdividing each triangle into four by connecting the midpoints of the edges. Each triangulation $\mathcal{T}_k$ induces a triangulation $\mathcal{G}_k$ of the boundary Γ in an obvious way.

We let the number of levels in the multigrid algorithm be determined by $J \geq 1$ and define $\mathcal{M}_k$ $(k = 1, \ldots, J)$ to be the space of functions which are piecewise constant with respect to the triangulation $\mathcal{G}_k$. Since the triangulations $\mathcal{T}_k$ are nested and consequently $\mathcal{G}_k$, it follows that

$$\mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots \subset \mathcal{M}_J .$$

For $k = 1, \ldots, J$, let t_i^k, $i = 1, \ldots, n_k$ be basis functions of $\mathcal{M}_k$, $h_i^k = \text{diam}\,(\text{supp}\,(t_i^k))$, $h_{\max}^k = \max\limits_i h_i^k$, and the coarsening matrix

$$P_k = \{p_{ij}^k\}_{i=1,n_k}^{j=1,n_{k+1}}, \quad p_{ij}^k \text{ be defined by } t_i^k = \sum_{j=1}^{n_{k+1}} p_{ij}^k\, t_j^{k+1} .$$

The space $\mathcal{S}_k^1$ is defined to be the space of functions which are affine between two successive midpoints with respect to the triangulation $\mathcal{G}_k$ and continuous on Γ. Also, we define $\mathcal{S}_k^2$ to be the space of splines of order three. A plot of the B-spline basis functions $\phi_i^{k,\mu}$ $(i = 1, \ldots, n_k)$ for $\mathcal{S}_k^\mu$ $(\mu = 1, 2)$ is given in Fig. 1. Note, that the basis functions $\phi_i^{k,\mu}$ form a decomposition of unity.

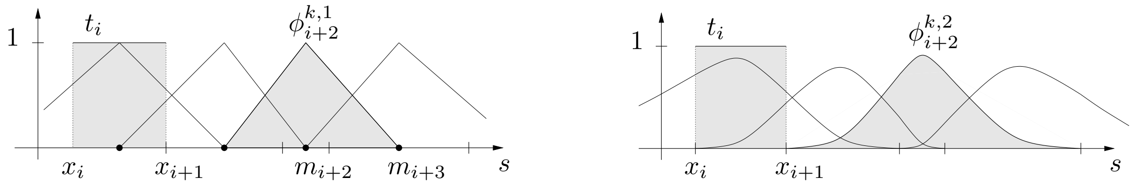

FIGURE 1. B-spline basis functions $\phi_i^{k,1}$ $(i = 1, \ldots, n_k)$ for $\mathcal{S}_k^1$ (left) and $\phi_i^{k,2}$ for $\mathcal{S}_k^2$ (right)

In the following we neglect the upper index to keep the notation simple. For $k = 1, \ldots, J$ we define

$$L_k := \operatorname{diag}\{h_1, \ldots, h_{n_k}\}, \quad D_{k,\mu} := \{\langle \left(1 + \frac{d^2}{ds^2}\right) \phi_i^{k,\mu}, \phi_j^{k,\mu} \rangle\}_{i,j=1,n_k} \tag{6.1}$$

$$M_{k,\mu} := \{\langle t_i^k, \phi_j^{k,\mu} \rangle\}_{i,j=1,n_k}, \quad V_k := \{ \langle V t_i^k, t_j^k \rangle \}_{i,j=1,n_k}. \tag{6.2}$$

Further, let χ_k^* be an upper bound of

$$\chi_k = \sup_{\boldsymbol{x} \in \mathbb{R}^{n_k}} \frac{\boldsymbol{x}^\top V_k \boldsymbol{x}}{\boldsymbol{x}^\top M_{k,\mu} D_{k,\mu}^{-1} M_{k,\mu}^\top \boldsymbol{x}} \tag{6.3}$$

Algorithm $\mathbf{Mg}^V$: Matrix form of multigrid V-cyle algorithm for single layer potential with ν pre and post smoothing steps.

Set $\mathbf{Mg}_1^V = V_1^{-1}$. Assume $\mathbf{Mg}_{k-1}^V$ has been defined and define $\mathbf{Mg}_k^V\, \boldsymbol{y} \in \mathbb{R}^{n_k}$ as follows:

i) Set $\boldsymbol{x}^0 = 0$ and define $\boldsymbol{x}^\ell$ for $\ell = 1, \ldots, \nu$ by
$$\boldsymbol{x}^\ell = \boldsymbol{x}^{\ell-1} + \frac{1}{\chi_k^*} M_{k,\mu}^{-\top} D_{k,\mu} M_{k,\mu}^{-1} (\boldsymbol{y} - V_k \boldsymbol{x}^{\ell-1}).$$
i) Define $\boldsymbol{x}^{\nu+1} = \boldsymbol{x}^\nu + P_{k-1}^\top \boldsymbol{q}$ where
$$\boldsymbol{q} = \mathbf{Mg}_{k-1}^V\, P_{k-1}(\boldsymbol{y} - V_k \boldsymbol{x}^\nu).$$
iii) Finally, set $\mathbf{Mg}_k^V\, \boldsymbol{y} = \boldsymbol{x}^{2\nu+1}$ where $\boldsymbol{x}^\ell$ for $\ell = \nu+2, \ldots, 2\nu+1$ are defined by
$$\boldsymbol{x}^\ell = \boldsymbol{x}^{\ell-1} + \frac{1}{\chi_k^*} M_{k,\mu}^{-\top} D_{k,\mu} M_{k,\mu}^{-1} (\boldsymbol{y} - V_k \boldsymbol{x}^{\ell-1}).$$

Remark 2. *i.) The smoother* $\frac{1}{\chi_k^*} M_{k,\mu}^{-\top} D_{k,\mu} M_{k,\mu}^{-1}$ *is symmetric and hence* $\mathbf{Mg}^V$ *is a symmetric operator.*
ii.) Note, that $M_{k,\mu}$ *is sparse and diagonal dominant, so an approximate inverse can be computed by the Jacobi iteration. Numerical examples in Section 7 indicate that only a few iterations will be needed to keep the contraction number of the multigrid method bounded. See Tab. 9 and Tab. 10 below where only 3 Jacobi iterations were used.*
iii.) The following numerical examples show, that we get also very good results if we choose $\mu = 1$ *and substitute* $M_{k,\mu}$ *just by the diagonal matrix* L_k.

7. Numerical Example

The following numerical experiments illustrates the sharpness of the bounds given by Theorem 2 in case of a FEM-BEM coupling problem (3.9), underlines the choices of β in $T_h = \beta M_h$, resp. γ (5.21) in $T_h = W_h + \gamma Id_h^\top D_h^{-1} Id_h$, the smoother in $\mathbf{Mg}^V$, and highlights the quasi-optimal convergence rate of the PCR-method, independent of the mesh size h and mesh-structure.

FIGURE 2. L-shaped domain of Example 1 and Z-shaped domain of Example 2.

Example 1: (L-shaped domain, uniform mesh).
Let Ω be the L-shaped domain with vertices $(0,0)$, $(s,0)$, (s,s), $(-s,s)$, $(-s,-s)$, $(0,-s)$ with $s = 0.25$. Therefore the single layer potential operator V is positive definite. On a shape-regular mesh with triangular elements we use piecewise linear functions in Ω and piecewise constant functions on Γ for discretization. The coarsest triangulation consists of six have squares with edge length $h_1 = 0.25$ as depicted in Fig. 2 (left). The implementation is realized in Matlab in the way like [1, 2, 18].

In Tab. 1 we give the extreme eigenvalues of the unpreconditioned matrix $\mathcal{A}_h$ (5.1) and the resulting behavior of the bounds in terms of the mesh size h.

TABLE 1. Extreme eigenvalues of the unpreconditioned matrix $\mathcal{A}_h$.

h/h_1	-ve ev's min / max		+ve ev's min / max	
1/1	-0.6194194	-0.014306039	0.4035585	9.280412
1/2	-0.2813471	-0.003824051	0.1689828	13.20866
1/4	-0.1216139	-0.000998319	0.0630067	15.08379
1/8	-0.0515328	-0.000255008	0.0214023	15.73712
1/16	-0.0219900	-0.000064265	0.0067335	15.92962
1/32	-0.0096163	-0.000016083	0.0019840	15.98179
	$\mathcal{O}(h)$	$\mathcal{O}(h^2)$	$\mathcal{O}(h^2)$	$\mathcal{O}(1)$

As $P_{A_h}^{-1}$ we choose a symmetric multigrid V-cycle with Gauss-Seidel smoother applied to the matrix $A_h + T_h$ and as $P_{V_h}^{-1}$ we take the modified multigrid V-cycle algorithm $\mathbf{Mg^V}$ as explained in Section 6 with $\chi_k^* = 2.5/2^k$ $(k = 1, \dots, J)$ and $\frac{1}{\chi^*} L_k^{-1} D_{k,1} L_k^{-1}$ as smoother. In both cases we use one pre and post smoothung step on each level.

