Dynamic Credit Portfolio Modelling in Structural Models with Jumps

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Dynamic credit portfolio modelling in structural models with jumps

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Abstract

We present a structural jump-diffusion credit-portfolio model which models the loss distribution and dependence structure of the portfolio dynamically. We are able to obtain the log-asset correlation analytically and precise estimates of the term-structure of default correlations within the model. The models allows the simultaneous pricing of bonds, CDS and portfolio derivatives across all maturities. We present an efficient algorithm for the calibration of our model, which makes the model suitable for practical applications. As an example we calibrate our model to *iTraxx* quotes of Collateral Debt Obligations (CDOs).

1 Introduction

Dynamic credit-portfolio models are needed for an appropriate risk management by financial institutions and for the pricing of (exotic) portfolio-credit derivatives. Triggered in particular by the need for realistic pricing models, several models based on a top-down approach within the reduced-form framework have been proposed, compare Schönbucher (2006), Albrecher, Ladoucette, and Schoutens (2007), Sidenius, Piterbag, and Andersen (2005). However, these models are mainly suited for a portfolio pricing approach, since they model the overall portfolio-loss process and then try to consistently include the individual risk. Specifics of the individual risk contributions can therefore hardly be taken into account. More traditional bottom-up models on the other hand have to balance computational tractability with simplifying assumptions such as a homogeneous portfolio structure or a static model in terms of loss distribution and dependence structure, examples are Laurent and Gregory (2003), Vasicek (1987).

In this paper we present a bottom-up credit portfolio model which is in the spirit of so-called structural (Merton) models. A default event is triggered by insufficient asset values of the respective firm in the portfolio. The asset value process is driven by idiosyncratic and common factors which are subject to diffusion and jump risk. This allows to explain the loss distribution and dependence structure of the portfolio at any point in time. Additionally, it allows the simultaneous

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pricing of bonds, CDS and portfolio derivatives across all maturities. Such a consistent arbitrage-free framework is especially important for risk management. We present an efficient algorithm for the calibration of our model, which makes the model suitable for practical applications. As an example we calibrate and price Collateral Debt Obligations (CDOs) within our framework.

Introducing our model more detailed, individually for each firm the model is a generalization of Zhou (2001b)'s jump-diffusion model. Compared to pure diffusion models in the tradition of Merton (1974), the advantage of supporting jumps (on a univariate level) is that the induced term structure of bond and CDS spreads has a positive limit at the short end and allows for a much better fit to observed spread curves. The first argument is made precise in Theorems 2.2 and 2.3, the impressive improvement in fitting capability is shown in Table 3. Additionally, we can use common jumps to introduce dependence to the modeled companies. This approach is combined with the popular concept of using correlated Brownian motions or common Brownian factors. In this sense, our model generalizes the models of Willemann (2007) and Hull, Pedrescu, and White (2006). Within our framework we are then able to analyse the dependence structure of the model. We are able to analytically calculate asset correlations for companies within the portfolio. Furthermore, we obtain the default correlation by simulation very precisely, thus extending results by Zhou (2001a). We compare the default correlation of our model to the default correlation in a pure diffusion setting and observe that short term correlations are almost entirely a result of the jump component.

To demonstrate the practical relevance of our model we apply our model to the pricing of CDOs, for which we need an efficient Monte Carlo procedure. We run a calibration exercise on *iTraxx* data to demonstrate the fitting capability of our model to individual CDS as well as the CDO structure.

Our article is organized as follows. The second Chapter introduces the jump-diffusion model on a univariate level, where we show stylized properties as a positive limit of spreads and explain two pricing algorithms for single-firm products. In Chapter 3 we show two approaches on how dependence may be introduced to the individual firms, both allow for an intuitive economic interpretation. The consequences of this construction on the term structure of firm-value and default correlation are derived in Chapter 4. A pricing algorithm for CDOs is given in Chapter 5, this algorithm is used in Chapter 6 for a calibration of the model. Finally, we summarize our findings in Chapter 7.

2 Our structural default model

2.1 Univariate case

We assume the value of the modeled company to start at some initial level $v_0 > 0$ and to evolve stochastically on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. We want to incorporate small and unsystematic changes of the value process, resulting from daily business activities, as well as sudden jumps due to unexpected events. Moreover, we assume the total value of all assets to remain positive. A model for the firm-value process $V = \{V_t\}_{t\geq 0}$ fulfilling these requirements is given by

$$V_t = v_0 \exp(X_t), \quad v_0 > 0, \, \forall t \geq 0,$$

where the process $X = \{X_t\}_{t\geq 0}$ is a jump-diffusion process with canonical decomposition

$$X_t = \gamma t + \sigma W_t + \sum_{i=1}^{N_t} Y_i. \tag{1}$$

Throughout this article, we assume as given the pricing measure $\mathbb{P}$. The filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ denotes the natural filtration of the firm-value process, i.e. with $\mathcal{N}$ the collection of all $\mathbb{P}$-null sets

$$\mathcal{F}_t = \sigma(V_s : 0 \leq s \leq t) \vee \mathcal{N} = \sigma(X_s : 0 \leq s \leq t) \vee \mathcal{N},$$

augmented to satisfy the usual conditions of completeness and right continuity. To exclude degenerated cases, we impose the constraints $\sigma > 0$, $\lambda > 0$ and for the jump measure $\mathbb{P}_Y \neq \delta_0$, where δ_x denotes the Dirac measure concentrated at x. Following Black and Cox (1976), we define τ as the first passage of the firm-value process below the debt level of the company, which we denote by d. Formally, the time of default is defined by

$$\tau = \inf\{t > 0 : V_t \leq d\}.$$

All model parameters are summarized in the table below.