By solving generalized eigenvalue problems using the QZ algorithm [20] we approximated numerically the constants θ, Θ, Δ, λ_1, λ_n, η_1, and η_m. The values are given in Tab. 2 for $T_h = 50\,M_h$ and in Tab. 3 for $T_h = W_h + \gamma I d_h^\top D_h^{-1} I d_h$ (For γ see (5.21)). All results underline for both choices of T_h the h-independence of the constants θ, Θ, Δ, λ_1, λ_n, η_1, and η_m.

TABLE 2. Constants θ, Θ, Δ, λ_1, λ_n, η_1 and η_m ($T_h = 50M_h$)

			η_1	η_m	Δ	θ^2	Θ^2
h/h_1	λ_1	λ_n	$T_h = 50M_h$				
1/1	1.00000	1	1.00000	1	1.31705	1.04108	2.21626
1/2	0.94793	1	0.73407	1	1.44296	0.90202	2.31344
1/4	0.96172	1	0.75646	1	1.54046	0.88215	2.38436
1/8	0.96266	1	0.75627	1	1.59630	0.87648	2.42884
1/16	0.96275	1	0.75631	1	1.63252	0.87492	2.45798
1/32	0.96276	1	0.75629	1	1.65821	0.87449	2.47739

In Tab. 4, resp. 5 we give the extreme eigenvalues of the preconditioned matrix $\mathcal{P}_h^{-1} \cdot \mathcal{A}_h$ and in brackets its estimates by Theorem 2 using values from Tab. 2, resp. 3. The bounds for the

TABLE 3. Constants θ, Θ, Δ, η_1 and η_m ($T_h = W_h + \gamma I d_h^\top D_h^{-1} I d_h$)

h/h_0	$\eta_{\min}$	$\eta_{\max}$	Δ	θ^2	Θ^2
	$T_h = W_h + \gamma I d_h^\top D_h^{-1} I d_h$				
1/1	1.00000	1	1	0.84419	1.03581
1/2	0.80521	1	1	0.82932	1.05373
1/4	0.75072	1	1	0.82016	1.07151
1/8	0.71708	1	1	0.81680	1.08549
1/16	0.69343	1	1	0.81564	1.09695
1/32	0.68075	1	1	0.81526	1.11419

negative eigenvalues are very close to the exact values (relative error between estimated and exact value $< 5\%$).

TABLE 4. Extreme eigenvalues of the preconditioned matrix $\mathcal{P}_h^{-1} \cdot \mathcal{A}_h$ ($T_h = A_h + 50M_h$)

h/h_1	-ve ev's min / max		+ve ev's min / max	
	in brackets the estimated bounds by Theorem 1			
1/1	-2.04598 (-2.07043)	-1.00000 (-1.00000)	0.88180 (0.63625)	1.75550 (2.04490)
1/2	-2.04571 (-2.10107)	-0.95193 (-0.94793)	0.79933 (0.52267)	1.74285 (2.18207)
1/4	-2.04671 (-2.12307)	-0.96621 (-0.96172)	0.74581 (0.45632)	1.77405 (2.27672)
1/8	-2.04718 (-2.13671)	-0.97127 (-0.96266)	0.70534 (0.42116)	1.79372 (2.33322)
1/16	-2.04754 (-2.14559)	-0.97275 (-0.96275)	0.68945 (0.39916)	1.80994 (2.37003)
1/32	-2.04777 (-2.15148)	-0.97286 (-0.96276)	0.67847 (0.39057)	1.82286 (2.39583)

TABLE 5. Extreme eigenvalues of the preconditioned matrix $\mathcal{P}_h^{-1} \cdot \mathcal{A}_h$ ($T_h = W_h + \gamma I d_h^\top D_h^{-1} I d_h$)

h/h_1	-ve ev's min / max		+ve ev's min / max	
	in brackets the estimated bounds by Theorem 1			
1/1	-1.63094 (-1.63393)	-1.00000 (-1.00000)	0.61307 (0.54603)	1.08685 (1.42681)
1/2	-1.60470 (-1.64180)	-0.95174 (-0.94793)	0.58339 (0.45212)	1.09834 (1.44107)
1/4	-1.59517 (-1.64956)	-0.96604 (-0.96172)	0.57321 (0.42653)	1.11860 (1.44518)
1/8	-1.59086 (-1.65563)	-0.97099 (-0.96266)	0.56680 (0.41058)	1.14644 (1.44992)
1/16	-1.58904 (-1.66058)	-0.97233 (-0.96275)	0.56343 (0.39967)	1.16660 (1.45391)
1/32	-1.58828 (-1.66798)	-0.97242 (-0.96276)	0.56176 (0.39386)	1.18543 (1.45990)

The rate of convergence of the PCR-method can be bounded by a function of $((\hat{b}\hat{c})/(\hat{a}\hat{d}))^{1/2}$ (see (5.4)). In Fig. 3 we show the dependence of $((\hat{b}\hat{c})/(\hat{a}\hat{d}))^{1/2}$ on β in (5.4) for $\mathcal{P}_h^{-1} \cdot \mathcal{A}_h$ and $T_h = A_h + \beta M_h$ and the rate of convergence of the PCR method. We get from the figure, that $\beta \approx 50$ minimizes the expression $((\hat{b}\hat{c})/(\hat{a}\hat{d}))^{1/2}$ for the computed meshsizes $h_1/h = 1, 2, 4, 8, 16$.

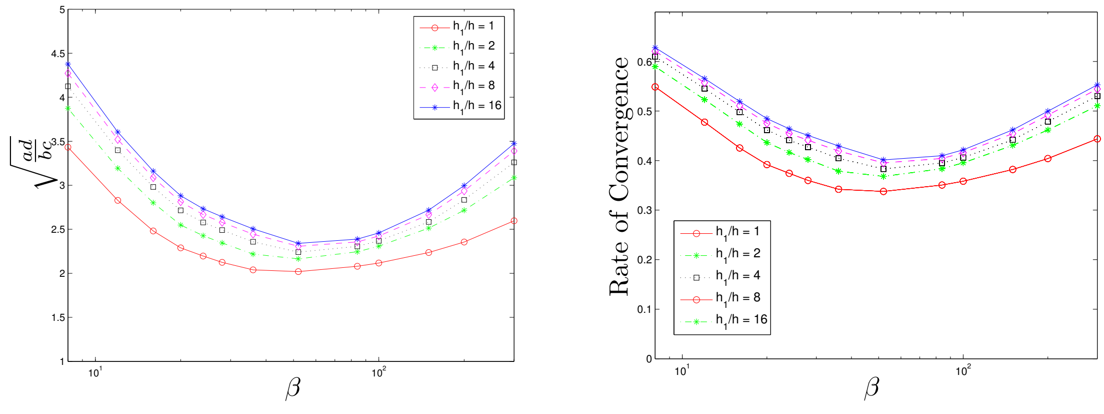

FIGURE 3. Dependence of the 'condition number' $((\hat{b}\hat{c})/(\hat{a}\hat{d}))^{1/2}$ of $\mathcal{P}_h^{-1} \cdot \mathcal{A}_h$ (left) and the rate of convergence of the PCR method (right) on β.

Example 2: (Z-shaped domain, quasi-uniform mesh).
In our next example we consider a domain with a stronger singularity at the reentrant corner and a non equally sized mesh as depicted in Fig. 2 (right). The vertices are $(0,0)$, $(0.5,0)$, $(0.15,0.22)$, $(0.6,0.2)$, $(0,0.4)$. We choose $P_{A_h}^{-1}$ and $P_{V_h}^{-1}$ to be the same multigrid V-cycle algorithms as in Example 1 and use $\nu = 1,3,5$ pre and post smoothing steps. For all computations we get $\lambda_n = \eta_m = 1$. In Tab. 6 and 7 we give the results for the Z-shaped domain.

TABLE 6. Lower bounds λ_1 and η_1 of the applied multigrid methods for different number of pre and postsmoothing steps ($T_h = 50M_h$ and $T_h = W_h + \gamma Id_h^\top D_h^{-1} Id_h$)

	λ_1			η_1 ($T_h = 50M_h$)			η_1 ($T_h = W_h + \gamma Id_h^\top D_h^{-1} Id_h$)		
h/h_1	$\nu = 1$	$\nu = 3$	$\nu = 5$	$\nu = 1$	$\nu = 3$	$\nu = 5$	$\nu = 1$	$\nu = 3$	$\nu = 5$
1/2	0.66961	0.94160	0.98419	0.46635	0.84235	0.92829	0.54526	0.89685	0.96725
1/4	0.64043	0.93618	0.98079	0.25122	0.68272	0.84646	0.32022	0.78705	0.92465
1/8	0.63512	0.93551	0.97887	0.22201	0.61625	0.80320	0.25767	0.69394	0.86920
1/16	0.63420	0.93543	0.97796	0.21464	0.59903	0.78006	0.23071	0.63883	0.82864
1/32	0.63405	0.93542	0.97753	0.21160	0.58992	0.77093	0.21915	0.61423	0.81013

The numerical experiments indicate that the contraction number of both V-cyles improves uniformly as the number of smoothing steps is increased and is bounded independently from the meshsize h. (For a prove see [10] for second order elliptic pde's.)

TABLE 7. Constants Δ, θ, and Θ, ($T_h = 50M_h$ and $T_h = W_h + \gamma Id_h^\top D_h^{-1} Id_h$)

	$T_h = 50M_h$			$T_h = W_h + \gamma Id_h^\top D_h^{-1} Id_h$		
h/h_1	Δ	θ^2	Θ^2	Δ	θ^2	Θ^2
1/1	1.62331	0.97108	11.22716	1	0.80743	3.32681
1/2	1.74237	0.94476	12.82658	1	0.79408	3.53328
1/4	1.81163	0.93656	14.10010	1	0.78772	3.82199
1/8	1.85523	0.93422	15.04025	1	0.78561	4.17655
1/16	1.88295	0.93359	15.73391	1	0.78496	4.63942
1/32	1.89515	0.93396	16.02164	1	0.78461	4.89112

All results underline for both choices of T_h the h-independence of the constants θ, Θ, Δ, λ_1, λ_n, η_1, and η_m.

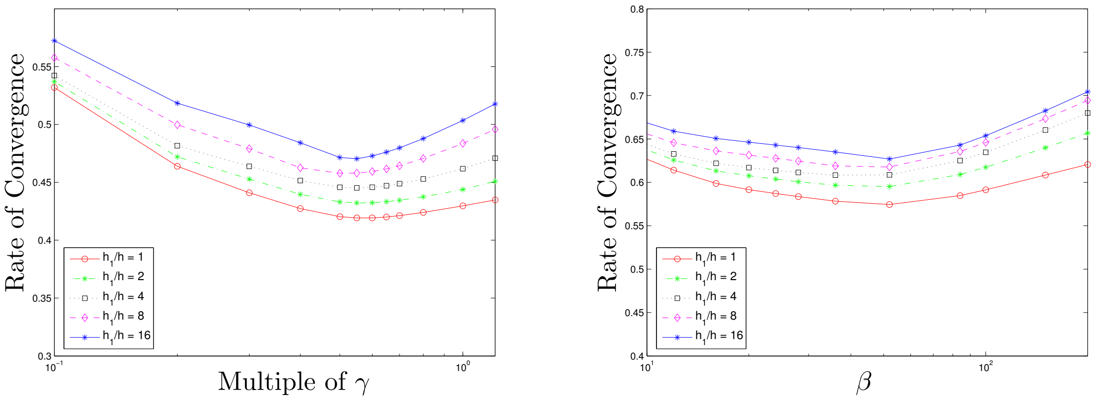

FIGURE 4. Dependence of the rate of convergence of the PCR method on γ using $T_h = W_h + \gamma Id_h^\top D_h^{-1} Id_h$ (left) resp. on β using $T_h = \beta M_h$ (right).

In Subsection 5.2 we stated the property

$$c_{19} \frac{\mathbf{1}^\top D_h \mathbf{1}}{\mathbf{1}^\top V_h \mathbf{1}} \leq \frac{\mathbf{1}^\top I_h V_h^{-1} I_h \mathbf{1}}{\mathbf{1}^\top I_h D_h^{-1} I_h \mathbf{1}} \leq c_{20} \frac{\mathbf{1}^\top D_h \mathbf{1}}{\mathbf{1}^\top V_h \mathbf{1}} = c_{20} |\Gamma| / (\mathbf{1}^\top V_h \mathbf{1})$$

For both examples and different meshsizes h the ratio between $(\mathbf{1}^\top I_h V_h^{-1} I_h \mathbf{1})/(\mathbf{1}^\top I_h V_h^{-1} I_h \mathbf{1})$ and $(\mathbf{1}^\top D_h \mathbf{1})/(\mathbf{1}^\top V_h \mathbf{1})$ is given in Tab. 8. Further numerical experiments on different domains show $c_{19} \le 0.9$ and $c_{20} \ge 1$.

Table 8. Ratio between $(\mathbf{1}^\top I_h V_h^{-1} I_h \mathbf{1})/(\mathbf{1}^\top I_h V_h^{-1} I_h \mathbf{1})$ and $(\mathbf{1}^\top D_h \mathbf{1})/(\mathbf{1}^\top V_h \mathbf{1})$

	$\frac{\mathbf{1}^\top I_h V_h^{-1} I_h \mathbf{1}}{\mathbf{1}^\top I_h V_h^{-1} I_h \mathbf{1}} / \frac{\mathbf{1}^\top D_h \mathbf{1}}{\mathbf{1}^\top V_h \mathbf{1}}$	
h/h_1	Example 1	Example 2
1/1	0.98215	0.95306
1/2	0.97590	0.94109
1/4	0.97256	0.93609
1/8	0.97127	0.93388
1/16	0.97075	0.93288
1/32	0.97055	0.93243

Example 3: (Different smoothers in $\mathbf{Mg}^V$).

In our numerical example we compare our three proposed smoother in the multigrid algorithm $\mathbf{Mg}^V$ with respect to the extreme eigenvalues λ_1, λ_n of $\mathbf{Mg}^V V_h$ for the geometries and meshes in Example 1 and 2. To execute the action $M_{k,1}^{-1}$, we use the Jacobi iteration; i.e. we replace $M_{k,1}^{-1}$ by $\widetilde{M}_{k,1}^{-1}$, which is the matrix representation of 3 Jacobi iterations applied to $M_{k,1}$. We computed for all smoothers the damping parameter χ_k (6.3) and the resulting extreme eigenvalues λ_1, λ_n. Numerical experiments show, that the χ_k given by (6.3) is not optimal. By bisection we computed an optimal parameter χ_k (given in the last row), which minimizes the contraction number of the multigrid method $\mathbf{Mg}^V$, i.e. maximizes λ_1 while $\lambda_n = 1$. The resulting minimal eigenvalue λ_1 is given in brackets.

In Tab. 9 we give the results for Example 1, resp. in Tab. 10 for Example 2. For all computations we get $\lambda_n = 1$.

Table 9. Constant λ_1 in Example 1 for different smoother.

	$\frac{1}{\chi_k} L_k^{-1} D_{k,1} L_k^{-1}$		$\frac{1}{\chi_k} \widetilde{M}_{k,1}^{-\top} D_{k,1} \widetilde{M}_{k,1}^{-1}$		$\frac{1}{\chi_k} \widetilde{M}_{k,2}^{-\top} D_{k,2} \widetilde{M}_{k,2}^{-1}$	
k = # level	$\chi_k/2^k$	λ_1	$\chi_k/2^k$	λ_1	$\chi_k/2^k$	λ_1
2	2.37308	0.96501 (0.96964)	1.32083	0.93219 (0.96674)	6.06185	0.79858 (0.89694)
3	2.30758	0.97893 (0.98045)	1.36793	0.92411 (0.96482)	6.33462	0.78011 (0.90203)
4	2.31010	0.98130 (0.98235)	1.38358	0.91275 (0.96688)	6.44986	0.76161 (0.89422)
5	2.30250	0.98218 (0.98332)	1.38792	0.90856 (0.96874)	6.49317	0.75410 (0.89076)
6	2.30178	0.98251 (0.98384)	1.38906	0.90719 (0.96910)	6.50737	0.75130 (0.88948)
optimal	2.10867		1.17063		4.88401	

All three smoothers used with $\mathbf{Mg}^V$ provide constant bounds for the spectrum, which do not depend on the meshsize h. All three smoothers can be used to construct an efficient multigrid method for the single layer potential V; however the smoothers $M_{k,\mu}^{-\top} D_{k,1} M_{k,\mu}^{-1}$ require twice the appilaction of $M_{k,\mu}^{-1}$, resp. $M_{k,\mu}^{-\top}$ compared with used a diagonal scaling when using $L_k^{-1} D_{k,1} L_k^{-1}$.