Parameter	Interpretation
$\gamma \in \mathbb{R}$	The linear growth rate of V.
$\sigma \in \mathbb{R}^+$	The volatility of the diffusion component.
$\lambda \in \mathbb{R}_0^+$	The jump intensity of the driving Poisson process.
$\mathbb{P}_Y$	The jump-size distribution.
$v_0 \in \mathbb{R}^+$	The initial value of the company.
$d \in \mathbb{R}^+$	The default threshold, satisfying $d < v_0$.

Table 1: The parameters of our univariate default model.

2.2 Bond and CDS prices in a jump-diffusion framework

Allowing jumps is a natural generalization of pure diffusion models. However, this generalization significantly complicates the computation of bond and CDS prices, as closed-form expressions of the distribution of first-passage times are not known in a general jump-diffusion scenario. To overcome this problem, different approaches are possible and introduced below.

The probabilistic approach

The probabilistic approach is to perform a Monte Carlo simulation to estimating survival probabilities and to price financial contracts. A naïve implementation requires one to sample the firm-value process on a discrete grid and test for default on this grid. Not only is this computationally expensive, it also implies a systematic discretisation bias, as a possible default in between two grid points is not considered. This is a shortcoming of the algorithm proposed by Zhou (2001b). A substantial improved algorithm is obtained if the nature of a jump-diffusion process is systematically exploited. Conditioned on the number N_T and location $0 < \tau_1 < \ldots < \tau_{N_T}$ of the jumps in the firm-value process, it is possible to compute default probabilities and prices of financial contracts in closed form. These jump times have a well known distribution which is efficient to simulate. Based on this observation it is possible to formulate an extremely fast and unbiased Monte Carlo algorithm, a detailed description of such an implementation is given by Scherer (2007). In the context of pricing barrier options, a similar idea is presented by Metwally and Atiya (2002). In the multidimensional implementation of the model, such an efficient implementation is indispensable to obtain prices of portfolio derivatives in acceptable time.

The numerical approximation

Monte Carlo simulations are fast enough for the pricing of bonds and CDS given a set of parameters. However, the calibration of the model, which essentially corresponds to a minimization over the multidimensional parameter space, requires faster pricing routines. Such a numerical pricing routine is constructed and tested in Scherer (2007). The mathematical ingredients are results on the Laplace transform of first-passage times in a jump-diffusion model with two-sided exponentially distributed jumps. This result was found by Kou and Wang (2003). It is then possible to recover the required default probabilities by inverting the Laplace transform numerically. Using these approximated default probabilities, pricing formulas of bond and CDS contracts are easily evaluated in fractions of a second.

2.3 The local default rate in a jump-diffusion scenario

In this section we present an elementary derivation of the local default rate in a jump-diffusion framework. This result is important from an economic perspective as it allows us to derive the exact limit of bond and CDS spreads at the short end of the term structure. It is well known that the local default rate in a pure diffusion model, as well as the limit of spreads in such a framework, are zero. This result is generalized below and used in Section 2.4 to shown that a positive limit of spreads is the result of supporting negative jumps.

Theorem 2.1 (The local default rate of τ)
We assume a continuous jump-size distribution with F_Y denoting its cumulative distribution function. At time zero, the distance to default for X is given by $x_0 = -\log(d/v_0)$. We then obtain

$$LDR_\tau = \lim_{h \searrow 0} \frac{1}{h} \mathbb{P}(\tau \leq h) = \lambda F_Y(-x_0) = \nu((-\infty, -x_0]),$$

where ν denotes the Lévy measure of X. This shows that the local default rate is determined by the Lévy measure of the logarithm of the firm-value process and the distance to default.

This result can be interpreted as follows. If a negative jump exceeds the distance to default with positive probability, that is $F_Y(-x_0) > 0$, then the local default rate is positive. The jump intensity, which specifies how many jumps are expected per unit time interval, is the second factor of the local default rate. Economically, a positive local-default rate means that defaults can occur within an infinitesimal amount of time, which is not possible in a continuous model. This agrees with the observation that the parameters of the diffusion component are not present in this formula.

Considering this, Theorem 2.1 is a natural extension of the pure diffusion result, as $\lambda = 0$ puts us back into a continuous setting. Summarized, the local default rate agrees with our intuition in the sense that it factors into the independent probabilities for a jump to occur and the probability that such a jump causes the firm-value process to fall below the default threshold.

Proof: We condition on the number N_h of jumps which occurred up to time h and denote the first jump time by $\tau(h)$. We obtain

$$\begin{aligned}
&\lim_{h\searrow 0}\frac{1}{h}\mathbb{P}(\tau \le h)\\
&= \lim_{h\searrow 0}\frac{1}{h}\sum_{n=0}^{\infty}\mathbb{P}(N_h = n)\mathbb{P}\Big(\inf_{0\le s\le h}\{X_s\} \le -x_0 \big| N_h = n\Big) \qquad (2)\\
&= \lim_{h\searrow 0}\frac{e^{-\lambda h}}{h}\mathbb{P}\Big(\inf_{0\le s\le h}\{\gamma s + \sigma W_s\} \le -x_0\Big) + \\
&\quad \lim_{h\searrow 0}\lambda e^{-\lambda h}\mathbb{P}\Big(\inf_{0\le s\le h}\big\{\gamma s + \sigma W_s + \mathbf{1}_{\{s\ge\tau(h)\}}Y_1\big\} \le -x_0\Big) + \\
&\quad \lim_{h\searrow 0}\frac{1}{h}\sum_{n=2}^{\infty}\frac{e^{-\lambda h}(\lambda h)^n}{n!}\mathbb{P}\Big(\inf_{0\le s\le h}\big\{\gamma s + \sigma W_s + \sum_{j=1}^{N_s}Y_j\big\} \le -x_0 \big| N_h = n\Big).
\end{aligned}$$

The first limit, representing a pure diffusion setup, is zero by l'Hospital's rule[1]. Considering the last limit, a dominated convergence argument allows us to interchange limit and summation, establishing that this limit also equals zero.