8. Inner–Outer Iteration

In this section we discuss the convergence of an inner-outer iteration, which can be written for a general system $\mathcal{A}x = b$ as follows:

TABLE 10. Constant λ_1 in Example 2 for different smoother.

k = # level	$\frac{1}{\chi_k} L_k^{-1} D_{k,1} L_k^{-1}$		$\frac{1}{\chi_k} \widetilde{M}_{k,1}^{-\top} D_{k,1} \widetilde{M}_{k,1}^{-1}$		$\frac{1}{\chi_k} \widetilde{M}_{k,2}^{-\top} D_{k,2} \widetilde{M}_{k,2}^{-1}$	
	$\chi_k/2^k$	λ_1	$\chi_k/2^k$	λ_1	$\chi_k/2^k$	λ_1
2	3.72814	0.49728 (0.63756)	1.88333	0.55408 (0.69623)	7.72710	0.54991 (0.70662)
3	3.52539	0.48804 (0.67501)	1.84899	0.53167 (0.71827)	7.81520	0.52436 (0.68449)
4	3.52403	0.48541 (0.66984)	1.84878	0.52720 (0.71923)	8.25247	0.50133 (0.67973)
5	3.52397	0.48493 (0.66894)	1.84876	0.52634 (0.71955)	8.46519	0.48847 (0.67886)
6	3.52397	0.48485 (0.66879)	1.84876	0.52620 (0.71956)	8.53971	0.48307 (0.67875)
optimal	2.31715		1.20125		5.47552	

Algorithm 1: Inner-Outer-Iteration applied to $\mathcal{A}x = b$

Let $\widetilde{\mathcal{A}}$ be 'close' to $\mathcal{A}$.
Choose initial guess $\boldsymbol{x}^0$
for $k = 0, 1, 2, \ldots$ until convergence do % begin outer iteration
  compute residual $\boldsymbol{r}^k = \boldsymbol{b} - \mathcal{A}\boldsymbol{x}^k$
  compute approximation $\widetilde{\boldsymbol{d}}^{k,s}$ to $\widetilde{\mathcal{A}}^{-1}\boldsymbol{r}^k$ by s iterations % inner iteration
  update $\boldsymbol{x}^{k+1} = \boldsymbol{x}^k + \widetilde{\boldsymbol{d}}^{k,s}$
end do % end outer iteration

The iteration was introduced by Axelsson and Vassilevski [3]. In the following we will analyze the convergence of this method for indefinite and symmetric matrix. First we give a general perturbation argument for perturbed linear systems of equations, which we will use in the sequel. We quote the convergence rate for the steepest descent method under consideration of perturbations. The steepest descent method will be used as inner iteration. After we have shown the convergence of the inner-outer iteration for indefinite and symmetric matrices in general, we prove the convergence of this method for discretized problems (3.9), independent of mesh size h.

8.1. **Preliminaries.** In the sequel we consider an indefinite and symmetric system of equations

$$\begin{pmatrix} A & B^\top \\ B & -C \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{pmatrix} \tag{8.1}$$

where $0 \le A \in \mathbb{R}^{m\times m}_{sym}$ and $0 < C \in \mathbb{R}^{n\times n}_{sym}$, $m-\mathrm{rank}(A) \le \mathrm{rank}(C)$, and $B \in \mathbb{R}^{n\times m}$ be given, s.t. the Schur complement $S := A + B^\top C^{-1} B$ is positive definite. Let $0 < P_C \in \mathbb{R}^{n\times n}_{sym}$ and $0 < P_{\widetilde{S}} \in \mathbb{R}^{m\times m}_{sym}$ 'close' to C, resp. $\widetilde{S} := A + B^\top P_C^{-1} B$ be given.

Now, let us motivate how to apply Algorithm 1 to system (8.1) with right hand side $\boldsymbol{b} = (\boldsymbol{b}_1, \boldsymbol{b}_2)$ and exact solution $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2)$ which is approximated after k outer iterations by $\boldsymbol{x}^k = (\boldsymbol{x}_1^k, \boldsymbol{x}_2^k)$ $(k = 0, 1, 2, \ldots)$. In each outer iteration the residuum $\boldsymbol{r} = (\boldsymbol{r}_1^k, \boldsymbol{r}_2^k)$ $(k = 0, 1, \ldots)$ given by

$$\begin{pmatrix} \boldsymbol{r}_1^k \\ \boldsymbol{r}_2^k \end{pmatrix} = \begin{pmatrix} \boldsymbol{b}_1 - A\boldsymbol{x}_1^k - B^\top \boldsymbol{x}_2^k \\ \boldsymbol{b}_2 - B\boldsymbol{x}_1^k + C\boldsymbol{x}_2^k \end{pmatrix} \tag{8.2}$$

is calculated and an approximation to the exact defect $(\boldsymbol{d}_1^k, \boldsymbol{d}_2^k) = (\boldsymbol{x}_1 - \boldsymbol{x}_1^k, \boldsymbol{x}_2 - \boldsymbol{x}_2^k)$ is calculated. Notice

$$\begin{pmatrix} \boldsymbol{r}_1^k \\ \boldsymbol{r}_2^k \end{pmatrix} = \begin{pmatrix} A(\boldsymbol{x}_1 - \boldsymbol{x}_1^k) + B^\top(\boldsymbol{x}_2 - \boldsymbol{x}_2^k) \\ B(\boldsymbol{x}_1 - \boldsymbol{x}_1^k) - C(\boldsymbol{x}_2 - \boldsymbol{x}_2^k) \end{pmatrix} = \begin{pmatrix} A & B^\top \\ B & -C \end{pmatrix} \begin{pmatrix} \boldsymbol{d}_1^k \\ \boldsymbol{d}_2^k \end{pmatrix} \tag{8.3}$$

Hence, $\boldsymbol{x}_i = \boldsymbol{d}_i^k + \boldsymbol{x}_i^k$ $(i = 1, 2)$ would give the exact solution. The inverse of $\left(\begin{smallmatrix} A & B^\top \\ B & -C \end{smallmatrix}\right)$ can be written as

$$\begin{pmatrix} A & B^\top \\ B & -C \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{r}_1^k \\ \boldsymbol{r}_2^k \end{pmatrix} = \begin{pmatrix} S^{-1}(\boldsymbol{r}_1^k + B^\top C^{-1}\boldsymbol{r}_2^k) \\ C^{-1}\left((BS^{-1}(\boldsymbol{r}_1^k + B^\top C^{-1}\boldsymbol{r}_2^k)) - \boldsymbol{r}_2^k\right) \end{pmatrix}. \tag{8.4}$$

Our modification of the inner-outer iteration to compute an approximation to $(\boldsymbol{d}_1^k, \boldsymbol{d}_2^k)$ resp. $(\boldsymbol{x}_1^k, \boldsymbol{x}_2^k)$ is to use two different approximations. Firstly, using a matrix P_C which is "close" to C. That gives an approximation $(\widetilde{\boldsymbol{d}}_1^k, \widetilde{\boldsymbol{d}}_2^k)$ satisfying

$$\text{(8.5)} \quad \begin{pmatrix} A & B^\top \\ B & -P_C \end{pmatrix} \begin{pmatrix} \widetilde{\boldsymbol{d}}_1^k \\ \widetilde{\boldsymbol{d}}_2^k \end{pmatrix} = \begin{pmatrix} \boldsymbol{r}_1^k \\ \boldsymbol{r}_2^k \end{pmatrix} \text{ resp. } \begin{pmatrix} \widetilde{\boldsymbol{d}}_1^k \\ \widetilde{\boldsymbol{d}}_2^k \end{pmatrix} = \begin{pmatrix} \widetilde{S}^{-1}(\boldsymbol{r}_1^k + B^\top P_C^{-1}\boldsymbol{r}_2^k) \\ P_C^{-1}\left((B\widetilde{S}^{-1}(\boldsymbol{r}_1^k + B^\top P_C^{-1}\boldsymbol{r}_2^k)) - \boldsymbol{r}_2^k\right) \end{pmatrix}.$$

Secondly, an approximation $(\widetilde{\boldsymbol{d}}_1^{k,s}, \widetilde{\boldsymbol{d}}_2^{k,s})$ to $(\widetilde{\boldsymbol{d}}_1^k, \widetilde{\boldsymbol{d}}_2^k)$ which will be calculated by s iterations of the steepest descent method, i.e. $\widetilde{\boldsymbol{d}}_1^{k,s}$ will be computed as approximation to $\widetilde{\boldsymbol{d}}_1^k = \widetilde{S}^{-1}(\boldsymbol{r}_1^k + B^\top P_C^{-1}\boldsymbol{r}_2^k)$ and $\widetilde{\boldsymbol{d}}_2^{k,s} = P_C^{-1}(B\widetilde{\boldsymbol{d}}_1^{k,s} - \boldsymbol{r}_2^k)$. Notice, $\boldsymbol{d}_i^k = \boldsymbol{x}_i - \boldsymbol{x}_i^k$ and $\boldsymbol{x}_i^{k+1} = \boldsymbol{x}_i^k + \widetilde{\boldsymbol{d}}_i^{k,s}$ $(i = 1,2)$. Hence, $\boldsymbol{x}_i - \boldsymbol{x}_i^{k+1} = \boldsymbol{d}_i^k - \widetilde{\boldsymbol{d}}_i^{k,s}$ and from (8.4) and (8.5) we get

$$\text{(8.6)} \qquad \begin{aligned} S\boldsymbol{d}_1^k &= \boldsymbol{r}_1^k + B^\top C^{-1}\boldsymbol{r}_2^k\,, & \boldsymbol{d}_2^k &= C^{-1}(B\boldsymbol{d}_1^k - \boldsymbol{r}_2^k) \\ \widetilde{S}\widetilde{\boldsymbol{d}}_1^k &= \boldsymbol{r}_1^k + B^\top P_C^{-1}\boldsymbol{r}_2^k\,, & \widetilde{\boldsymbol{d}}_2^k &= P_C^{-1}(B\widetilde{\boldsymbol{d}}_1^k - \boldsymbol{r}_2^k) \end{aligned}$$

Algorithm 2 applied to the system (8.1) with s inner iterations of the steepest descent method and the specific choice of approximations P_C and $P_{\widetilde{S}}$ can be written as pseudocode as follows

Algorithm 2: Inner-Outer-Iteration applied to (8.1)

Choose initial guess $\boldsymbol{x}^0 = (\boldsymbol{x}_1^0, \boldsymbol{x}_2^0)$
for $k = 0,1,2,\ldots$ % begin outer iteration
  $\boldsymbol{r}_1^k = \boldsymbol{b}_1 - A\boldsymbol{x}_1^k - B^\top\boldsymbol{x}_2^k$
  $\boldsymbol{r}_2^k = \boldsymbol{b}_2 - B\boldsymbol{x}_1^k + C\boldsymbol{x}_2^k$
  check convergence; continue if necessary
  $\boldsymbol{z}_1^0 = 0$
  $\boldsymbol{p}_1^0 = \boldsymbol{r}_1^k + B^\top P_C^{-1}\boldsymbol{r}_2^k$
  for $l = 0,1,2,\ldots,s-1$ **do** % begin inner iteration
    $\boldsymbol{q}_1^\ell = P_{\widetilde{S}}^{-1}\boldsymbol{p}_1^\ell$
    $\boldsymbol{a}_1^\ell = (A + B^\top P_C^{-1}B)\boldsymbol{q}_1^\ell$
    $\beta_\ell = (\boldsymbol{p}_1^\ell, \boldsymbol{q}_1^\ell)/(\boldsymbol{a}_1^\ell, \boldsymbol{q}_1^\ell)$
    $\boldsymbol{z}_1^{\ell+1} = \boldsymbol{z}_1^\ell + \beta_\ell\boldsymbol{q}_1^\ell$
    $\boldsymbol{p}_1^{\ell+1} = \boldsymbol{p}_1^\ell - \beta_\ell\boldsymbol{a}_1^\ell$
  end do % end inner iteration
  $\boldsymbol{x}_1^{k+1} = \boldsymbol{x}_1^k + \boldsymbol{z}_1^s$
  $\boldsymbol{x}_2^{k+1} = \boldsymbol{x}_2^k + P_C^{-1}(B\boldsymbol{z}_1^s - \boldsymbol{r}_2^k)$
end do % end outer iteration

Theorem 3. *Let the positive constants ξ_1, ξ_m, λ_1, λ_n, and τ be given such that the following inequalities hold for all $\boldsymbol{x} \in \mathbb{R}^m \setminus \{0\}$*

$$\text{(8.7)} \qquad \xi_1 \le \frac{\boldsymbol{x}^\top\widetilde{S}\boldsymbol{x}}{\boldsymbol{x}^\top P_{\widetilde{S}}\boldsymbol{x}} \le \xi_m$$

resp. $\boldsymbol{x} \in \mathbb{R}^n \setminus \{0\}$

$$\text{(8.8)} \qquad \lambda_1 \le \frac{\boldsymbol{x}^\top C\boldsymbol{x}}{\boldsymbol{x}^\top P_C\boldsymbol{x}} \le \lambda_n\,, \qquad \frac{\boldsymbol{x}^\top(BS^{-1}B^\top)\boldsymbol{x}}{\boldsymbol{x}^\top C\boldsymbol{x}} \le \tau^2\,.$$

Furthermore, let $[\lambda_1, \lambda_n] \subset [\alpha^{-1}, \alpha]$ with $\alpha > 1$. Then, the iterates $(\boldsymbol{x}_1^k, \boldsymbol{x}_2^k)$ $(k = 0,1,2,\ldots)$ of Algorithm 2 satisfy

$$\text{(8.9)} \quad \|\, S^{1/2}(\boldsymbol{x}_1 - \boldsymbol{x}_1^{k+1}) \,\|_2 + \|\, C^{1/2}(\boldsymbol{x}_2 - \boldsymbol{x}_2^{k+1}) \,\|_2 \le \rho\left(\|\, S^{1/2}(\boldsymbol{x}_1 - \boldsymbol{x}_1^k) \,\|_2 + \|\, C^{1/2}(\boldsymbol{x}_2 - \boldsymbol{x}_2^k) \,\|_2\right)$$

with rate of convergence

$$\rho \quad = 2\alpha(1+\tau\,\alpha)\left(\alpha-1+\left(\tfrac{\xi_m-\xi_1}{\xi_m+\xi_1}\right)^s\right)+(\alpha-1)\,\alpha\left(1+2\tau\,+2\tau\,\left(\tfrac{\xi_m-\xi_1}{\xi_m+\xi_1}\right)^s\right). \tag{8.10}$$

Remark 3. *i.) If s is large enough and α sufficiently close to one we get $0 \le \rho \le c_{16} < 1$.*
ii.) Let $\delta = \alpha - 1 > 0$ and s such that $[(\xi_m - \xi_1)/(\xi_m + \xi_1)]^s \le \delta$. Then, we get from (8.10) *$\rho \le \delta(1+\delta)(5+6\tau(1+\delta))$.*

Proof. Before we give an estimate for the convergence rate of Algorithm 2 we show some technical estimates first. They will be used in (8.23) later.

The matrix $\widetilde{S} = A + B^\top P_C^{-1} B$ is symmetric and positive definite, since we obtain from (8.8)

$$\lambda_1 \le \frac{\boldsymbol{x}^\top P_C^{-1}\boldsymbol{x}}{\boldsymbol{x}^\top C^{-1}\boldsymbol{x}} \le \lambda_n \quad \text{and} \quad \min\{1,\lambda_1\} \le \frac{\boldsymbol{x}^\top \widetilde{S}\boldsymbol{x}}{\boldsymbol{x}^\top S\boldsymbol{x}} \le \max\{1,\lambda_n\} \quad (\boldsymbol{x} \in \mathbb{R}^n \setminus \{0\}). \tag{8.11}$$

The second last inequality implies

$$\begin{aligned}\max_{\boldsymbol{x}\in\mathbb{R}\setminus\{0\}} \|\,(I - C^{1/2}P_C^{-1}C^{1/2})\boldsymbol{x}\,\|_2^2 &= \max_{\boldsymbol{x}\in\mathbb{R}\setminus\{0\}} \frac{\boldsymbol{x}^\top (I - C^{1/2}P_C^{-1}C^{1/2})^\top (I - C^{1/2}P_C^{-1}C^{1/2})\boldsymbol{x}}{\boldsymbol{x}^\top\boldsymbol{x}} \\ &= \left(\max_{\boldsymbol{x}\in\mathbb{R}\setminus\{0\}} \frac{\boldsymbol{x}^\top (I - C^{1/2}P_C^{-1}C^{1/2})\boldsymbol{x}}{\boldsymbol{x}^\top\boldsymbol{x}}\right)^2 \le \max\{(1-\lambda_1)^2,(1-\lambda_n)^2\} =: \kappa^2\end{aligned} \tag{8.12}$$

and similar

$$\max_{\boldsymbol{x}\in\mathbb{R}\setminus\{0\}} \|\,(I - P_C^{1/2}C^{-1}P_C^{1/2})\boldsymbol{x}\,\|_2^2 \le \max\{(1-\lambda_1^{-1})^2,(1-\lambda_n^{-1})^2\} =: \sigma^2 \tag{8.13}$$

Since S and $\widetilde{S}$ are symmetric and positive definite, inequality (8.11) implies

$$\begin{aligned}\max_{\boldsymbol{x}\in\mathbb{R}\setminus\{0\}} \|\,(I - \widetilde{S}^{1/2}S^{-1}\widetilde{S}^{1/2})\boldsymbol{x}\,\|_2^2 &= \left(\max_{\boldsymbol{x}\in\mathbb{R}\setminus\{0\}} \frac{\boldsymbol{x}^\top (I - \widetilde{S}^{1/2}S^{-1}\widetilde{S}^{1/2})\boldsymbol{x}}{\boldsymbol{x}^\top\boldsymbol{x}}\right)^2 \\ &\le \max\{(1-\min\{1,\lambda_1\})^2,(1-\max\{1,\lambda_n\})^2\} = \max\{(1-\lambda_1)^2,(\lambda_n-1)^2\} = \kappa^2\end{aligned} \tag{8.14}$$