We now examine the second limit, the case of exactly one jump, where we additionally condition on whether this jump is negative or not. Obviously, only the case of a negative jump is of interest. We let $B_t = \gamma t + \sigma W_t$ and find

$$\begin{aligned}
&\mathbb{P}\big(B_h + Y_1 \le -x_0 | Y_1 < 0\big)\\
&\quad \le \mathbb{P}\Big(\inf_{0\le s\le h}\big\{B_s + \mathbf{1}_{\{s\ge\tau_1\}}Y_1\big\} \le -x_0 \big| N_h = 1, Y_1 < 0\Big)\\
&\quad \le \mathbb{P}\Big(\inf_{0\le s\le h}\big\{B_s + Y_1\big\} \le -x_0 \big| Y_1 < 0\Big).
\end{aligned}$$

The sequence of events $A_h = \{\omega \in \Omega : \inf_{0\le s\le h}\big\{B_s + Y_1\big\} \le -x_0,\ Y_1 < 0\}$ is decreasing in h. Therefore, by the continuity of the probability measure, we obtain the following result for the limit of upper bounds

$$\lim_{h\searrow 0}\mathbb{P}(A_h) = \mathbb{P}(A_0). \qquad (3)$$

From Equation (3), it follows that

$$\lim_{h\searrow 0}\mathbb{P}\Big(\inf_{0\le s\le h}\big\{B_s + Y_1\big\} \le -x_0 \big| Y_1 < 0\Big) = \mathbb{P}(Y_1 \le -x_0 | Y_1 < 0).$$

Showing that this limit agrees with the limit of lower bounds is straightforward if $B_h + Y_1$ conditioned on $Y_1 < 0$ has a closed-form expression, which holds for instance if the jump size distribution $\mathbb{P}_Y$ is a two-sided exponential distribution. In general, this result is shown as follows. For arbitrary

[1] In this continuous setup, the default probability is a well known continuously differentiable function of t.

$c > 0$, we have

$$\begin{aligned}
&\liminf_{h\searrow 0} \mathbb{P}\big(B_h + Y_1 \leq -x_0 | Y_1 < 0\big) \\
&\quad \geq \liminf_{h\searrow 0} \mathbb{P}\big(B_h + Y_1 \leq -x_0, B_h \leq \gamma h + c\sqrt{h\sigma^2} \big| Y_1 < 0\big) \\
&\quad \geq \liminf_{h\searrow 0} \mathbb{P}\big(Y_1 \leq -x_0 - \gamma h - c\sqrt{h\sigma^2} \big| Y_1 < 0\big) \mathbb{P}\big(B_h \leq \gamma h + c\sqrt{h\sigma^2}\big) \\
&\quad = \mathbb{P}(Y_1 \leq -x_0 | Y_1 < 0)\Phi(c),
\end{aligned}$$

where $\Phi(x)$ denotes the cdf of the standard normal distribution. Since $c > 0$ was chosen arbitrarily, we are allowed to let c tend to infinity and obtain the result. $\diamondsuit$

2.4 The limit of bond and CDS spreads

Pricing formulas for bonds and CDS contracts

We assume as given the pricing measure $\mathbb{P}$ and a constant (continuously compounding) interest rate $r > 0$. For the price of a risky zero-coupon bond with recovery rate R and a CDS contract written on this bond, we obtain the following formulas.

Lemma 2.1 (Price of a zero-coupon bond)
We denote by $\phi(0,T)$ the fair price at time zero of a defaultable zero-coupon bond with maturity T, unit face value and recovery rate R. This price satisfies

$$\phi(0,T) = e^{-rT}\mathbb{P}(\tau > T) + R\int_0^T e^{-rt} d\mathbb{P}(\tau \leq t). \tag{4}$$

The credit spread corresponding to $\phi(0,T)$ is denoted as η_T, it is the real number solving

$$\phi(0,T) = \exp\left(-(r+\eta_T)T\right). \tag{5}$$

For a CDS written on this bond, we assume the insurance buyer to continuously pay the spread c as long as the reference entity is solvent, whereas the insurance seller indemnifies the insurance buyer by paying the difference of face value minus recovery in the event of credit default. We again discount all future payments and obtain the following expression for the price of a contract with notional one, continuous premium payments c and maturity T.

$$\begin{aligned}
CDS(0,T) &= \mathbb{E}\left[e^{-r\tau}(1-R)\mathbf{1}_{\{\tau\leq T\}} - \int_0^T c e^{-rt}\mathbf{1}_{\{\tau > t\}} dt\right] \\
&= (1-R)\int_0^T e^{-rt} d\mathbb{P}(\tau \leq t) - c\int_0^T e^{-rt}\mathbb{P}(\tau > t) dt. \qquad (6)
\end{aligned}$$

Formula (6) reflects the view of the insurance buyer, the insurance seller uses the same formula with opposite signs. Market prices for CDS contracts are typically quoted in terms of the spread which allows both parties to enter the contract at zero cost. This par spread is obtained from solving Equation (6) for c and therefore given by

$$c_T = \frac{(1-R)\int_0^T e^{-rt} d\mathbb{P}(\tau \leq t)}{\int_0^T e^{-rt}\mathbb{P}(\tau > t) dt}. \tag{7}$$

The limit of credit spreads

Zhou (2001b) presents an intuitive argument why the limit of credit spreads in a jump-diffusion model should remain positive. He argues: *"Because a diffusion process is almost unlikely to cause a default in a short period of time, the defaults of short-term bonds are usually caused by the jump component of the firm value."*[2] He derives this argument by comparing the vanishing probability of a default by diffusion with the probability of a default caused by a single jump. We make this argument precise in Theorem 2.2, where we show that the local default rate of τ, which does not depend on the diffusion component of X, is the crucial component for the limit of spreads. Moreover, we also present the exact limit of CDS spreads in Theorem 2.3.

Theorem 2.2 (Credit spreads at time zero)
The limit of credit spreads at time zero is given by

$$\lim_{h \searrow 0} \eta_h = (1-R) LDR_\tau.$$

Proof: At first, we find an upper and lower bound for the bond price $\phi(0,h)$. With

$$\begin{aligned} \phi^{low}(0,h) &= e^{-rh}\mathbb{P}(\tau > h) + Re^{-rh}\mathbb{P}(\tau \leq h), \\ \phi^{up}(0,h) &= e^{-rh}\mathbb{P}(\tau > h) + R\mathbb{P}(\tau \leq h), \end{aligned}$$

we observe $\phi^{low}(0,h) \leq \phi(0,h) \leq \phi^{up}(0,h)$[3]. Clearly, the opposite relation then holds for the corresponding credit spreads, i.e.

$$\eta_h^{low} \geq \eta_h \geq \eta_h^{up}.$$

Using the credit spread's definition and elementary inequalities, we obtain

$$\begin{aligned} \eta_h^{up} &= -\frac{1}{h}\log\left(\phi^{up}(0,h)\right) - r \\ &\geq -\frac{1}{h}\left(\phi^{up}(0,h) - 1\right) - r \\ &= -\frac{1}{h}\mathbb{P}(\tau \leq h)(R-1) - \frac{1}{h}\left(e^{-rh} - 1\right)\mathbb{P}(\tau > h) - r \\ &\quad \to -\mathrm{LDR}_\tau(R-1) + r - r, \quad (h \searrow 0). \end{aligned}$$

A similar bound is obtained from

$$\begin{aligned} \eta_h^{low} &= -\frac{1}{h}\log\left(\phi^{low}(0,h)\right) - r \\ &= -\frac{1}{h}\log\left(e^{-rh}\left(1 - \mathbb{P}(\tau \leq h) + R\mathbb{P}(\tau \leq h)\right)\right) - r \\ &= -\frac{1}{h}(-rh) - \frac{1}{h}\log\left(\mathbb{P}(\tau \leq h)(R-1) + 1\right) - r \\ &\leq -\frac{1}{h}\frac{\mathbb{P}(\tau \leq h)(R-1)}{\mathbb{P}(\tau \leq h)(R-1) + 1} \\ &\quad \to \mathrm{LDR}_\tau(1-R), \quad (h \searrow 0). \end{aligned}$$

[2]Zhou (2001b), *Journal of Banking and Finance* 25, page 2027.
[3]Let us remark that ϕ^{low} and ϕ^{up} can both be interpreted as bonds with alternative recovery scheme.

Obviously, the same limit must then hold for η_h. $\diamondsuit$

We found that the limit of credit spreads at the short end of the term structure is the product of the local default rate of τ and the fractional loss given default. This is economically reasonable, as the potential loss at default is decreasing in the recovery rate, which implies smaller credit spreads. Moreover, the local default rate of τ approximates the probability of default within small intervals of time. Therefore, credit spreads of bonds with small maturities merely depend on the probability of a sudden default. In other words, their credit spreads are increasing in the local default rate.

The limit of CDS spreads

In a jump-diffusion model for the firm-vaule process, the limit of CDS spreads at the short end of the term structure is also positive and can be found using the local default rate of τ. Moreover, this limit agrees with the limit of bond spreads computed above.

Theorem 2.3 (The limit of CDS spreads with jumps)
Within the setup of our jump-diffusion model, the limit of CDS spreads at the short end of the term structure agrees with the limit of bond spreads. Hence, it is given by

$$\lim_{T \searrow 0} c_T = (1-R)LDR_\tau.$$

Proof: Using integrations by part, we rewrite c_h as

$$c_h = \frac{\frac{1}{h}(1-R)\left(e^{-rh}\mathbb{P}(\tau \le h) + 1 - e^{-rh} + r\int_0^h \mathbb{P}(\tau > s)e^{-rs}ds\right)}{\frac{1}{h}\int_0^h e^{-rs}\mathbb{P}(\tau > s)ds}.$$

As $\mathbb{P}(\tau > s)$ is continuous in 0 with $\mathbb{P}(\tau > 0) = 1$, it follows that

$$\frac{1}{h}\int_0^h e^{-rs}\mathbb{P}(\tau > s)ds \to 1, \quad (h \searrow 0).$$

Using this, it is easily deduced that

$$\lim_{h \searrow 0} c_h = (1-R)(r + 1 \cdot \mathrm{LDR}_\tau - r) = (1-R)\mathrm{LDR}_\tau.$$

$\diamondsuit$

3 The multivariate default model

We consider a portfolio consisting of I credit-risky assets, indexed by $i \in \{1, \ldots, I\}$, whose payment streams depend on the default status of I different firms. To specify the default times $\tau^1, \ldots, \tau^I$ we again model each individual firm-value process as the exponential of a jump-diffusion process. Undisputedly, default events are not independent in reality. In what follows, we introduce different sources of dependence to the firm-value processes to account for this empirical observation. Also, we show how CDOs are priced within our multivariate framework to demonstrate that our model is analytically tractable.

3.1 Dependence via a market factor and joint jumps

At first, we introduce dependence through a common market factor and via jumps that are triggered by a joint Poisson process. The common market factor accounts for the observation that most companies are sensitive to business cycles. The current macroeconomic situation is modeled using a Brownian motion of the market, denoted by W_t^M. The individual Brownian motion driving the firm value process of company i in the univariate case is replaced by the weighted sum $a_i W_t^M + \sqrt{1-a_i^2} W_t^i$. The factor $a_i \in (-1,1)$ assesses the degree of systematic dependence of company i to the market and W_t^i models the idiosyncratic risk of company i. Note that there is no restriction to a single credit-risk factor, our model is easily generalized to multiple risk factors such as macroeconomic variables, commodity prices, interest rates, etc.