From inequalities (8.11), (8.8), and (8.12) we get for all $\boldsymbol{x} \in \mathbb{R}^n \setminus \{0\}$

$$\begin{aligned}&\|\,\widetilde{S}^{-1/2}B^\top(P_C^{-1} - C^{-1})x\,\|_2 \\ &\le \max\{1,\lambda_1^{-1/2}\}\|\,S^{-1/2}B^\top C^{-1/2}(C^{1/2}P_C^{-1}C^{1/2} - I)C^{-1/2}x\,\|_2 \\ &\le \max\{1,\lambda_1^{-1/2}\}\tau\,\kappa\,\|\,C^{-1/2}x\,\|_2\end{aligned} \tag{8.15}$$

From the last inequality we obtain

$$\begin{aligned}&\|\,\widetilde{S}^{-1/2}B^\top(P_C^{-1} - C^{-1})\boldsymbol{r}_2^k\,\|_2 \le \max\{1,\lambda_1^{-1/2}\}\tau\,\kappa\,\|\,C^{-1/2}\boldsymbol{r}_2^k\,\|_2 \\ &\le \max\{1,\lambda_1^{-1/2}\}\tau\,\kappa\,\|\,C^{-1/2}(B(\boldsymbol{x}_1 - \boldsymbol{x}_1^k) - C(\boldsymbol{x}_2 - \boldsymbol{x}_2^k))\,\|_2 \\ &\le \max\{1,\lambda_1^{-1/2}\}\tau\,\kappa\,\Big(\|\,S^{1/2}(\boldsymbol{x}_1 - \boldsymbol{x}_1^k)\,\|_2 + \|\,C^{1/2}(\boldsymbol{x}_2 - \boldsymbol{x}_2^k)\,\|_2\Big)\end{aligned} \tag{8.16}$$

Using $S(\boldsymbol{x}_1 - \boldsymbol{x}_1^k) = S\boldsymbol{d}_1^k = \boldsymbol{r}_1^k + B^\top C^{-1}\boldsymbol{r}_2^k$ in (8.6) and inequality (8.11) we obtain

$$\|\,\widetilde{S}^{-1/2}(\boldsymbol{r}_1^k + B^\top C^{-1}\boldsymbol{r}_2^k)\,\|_2 = \|\,\widetilde{S}^{-1/2}S(\boldsymbol{x}_1^* - \boldsymbol{x}_1^k)\,\|_2 \le \max\{1,\lambda_1^{-1/2}\}\|\,S^{1/2}(\boldsymbol{x}_1^* - \boldsymbol{x}_1^k)\,\|_2\,. \tag{8.17}$$

For the s-iterate $\widetilde{\boldsymbol{d}}_1^{k,s}$ of the preconditioned steepest descent method applied to $\widetilde{S}\widetilde{\boldsymbol{d}}_1^k = \boldsymbol{r}_1^k + B^\top C^{-1}\boldsymbol{r}_2^k$ with coefficient matrix $\widetilde{S}$ and preconditioner $P_{\widetilde{S}}$ and ξ_1, ξ_m from (8.7) there holds the following convergence result [21]

$$\|\,\widetilde{S}^{1/2}(\widetilde{\boldsymbol{d}}_1^k - \widetilde{\boldsymbol{d}}_1^{k,s})\,\| \le \left(\frac{\xi_m - \xi_1}{\xi_m + \xi_1}\right)^s \|\,\widetilde{S}^{1/2}\,\widetilde{\boldsymbol{d}}_1^k\,\|\,. \tag{8.18}$$

By adding $\pm\widetilde{S}^{-1/2}B^\top C^{-1}\boldsymbol{r}_2^k$, (8.6), and the triangle inequality we get

$$\|\,\widetilde{S}^{1/2}\widetilde{\boldsymbol{d}}_1^k\,\|_2 \le \|\,\widetilde{S}^{-1/2}(\boldsymbol{r}_1^k + B^\top C^{-1}\boldsymbol{r}_2^k)\,\|_2 + \|\,\widetilde{S}^{-1/2}B^\top(P_C^{-1} - C^{-1})\boldsymbol{r}_2^k\,\|_2\,. \tag{8.19}$$

By the same technique (adding $\pm\widetilde{S}^{-1/2}B^\top C^{-1}\boldsymbol{r}_2^k$) and inequality (8.14) we obtain

$$
\begin{aligned}
&\|\,\widetilde{S}^{1/2}(\widetilde{\boldsymbol{d}}_1^k-\boldsymbol{d}_1^k)\,\|_2 \le \|\,\widetilde{S}^{1/2}(S^{-1}-\widetilde{S}^{-1})(\boldsymbol{r}_1^k+B^\top C^{-1}\boldsymbol{r}_2^k)\,\|_2+\|\,\widetilde{S}^{-1/2}B^\top(C^{-1}-P_C^{-1})\boldsymbol{r}_2^k\,\|_2\\
&\le \|\,(\widetilde{S}^{1/2}S^{-1}\widetilde{S}^{1/2}-I)\widetilde{S}^{-1/2}(\boldsymbol{r}_1^k+B^\top C^{-1}\boldsymbol{r}_2^k)\,\|_2+\|\,\widetilde{S}^{-1/2}B^\top(C^{-1}-P_C^{-1})\boldsymbol{r}_2^k\,\|_2\\
&\le \kappa\|\,\widetilde{S}^{-1/2}(\boldsymbol{r}_1^k+B^\top C^{-1}\boldsymbol{r}_2^k)\,\|_2+\|\,\widetilde{S}^{-1/2}B^\top(C^{-1}-P_C^{-1})\boldsymbol{r}_2^k\,\|_2
\end{aligned}
\tag{8.20}
$$

From (8.18), (8.19) together with (8.16) and (8.17)

$$
\begin{aligned}
&\|\,\widetilde{S}^{1/2}(\widetilde{\boldsymbol{d}}_1^{k,s}-\boldsymbol{d}_1^k)\,\|_2 \le \|\,\widetilde{S}^{1/2}(\widetilde{\boldsymbol{d}}_1^{k,s}-\widetilde{\boldsymbol{d}}_1^k)\,\|_2+\|\,\widetilde{S}^{1/2}(\widetilde{\boldsymbol{d}}_1^k-\boldsymbol{d}_1^k)\,\|_2\\
&\le (\kappa+\left(\frac{\xi_m-\xi_1}{\xi_m+\xi_1}\right)^s)\|\,\widetilde{S}^{-1/2}(\boldsymbol{r}_1^k+B^\top C^{-1}\boldsymbol{r}_2^k)\,\|_2+(1+\left(\frac{\xi_m-\xi_1}{\xi_m+\xi_1}\right)^s)\,\|\,\widetilde{S}^{-1/2}B^\top(C^{-1}-P_C^{-1})\boldsymbol{r}_2^k\,\|_2\\
&\le \left(\kappa(1+\tau)+(1+\tau\kappa)\left(\frac{\xi_m-\xi_1}{\xi_m+\xi_1}\right)^s\right)\max\{1,\lambda_1^{-1/2}\}\,\|\,S^{1/2}(\boldsymbol{x}_1-\boldsymbol{x}_1^k)\,\|_2\\
&\qquad+\kappa\,\tau\,(1+\left(\frac{\xi_m-\xi_1}{\xi_m+\xi_1}\right)^s)\max\{1,\lambda_1^{-1/2}\}\|\,C^{1/2}(\boldsymbol{x}_2-\boldsymbol{x}_2^k)\,\|_2\,.
\end{aligned}
\tag{8.21}
$$

Using $B\boldsymbol{d}_1^k-\boldsymbol{r}_2^k=C(\boldsymbol{x}_2-\boldsymbol{x}_2^k)$, A is assumed to be positive semi-definite, and (8.6), (8.13), (8.11) we bound the error $\boldsymbol{x}_2-\boldsymbol{x}_2^{k+1}$ as follows

$$
\begin{aligned}
&\|\,P_C^{1/2}(\boldsymbol{x}_2-\boldsymbol{x}_2^{k+1})\,\|_2=\|\,P_C^{1/2}(\widetilde{\boldsymbol{d}}_2^{k,s}-\boldsymbol{d}_2^k)\,\|_2=\|\,P_C^{1/2}(\widetilde{\boldsymbol{d}}_2^{k,s}-P_C^{-1}B\boldsymbol{d}_1^k+P_C^{-1}B\boldsymbol{d}_1^k-\boldsymbol{d}_2^k)\,\|_2\\
&\qquad=\|\,P_C^{-1/2}B(\widetilde{\boldsymbol{d}}_1^{k,s}-\boldsymbol{d}_1^k)\,\|_2+\|\,P_C^{1/2}(P_C^{-1}-C^{-1})(B\boldsymbol{d}_1^k-\boldsymbol{r}_2^k)\,\|_2\\
&\qquad\le\|\,\widetilde{S}^{1/2}(\widetilde{\boldsymbol{d}}_1^{k,s}-\boldsymbol{d}_1^k)\,\|_2+\|\,(I-P_C^{1/2}C^{-1}P_C^{1/2})P_C^{-1/2}(B\boldsymbol{d}_1^k-\boldsymbol{r}_2^k)\,\|_2\\
&\le\|\,\widetilde{S}^{1/2}(\widetilde{\boldsymbol{d}}_1^{k,s}-\boldsymbol{d}_1^k)\,\|_2+\sigma\|\,P_C^{-1/2}(B\boldsymbol{d}_1^k-\boldsymbol{r}_2^k)\,\|_2\le\|\,\widetilde{S}^{1/2}(\widetilde{\boldsymbol{d}}_1^{k,s}-\boldsymbol{d}_1^k)\,\|_2+\sigma\,\lambda_n^{1/2}\,\|\,C^{1/2}(\boldsymbol{x}_2-\boldsymbol{x}_2^k)\,\|_2.
\end{aligned}
\tag{8.22}
$$