Our mechanism of triggering joint jumps is motivated by the following economic interpretation. Intuitively, jumps of the firm-value process of some company are triggered if unexpected information or events are revealed. Mathematically, this may be translated into a Poisson process N_t whose jumps are interpreted as the arrival of new information. Since not all information is relevant for company i, we introduce the factor $b_i \in (0,1]$ to represent the probability of company i to respond with a jump in its firm-value process to a jump of the ticker process N_t. In mathematical terms, this construction corresponds to a thinned-out Poisson process with intensity λ and thinning probability $(1-b_i)$. In what follows, this process is denoted by $N_t(b_i)$. Finally, we obtain the following model for the firm-value process of company i:

$$V_t^i = v_0^i \exp\left(X_t^i\right), \quad X_t^i = \gamma_i t + \sigma_i\big(a_i W_t^M + \sqrt{1-a_i^2} W_t^i\big) + \sum_{j=1}^{N_t(b_i)} Y_j^i. \tag{8}$$

All Brownian motions W_t^M and W_t^i, as well as all random variables defining the jump components, are assumed to be mutually independent in Equation (8). The first consequence of this construction is the following lemma.

Lemma 3.1 (Original jump-diffusion model for all margins)
In distribution, the firm-value process of company i agrees with v_0^i times the exponential of a jump-diffusion process with diffusion volatility σ_i, jump intensity $b_i\lambda$ and jump-size distribution $\mathbb{P}_{Y^i}$.

Lemma 3.1 results from the fact that W_t^M and W_t^i being independent, their weighted sum $a_i W_t^M + \sqrt{1-a_i^2} W_t^i$ agrees in distribution with a Brownian motion at time t. Moreover, it is deduced from the definition that the thinned-out Poisson process $N(b_i)$ agrees in distribution with a regular Poisson process with intensity $b_i\lambda$.

A common market factor and dependent jumps

So far, we assumed jumps of all firm-value processes to be mutually independent. However, it seems reasonable that most news affect different firms in a similar manner. Therefore, we propose to classify new information as being good or bad for the economy. Based on the observation that markets are highly correlated in extreme events we may give up the assumption of independent jumps, assuming jumps of all companies at the same jump time to have a common sign, instead. However, we do not impose identical jump sizes. The sign of all jumps at some jump time τ_l is determined by an initial Bernoulli experiment with success probability $p = 50\%$. The outcome of this experiment specifies whether an information is considered to be good or bad. In order to not changing the individual default probabilities, we have to restrict the set of jump-size distributions of each individual firm-value process by the condition

$$\mathbb{P}_{Y^i}\left(Y_l^i > 0\right) = \mathbb{P}_{Y^i}\left(Y_l^i < 0\right) = 0.5, \quad \forall i \in \{1, \dots, I\}, \forall l \in \mathbb{N}.$$

Let us remark that neither does the choice $p = 50\%$ imply symmetric jump-size distributions nor are we restricted to identical probabilities for up- and downward jumps, as we can assign positive probability to up- or downward jumps of neglectable size. Still, implementing this variant of the model is simplified considerably if a jump-size distribution is chosen where up and downward jumps are easy to distinguish. Examples are two-sided exponentially distributed jumps and normally distributed jumps with zero mean.

3.2 Consequences of dependent jumps

Our numerical investigations in Section 4.2 show that jumps in the same direction significantly increase the default correlation among the firms. This implies that tail events such as multiple defaults become more likely. Thus we need a higher risk capital to cover for potential losses and, for CDOs, larger spreads in the senior tranches are implied, as it becomes more likely for such a tranche to suffer from a loss. A sample plot of three dependent firm-value processes is given in Figure 1.

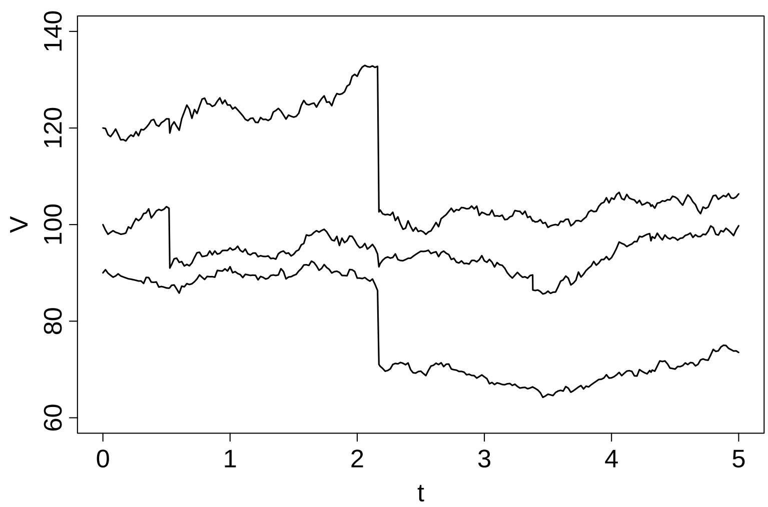

Figure 1: The sample paths of three dependent firm-value processes.

In the example of Figure 1, the first jump at $\tau_1 = 0.53$ only affects one firm. In contrast, the second jump at $\tau_2 = 2.17$ affects the other two firms. Jumps are forced in the same direction but have different sizes. The diffusion components are coupled to the market using $a_i = 0.5$ for $i = 1, 2, 3$. Following each negative jump it is likely that several firms default simultaneously. This matches the empirical observation of default clusters.