We are now able to bound the error of the $(k+1)$-th iterate by using (8.11), (8.8), (8.22), (8.21), identities $\boldsymbol{x}_i-\boldsymbol{x}_i^{k+1}=\widetilde{\boldsymbol{d}}_i^{k,s}-\boldsymbol{d}_i^k$ and σ defined by (8.13)

$$
\begin{aligned}
&\|\,S^{1/2}(\boldsymbol{x}_1-\boldsymbol{x}_1^{k+1})\,\|_2+\|\,C^{1/2}(\boldsymbol{x}_2-\boldsymbol{x}_2^{k+1})\,\|_2\\
&\qquad\le\max\{1,\lambda_1^{-1/2}\}\|\,\widetilde{S}^{1/2}(\widetilde{\boldsymbol{d}}_1^{k,s}-\boldsymbol{d}_1^k)\,\|_2+\lambda_n^{1/2}\|\,P_C^{1/2}(\widetilde{\boldsymbol{d}}_2^{k,s}-\boldsymbol{d}_2^k)\,\|_2\\
&\qquad\le(\lambda_n^{1/2}+\max\{1,\lambda_1^{-1/2}\})\,\|\,\widetilde{S}^{1/2}(\widetilde{\boldsymbol{d}}_1^{k,s}-\boldsymbol{d}_1^k)\,\|_2+\sigma\lambda_n\|\,C^{1/2}(\boldsymbol{x}_2-\boldsymbol{x}_2^{(k+1)})\,\|_2\\
&\le\left(\kappa(1+\tau)+(1+\tau\kappa)\left(\frac{\xi_m-\xi_1}{\xi_m+\xi_1}\right)^s\right)(\lambda_n^{1/2}+\max\{1,\lambda_1^{-1/2}\})(\max\{1,\lambda_1^{-1/2}\}\,\|\,S^{1/2}(\boldsymbol{x}_1-\boldsymbol{x}_1^k)\,\|_2\\
&\quad+\left(\sigma\lambda_n+\kappa\,\tau(\lambda_n^{1/2}+\max\{1,\lambda_1^{-1/2}\})(1+\left(\frac{\xi_m-\xi_1}{\xi_m+\xi_1}\right)^s)(\max\{1,\lambda_1^{-1/2}\})\right)\|\,C^{1/2}(\boldsymbol{x}_2-\boldsymbol{x}_2^k)\,\|_2
\end{aligned}
\tag{8.23}
$$

Let $[\lambda_1,\lambda_n]\subset[\alpha^{-1},\alpha]$, with $\alpha>1$. Hence, we get $\kappa,\sigma\le\alpha-1$ and

$$
\begin{aligned}
&\|\,S^{1/2}(\boldsymbol{x}_1-\boldsymbol{x}_1^{k+1})\,\|_2+\|\,C^{1/2}(\boldsymbol{x}_2-\boldsymbol{x}_2^{k+1})\,\|_2\\
&\le\left((\alpha-1)(1+\tau)+(1+\tau\alpha)\left(\frac{\xi_m-\xi_1}{\xi_m+\xi_1}\right)^s\right)(\alpha^{1/2}+\alpha^{1/2})\,\alpha^{1/2}\,\|\,S^{1/2}(\boldsymbol{x}_1-\boldsymbol{x}_1^k)\,\|_2\\
&\quad+\left((\alpha-1)\alpha+(\alpha-1)\,\tau\,(\alpha^{1/2}+\alpha^{1/2})(1+\left(\frac{\xi_m-\xi_1}{\xi_m+\xi_1}\right)^s\alpha^{1/2}\right)\,\|\,C^{1/2}(\boldsymbol{x}_2-\boldsymbol{x}_2^k)\,\|_2\\
&\le\rho\left(\|\,S^{1/2}(\boldsymbol{x}_1-\boldsymbol{x}_1^k)\,\|+\|\,C^{1/2}(\boldsymbol{x}_2-\boldsymbol{x}_2^k)\,\|_2\right)
\end{aligned}
\tag{8.24}
$$

where

$$
\rho:=2\alpha(1+\tau\,\alpha)\left(\alpha-1+\left(\frac{\xi_m-\xi_1}{\xi_m+\xi_1}\right)^s\right)+(\alpha-1)\,\alpha\left(1+2\tau+2\tau\left(\frac{\xi_m-\xi_1}{\xi_m+\xi_1}\right)^s\right).
$$

□

8.2. **The Inner-Outer Iteration Applied to the Coupling Matrix $\mathcal{A}$.** We consider now the inner-outer iteration Algorithm 2 applied to the linear system (5.1).

Since $P_{\widetilde{S}_h}$ and P_{V_h} are both assumed to be symmetric and positive definite, we can define positive constants depending on the mesh size $h = h(m)$ (resp. $h = h(n)$) to bound the extreme eigenvalues satisfying

$$0 < \eta_{\min}(h) \le \frac{\boldsymbol{x}^\top (A_h + T_h)\boldsymbol{x}}{\boldsymbol{x}^\top P_{\widetilde{S}_h} \boldsymbol{x}} \le \eta_{\max}(h) \qquad (\boldsymbol{x} \in \mathbb{R}^m \setminus \{0\})\,. \tag{8.25}$$

and

$$0 < \lambda_{\min}(h) \le \frac{\boldsymbol{x}^\top V_h \boldsymbol{x}}{\boldsymbol{x}^\top P_{V_h} \boldsymbol{x}} \le \lambda_{\max}(h) \qquad (\boldsymbol{x} \in \mathbb{R}^n \setminus \{0\})\,. \tag{8.26}$$

Therefore by applying Section 5 we obtain the estimates

$$\omega \le \frac{\boldsymbol{x}^\top \{A_h + W_h + (K_h^\top - I_h^\top)V_h^{-1}(K_h - I_h)\}\boldsymbol{x}}{\boldsymbol{x}^\top (A_h + T_h)\boldsymbol{x}} \le \Omega \qquad (\boldsymbol{x} \in \mathbb{R}^m \setminus \{0\}) \tag{8.27}$$

where ω, Ω are positive constants independent of the mesh size h (Let $\omega = \theta^2$ with θ from (5.8) and $\Omega = \max\{\Delta, \Theta^2\}$ with Δ from (5.8), Θ from (5.9), and notice Subsections 5.1 and 5.2 where the existence is proven.) and

$$\xi_{\min}(h) \le \frac{\boldsymbol{x}^\top \{A_h + W_h + (K_h^\top - I_h^\top)V_h^{-1}(K_h - I_h)\}\boldsymbol{x}}{\boldsymbol{x}^\top P_{\widetilde{S}_h} \boldsymbol{x}} \le \xi_{\max}(h) \qquad (\boldsymbol{x} \in \mathbb{R}^m \setminus \{0\}) \tag{8.28}$$

which is analogous to (8.7) (Let $\xi_{\min}(h) = \omega\, \eta_{\min}(h)$ and $\xi_{\max}(h) = \Omega\, \eta_{\max}(h)$ with $\eta_{\min}(h)$, $\eta_{\max}(h)$ from (8.25) and ω, Ω from (8.27).). Then, with

$$\boldsymbol{\phi}^\top (K_h - 1)(A_h + T_h)^{-1}(K_h^\top - 1)\boldsymbol{\phi} \le c_{17} \| (K^* - 1)\phi \|_{H^{-\frac{1}{2}}(\Gamma)} \le c_{18} \| \phi \|^2_{H^{-\frac{1}{2}}(\Gamma)}$$

and (8.27) we see that there is a constant $\tau \in \mathbb{R}$ with

$$\frac{\boldsymbol{x}^\top (K_h - I_h)(A_h + W_h + (K_h^\top - I_h^\top)V_h^{-1}(K_h - I_h))^{-1}(K_h^\top - I_h^\top)\boldsymbol{x}}{\boldsymbol{x}^\top V_h \boldsymbol{x}} \le \tau^2 \qquad (x \in \mathbb{R}^n \setminus \{0\}) \tag{8.29}$$

which is analogous to (8.8).

If Δ, θ, and Θ are independent from the mesh size h, there exists also a constant τ (8.29) independent from the mesh size h, and Theorem 3 gives bounds for convergence rate of Algorithm 2 applied to (8.1) depending only on the extreme eigenvalues of $P_{\widetilde{S}_h}^{-1} \widetilde{S}_h$ (8.28) and $P_{V_h}^{-1} V_h$ (8.26). With Lemma 2, we get from Theorem 3 the following result.

Theorem 4. *Let $P_{\widetilde{S}_h}$, P_{V_h} be spectral equivalent to $A_h + T_h$ and V_h, respectively. Then, for sufficiently large s and sufficiently small spectrum of $P_{V_h}^{-1} V_h$ around one, Algorithm 2 applied to* (8.1) *converges and the convergence rate is bounded above independently of the mesh size h.*

Remark 4. *Notice that there holds*

$$\| S^{\frac{1}{2}} \boldsymbol{x} \|_2 \cong \| x \|_{H^1(\Omega_1)}, \quad \| V^{\frac{1}{2}} \boldsymbol{\phi} \|_2 \cong \| \phi \|_{H^{-\frac{1}{2}}(\Gamma)}\,.$$

Therefore Theorem 4 implies convergence in the energy norm.

9. Numerical Example (Part 2)

We continue our numerical example of Section 7. We choose $P_{\widetilde{S}_h}^{-1}$ and $P_{V_h}^{-1}$ to be the same multigrid V-cycle algorithms as in Example 1, resp. 2 and use $\nu = 1$ pre and post smoothing step in Example 1 and 2 to compute $P_{V_h}^{-1}$, and $\nu = 1$ pre and post smoothing step in Example 1 resp. $\nu = 1, 3, 5$ in Example 2 to compute $P_{\widetilde{S}_h}^{-1}$. The computed eigenvalues and singular values were computed with Matlab-routines as explained in Section 7. In Tab. 11, there are given computed bounds for ξ_1, ξ_m, and τ^2 for Example 1 and 2. These reslts confirm in both examples our theory, namely that the lower resp. upper bounds for ξ_1, ξ_m, and τ^2 do not dependent on the meshsize h.

TABLE 11. Constants ξ_1, ξ_m, and τ^2

	Example 1			Example 2				
				ξ_1				
h/h_1	ξ_1	ξ_m	τ^2	$\nu = 1$	$\nu = 3$	$\nu = 5$	ξ_m	τ^2
1/2	0.86070	1	1	0.54705	0.92575	0.98770	1	1
1/4	0.75882	1	1	0.35954	0.84660	0.95849	1	1
1/8	0.68734	1	1	0.27591	0.72817	0.89298	1	1
1/16	0.64411	1	1	0.23706	0.65159	0.83989	1	1
1/32	0.62739	1	1	0.22196	0.61774	0.80658	1	1

In Tab. 12 we present results for solving the linear system $\mathcal{A}\boldsymbol{x} = \boldsymbol{b}$ arising form the geometry and meshes in Example 1 for 20 randomly choosen right hand sides $\boldsymbol{b}$ and $s = 1, 3$ number of inner iterations. The average number of iterations and minimimal/maximal number (in brackets) are given. The computations were done on a Laptop PC (3.2 GHz) using Matlab (Release 14). We used the stopping criterion

$$\| \widetilde{S}_h^{\frac{1}{2}}(x_1 - x_1^k) \|_2 \leq tol \| \widetilde{S}_h^{\frac{1}{2}}(x_1 - x_1^0) \|_2 \tag{9.1}$$

with $tol = 10^{-8}$. Here, $\| \cdot \|_2$ is the Euclidean-norm, (x_1, x_2) the exact solution of our discretized problem and $(x_1^0, x_2^0) = 0$ the starting vector.

TABLE 12. Number of iterations and CPU-time for preconditioned Inner-Outer-method (with $P_{\widetilde{S}_h}^{-1} \approx$ V-cycle multigrid method applied to $A_h + W_h + \gamma Id_h^\top D_h^{-1} Id_h$) and different number of inner iterations ($s = 1, 3$)

h_0/h	# unknowns	# iterations average (min/max)		CPU-time [s]	# iterations average (min/max)		CPU-time [s]
		$s = 1$			$s = 3$		
1/1	16	13.6	(12/14)	0.0101	5.45	(5/6)	0.0094
1/2	37	14.8	(14/15)	0.0328	14.2	(13/15)	0.0516
1/4	97	15.3	(14/16)	0.0703	14.0	(14/14)	0.1126
1/8	289	17.1	(16/18)	0.1875	13.8	(13/14)	0.2625
1/16	961	17.95	(17/19)	0.6375	13.05	(13/14)	0.8523
1/32	3457	19	(19/19)	2.5250	13	(13/13)	3.2015
1/64	13057	19.45	(19/20)	10.986	13	(13/13)	13.273
1/128	50689	20	(20/20)	47.672	12.95	(12/13)	55.208
1/256	199681	21	(21/21)	199.03	12.05	(12/13)	210.91

Both, the number of iterations and the CPU-time indicate the efficiency of the preconditioned inner-outer iteration.

For comparision we computed also the number of PCR-iterates (x_1^k, x_2^k) where we used the stopping criterion

$$\| A_h^{\frac{1}{2}}(x_1 - x_1^k) \|_2 + \| V_h^{\frac{1}{2}}(x_2 - x_2^k) \|_2 \leq tol \left(\| A_h^{\frac{1}{2}}(x_1 - x_1^0) \|_2 + \| V_h^{\frac{1}{2}}(x_2 - x_2^0) \|_2 \right) \tag{9.2}$$

with $tol = 10^{-8}$.

All results presented here underline the efficiency of both preconditioned iterative methods to solve indefinite linear systems of equations arising from symmetric coupling of finite elements and boundary elements and confirm our theory. Both methods are optimal in the sense, that the number of iterations is bounded independently by the meshsize h. The preconditioned conjugate residual method is in our example twice as fast as the preconditioned inner-outer iteration.

Table 13. Number of iterations and CPU-time for PCR-method with block-diagonal preconditioning

h_0/h	# unknowns	# iterations average (min/max)		CPU-time [s]	# iterations average (min/max)		CPU-time [s]
		$T_h = \beta M_h$			$T_h = W_h + \gamma Id_h^\top D_h^{-1} Id_h$		
1/1	16	17	(17/17)	0.0078	16.35	(16/17)	0.0063
1/2	37	33.6	(32/35)	0.0273	23.7	(23/25)	0.0204
1/4	97	41.6	(41/43)	0.0648	26.85	(26/27)	0.0437
1/8	289	46	(46/46)	0.1680	28	(28/28)	0.1156
1/16	961	50	(50/50)	0.5226	28.4	(28/30)	0.3725
1/32	3457	53	(53/53)	1.9141	29.5	(28/30)	1.4222
1/64	13057	55.2	(55/57)	7.7930	30	(30/30)	5.7105
1/128	50689	57.2	(57/59)	32.050	30	(30/30)	23.341
1/256	199681	58.05	(57/59)	133.32	30	(30/30)	101.31

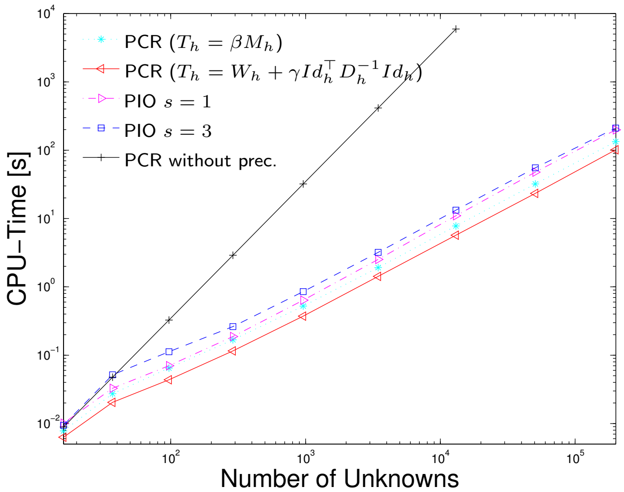

Figure 5. CPU-time [s] for different solvers.

10. Conclusion

All the preconditioned iteration schemes presented here are optimal in the sense, that the number of iterations is bounded independently by the meshsize h. The best result with respect to cpu-time (and matrix-vector multplication) was obtained by using the block-diagonal precondioned conjugate residual method and $T_h = W_h + \gamma Id_h^\top D_h^{-1} Id_h$. (Notice, one complete inner-outer iteration needs a least twice as much operations as one PCR-iteration.) All smoothers used in the multigrid-algorithm for the single layer potential provide constant bounds for the spectrum of $\mathbf{Mg}^V V_h$, which do not depend on the meshsize h. In all examples is the damping parameter χ_k^* less than χ_k in (6.3) to get the optimal contraction rate of $\mathbf{Mg}^V$.

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