3.3 Segmentation by industry sector

This approach on coupling the individual firm-value processes is inspired by mapping each company to a specific branche. For instance, each company of the European *iTraxx* portfolio is assigned to one of the six branches *Auto*, *Consumer*, *Energy*, *Financial*, *Industrial* and *TMT*. More abstractly, we may assume S different industry sectors, indexed by $s \in \{1, \ldots, S\}$. We can now introduce a factor for the market W_t^M and a factor for each industry sector W_t^s. The Brownian motion of each company is replaced by a weighted sum consisting of its individual Brownian motion W_t^i, the Brownian motion of the respective industry sector W_t^s and the Brownian motion of the market W_t^M. As abbreviation, we introduce

$$\tilde{W}_t^i(a_i, c_i) = a_i W_t^M + c_i W_t^s + \sqrt{1 - a_i^2 - c_i^2} W_t^i, \quad a_i, c_i \in (-1, 1),\ a_i^2 + c_i^2 \leq 1.$$

Incorporating dependence via jumps in this framework combines the idea of using common factors with the idea of supporting joint jumps from Section 3.1. More precisely, we assume some information to be relevant to all companies and other to affect only a specific industry sector. Finally, some news are only relevant to an individual company. The ticker processes reporting these pieces of information are independent Poisson processes denoted by N_t^M, N_t^s and N_t^i, respectively, their intensities are denoted by λ^M, λ^s and λ^i. For company i, relevant news are therefore reported by the superposition of market, sector and individual Poisson process. This superposition is abbreviated as

$$\tilde{N}_t^i = N_t^M + N_t^s + N_t^i.$$

The model of the firm-value process of company i is then given as

$$V_t^i = v_0^i \exp\left(X_t^i\right), \quad X_t^i = \gamma_i t + \sigma_i \tilde{W}_t^i(a_i, c_i) + \sum_{j=1}^{\tilde{N}_t^i} Y_j^i. \tag{9}$$

Again, we find that the univariate margins agree with the single-firm model of Chapter 2.

Lemma 3.2 (Original jump-diffusion model for all margins)
The firm-value process of company i, belonging to sector s, agrees in distribution with v_0^i times the exponential of a jump-diffusion process with diffusion volatility σ_i, jump intensity $\lambda_i = \lambda^M + \lambda^s + \lambda^i$ and jump-size distribution $\mathbb{P}_{Y^i}$.

Using basic properties of the normal distribution and independence of the Brownian motions W_t^M, W_t^s and W_t^i, it is easily seen that the weighted sum $\tilde{W}_t^i(a_i, c_i)$ agrees in distribution with a Brownian motion at time t. Moreover, the superposition of independent Poisson processes is again a Poisson process and the intensity of the superposition is the sum of the intensities of its summands.

In Section 5 we introduce an algorithm for the pricing of CDOs within the model of Section 3.1. Altering this algorithm for an implementation of our model with different industry sectors only requires minor changes. However, we focus on the first version of the model as we do not have sufficient market data to fit this latter variant.

3.4 Properties and applications of the model

One important feature of our model is the possibility of modeling different firms with different sets of firm-value parameters. This distinguishes our approach from models which accept the simplification of a homogeneous portfolio, e.g. Vasicek (1987). Moreover, we do not rely on the popular conditional independence assumption, compare Hull and White (2004), Laurent and Gregory (2003) and others, as dependence is introduced via jumps and via diffusion in our model. Also, we explicitly model the evolution of each firm-value process over time, which allows the simultaneous pricing across all maturities. Finally, we continuously test for default which generalizes structural models such as Willemann (2007). In this section, we briefly comment on consequences and possible applications of these properties.

1. **Creating sub-portfolios**: Given identical companies, the loss distribution of each sub-portfolio only depends on its size. In contrast, sub-portfolios in our model are automatically equipped with a realistic default structure, if the individual firms are calibrated appropriately. An application of this property is the simultaneous pricing of CDS sector indices, that are sub-portfolios consisting only of companies of one industry sector.

2. **Sensitivity to defaults**: If a company defaults, it is likely that this company was rated below the average rating of the portfolio. In a model with identical companies, CDO spreads of the remaining portfolio remain unchanged. In our model, the average default probability of the remaining portfolio decreases, if one of the substandard companies defaults. The result is that spreads of a newly issued CDO contract, based on the remaining companies, are decreasing. We interpret this as the relief of the market that one of the substantial risk factors is removed from the portfolio. The opposite holds, if a company defaults which was considered to be a safe investment. Then, the average default probability of the remaining portfolio increases, and so do spreads of a new CDO contract on the remaining portfolio.

3. **Changes in the portfolio's constitution**: For instance, the composition of the *iTraxx* portfolio changes twice per year. Several companies are delisted from the portfolio and replaced by new firms. The default structure of the updated portfolio obviously depends on the relative creditworthiness of the new companies compared to the old firms. Capturing this feature also requires heterogeneous companies.

4. **Simultaneously describing single and multi-name derivatives**: Our model is able to describe the term structure of default probabilities of each company in the portfolio, as implied by individual CDS spreads for different maturities. This property is presumably the major improvement to pure dependence models for practical applications. When it comes to a calibration of the model, this feature is a burden (as a large number of parameters have to be adjusted) and an advantage (as it allows the use of a vast quantity of market information as input for the calibration) at the same time. We demonstrate this in detail in Section 6.

5. **A time consistent framework**: Another important aspect is the consistency of our model with respect to time, as we explicitly model the evolution of each firm-value process. Therefore, the model specifies the complete term structure of portfolio loss distributions, which allows to price portfolio derivatives with different maturities within a consistent framework.

4 Dependence among the companies in our model

4.1 Asset value correlation

In this section we present a closed-form expression of the correlation of X_t^i and X_t^j, the exponents of the respective firm-value processes V_t^i and V_t^j. If both processes are continuous, that is

$$X_t^l = \gamma_l t + \sigma_l \big(a_l W_t^M + \sqrt{1-a_l^2} W_t^l\big), \quad l \in \{i,j\},$$

then $\mathrm{Cov}\left(X_t^i, X_t^j\right) = \sigma_i \sigma_j a_i a_j t$ is deduced from independence of all Brownian motions and properties of the covariance. Therefore, the correlation of X_t^i and X_t^j is given by

$$\mathrm{Corr}\left(X_t^i, X_t^j\right) = a_i a_j. \tag{10}$$

As the continuous model is a special case of our jump-diffusion framework, a general result necessarily reduces to Equation (10) if we set $\lambda = 0$, $b_i = b_j = 0$ or $\mathbb{P}_Y^i = \mathbb{P}_Y^j = \delta_0$. This necessary property is easily verified in Theorems 4.1 and 4.2 presented below.

Theorem 4.1 (Asset-value correlation, independent jumps)
Given the model of Section 3.1 with independent jumps and square integrable jump-size distributions $\mathbb{P}_Y^i$ and $\mathbb{P}_Y^j$, the correlation of X^i and X^j satisfies

$$\rho^X = Corr\left(X_t^i, X_t^j\right) = \frac{\sigma_i \sigma_j a_i a_j + \lambda b_i b_j \mathbb{E}[Y^i]\mathbb{E}[Y^j]}{\sqrt{\sigma_i^2 + \lambda b_i \mathbb{E}\left[(Y^i)^2\right]}\sqrt{\sigma_j^2 + \lambda b_j \mathbb{E}\left[(Y^j)^2\right]}}.$$

Proof: By independence of both diffusion and jump components and by repeating the arguments of the pure diffusion case above, we obtain

$$\begin{aligned}\mathrm{Cov}\left(X_t^i, X_t^j\right) &= \sigma_i \sigma_j \mathrm{Cov}\left(a_i W_t^M, a_j W_t^M\right) + \mathrm{Cov}\left(CP_t^i, CP_t^j\right) \\ &= \sigma_i \sigma_j a_i a_j t + \lambda t b_i b_j \mathbb{E}[Y^i]\mathbb{E}[Y^j],\end{aligned} \tag{11}$$

where the abbreviation $CP_t^l = \sum_{k=1}^{N_t(b_l)} Y_k^l$ for $l \in \{i,j\}$ is used. To justify Equation (11) we have to derive the covariance of the jump components. This is achieved by first conditioning on the number of information $N_t = k$, then on the number of jumps $N_t(b_i) = l^i$ and $N_t(b_j) = l^j$, respectively. Given the Poisson distributed random variable $N_t \sim \mathrm{Poi}(\lambda t)$, the number of jumps of the thinned-out Poisson processes $N_t(b_i)$ and $N_t(b_j)$ follow binomial distributions with respective parameters. We identify the sums involved in the following computation as the expectation of a binomial distribution and the second moment of a Poisson distribution. Therefore, we find

$$\begin{aligned}\mathbb{E}\left[CP_t^i CP_t^j\right] &= \mathbb{E}\left[\mathbb{E}[CP_i CP_j | \sigma(N_t, N_t(b_i), N_t(b_j))]\right] \\ &= \mathbb{E}\Bigg[\sum_{k=0}^{\infty}\Bigg(\sum_{l^i=0}^{k}\sum_{l^j=0}^{k}\Bigg(\sum_{l=1}^{l^i} Y_l^i\Bigg)\Bigg(\sum_{l=1}^{l^j} Y_l^j\Bigg)\cdot \\ &\qquad \binom{k}{l^i} b_i^{l^i}(1-b_i)^{k-l^i}\binom{k}{l^j} b_j^{l^j}(1-b_j)^{k-l^j}\Bigg)\frac{(\lambda t)^k e^{-\lambda t}}{k!}\Bigg] \\ &= b_i b_j \mathbb{E}[Y^i]\mathbb{E}[Y^j]\sum_{k=0}^{\infty} k^2 \frac{(\lambda t)^k e^{-\lambda t}}{k!} \\ &= b_i b_j \mathbb{E}[Y^i]\mathbb{E}[Y^j]\left((\lambda t)^2 + \lambda t\right).\end{aligned}$$

The claim follows since $I\!E[CP_t^l] = t\lambda b_l I\!E[Y^l]$ for $l \in \{i,j\}$, and the variances of the jump-diffusion processes X_t^l are given by $\mathrm{Var}(X_t^l) = t\left(\sigma_l^2 + \lambda b_l I\!E[(Y^l)^2]\right)$, for $l \in \{i,j\}$ respectively. $\diamondsuit$

The assumption of independent jumps considerably simplified the main computation of this proof. If jumps in the same direction are imposed instead, then one has to additionally condition on the number of jumps of both companies in either direction. Under some mild conditions on the symmetry of the jump-size distributions, we are able to obtain the corresponding result in the context of Section 3.1 with jumps in the same direction.

Theorem 4.2 (Asset-value correlation, dependent jumps)
We consider the model of Section 3.1 with jumps in the same direction. We define $Y_\oplus^l$ and $Y_\ominus^l$, for $l \in \{i,j\}$, to be the size of a positive and negative jump, respectively, given the sign of the jump Y^l. More precisely, the distributions of $Y_\oplus^l$ and $Y_\ominus^l$ are given for $x > 0$ by $I\!P(Y_\oplus^l \le x) = I\!P(Y^l \le x | Y^l > 0)$ and $I\!P(Y_\ominus^l \le x) = I\!P(-Y^l \le x | Y^l < 0)$, respectively. Again we assume square integrable jump-size distributions $I\!P_{Y^l}$ for $l \in \{i,j\}$. Further, we impose the following assumptions on the symmetry of the jump-size distributions

$$I\!P_{Y^l}\left(Y^l > 0\right) = I\!P_{Y^l}\left(Y^l < 0\right) = 0.5, \quad l \in \{i,j\},$$

and

$$I\!E\left[Y_\oplus^i\right] = I\!E\left[Y_\ominus^i\right] = I\!E\left[|Y^i|\right], \quad I\!E\left[Y_\oplus^j\right] = I\!E\left[Y_\ominus^j\right] = I\!E\left[|Y^j|\right].$$

Then, the processes X^i and X^j satisfy

$$\rho^X = Corr\left(X_t^i, X_t^j\right) = \frac{\sigma_i\sigma_j a_i a_j + \lambda b_i b_j I\!E[|Y^i|] I\!E[|Y^j|]}{\sqrt{\sigma_i^2 + \lambda b_i I\!E\left[(Y^i)^2\right]}\sqrt{\sigma_j^2 + \lambda b_j I\!E\left[(Y^j)^2\right]}}.$$

Proof: Large parts of this proof are similar to the proof of Theorem 4.1. The main difference is the more complicated computation of the expectation of the product of the jump components, i.e. $I\!E[CP_t^i CP_t^j]$, as jumps at common jump times are forced to have the same direction. We again start by conditioning on the amount of information up to time t. By construction, this is an upper bound for the number of jumps of the processes X^i and X^j up to time t. Following a $\mathrm{Poi}(\lambda t)$ distribution, we have $I\!P(N_t = k) = (\lambda t)^k e^{-\lambda t}/k!$. Given $N_t = k$, we additionally condition on how much of this news is positive. This yields

$$I\!P(l|k) = I\!P(\, l \text{ positive news } | N_t = k) = \binom{k}{l} 0.5^l (1-0.5)^{k-l}, \quad 0 \le l \le k.$$

Given k news from which l are classified as good, the number of bad news is given by $k-l$. The conditional probabilities of X^i to have exactly $l_\oplus^i$ upward and $l_\ominus^i$ downward jumps are therefore given and abbreviated as

$$\begin{aligned} I\!P(l_\oplus^i | l) &= \binom{l}{l_\oplus^i} b_i^{l_\oplus^i} (1-b_i)^{l-l_\oplus^i}, \quad 0 \le l_\oplus^i \le l, \\ I\!P(l_\ominus^i | l) &= \binom{k-l}{l_\ominus^i} b_i^{l_\ominus^i} (1-b_i)^{k-l-l_\ominus^i}, \quad 0 \le l_\ominus^i \le k-l. \end{aligned}$$

The probabilities $\mathbb{P}(l^j_\oplus|l)$ and $\mathbb{P}(l^j_\ominus|l)$ are defined similarly for X^j. Using these abbreviations we rewrite $\mathbb{E}\left[CP^i_t CP^j_t\right]$ as

$$\mathbb{E}\Big[\sum_{k=0}^{\infty}\big(\sum_{l=0}^{k}\mathrm{IS}^i(l,k)\cdot\mathrm{IS}^j(l,k)\mathbb{P}(l|k)\big)\mathbb{P}(N_t=k)\Big], \tag{12}$$

where the inner sums $\mathrm{IS}^i(l,k)$ and $\mathrm{IS}^j(l,k)$ are defined as

$$\begin{aligned}\mathrm{IS}^i(l,k) &= \sum_{l^i_\oplus=0}^{l}\sum_{l^i_\ominus=0}^{k-l}\left(\sum_{h=1}^{l^i_\oplus}Y^i_{\oplus_h}-\sum_{h=1}^{l^i_\ominus}Y^i_{\ominus_h}\right)\mathbb{P}(l^i_\oplus|l)\mathbb{P}(l^i_\ominus|l),\\ \mathrm{IS}^j(l,k) &= \sum_{l^j_\oplus=0}^{l}\sum_{l^j_\ominus=0}^{k-l}\left(\sum_{h=1}^{l^j_\oplus}Y^j_{\oplus_h}-\sum_{h=1}^{l^j_\ominus}Y^j_{\ominus_h}\right)\mathbb{P}(l^j_\oplus|l)\mathbb{P}(l^j_\ominus|l).\end{aligned}$$

We take the expectation inside in Equation (12) and use that $\mathbb{E}\left[|Y^i|\right]=\mathbb{E}\left[Y^i_\oplus\right]=\mathbb{E}\left[Y^i_\ominus\right]$. Using the expectation of a binomial distribution with l, respectively $k-l$, experiments and success probability b_i, we find

$$\begin{aligned}\mathbb{E}\left[\mathrm{IS}^i(l,k)\right] &= \sum_{l^i_\oplus=0}^{l}\sum_{l^i_\ominus=0}^{k-l}\left(l^i_\oplus\mathbb{E}[|Y^i|]-l^i_\ominus\mathbb{E}[|Y^i|]\right)\mathbb{P}(l^i_\oplus|l)\mathbb{P}(l^i_\ominus|l)\\ &= (2l-k)b_i\mathbb{E}\left[|Y^i|\right].\end{aligned}$$

Similarly, it holds that $\mathbb{E}\left[\mathrm{IS}^j(l,k)\right]=(2l-k)b_j\mathbb{E}\left[|Y^j|\right]$. We further observe that

$$\sum_{l=0}^{k}(2l-k)^2 b_i b_j\mathbb{E}[|Y^j|]\mathbb{E}[|Y^i|]\mathbb{P}(l|k)=b_i b_j\mathbb{E}[|Y^j|]\mathbb{E}[|Y^i|]k.$$

This holds, since the sum allows the interpretation of being $4b_ib_j\mathbb{E}[|Y^j|]\mathbb{E}[|Y^i|]$ times the variance of a binomial distribution with k experiments and success probability 50%. Finally, the outer sum is $b_ib_j\mathbb{E}[|Y^j|]\mathbb{E}[|Y^i|]$ times the expectation of a $\mathrm{Poi}(\lambda t)$ distribution. Therefore, we find

$$\mathbb{E}[CP^i_tCP^j_t]=\lambda t b_i b_j\mathbb{E}[|Y^j|]\mathbb{E}[|Y^i|].$$

At the same time, this is the covariance of the jump components, since $\mathbb{E}[CP^i_t]=\mathbb{E}[CP^j_t]=0$, by the initial assumptions on the symmetry of $\mathbb{P}_{Y^i}$ and $\mathbb{P}_{Y^j}$. $\diamondsuit$

Let us finish this section with a brief remark on the results of Theorems 4.1 and 4.2. First of all, the postulated correlations are within the required range of $[-1,1]$, due to the Cauchy-Schwarz inequality. Moreover, both results contain the pure diffusion model as a special case. The fact that allowing common jumps does not necessarily increase the asset correlation of two firms is remarkable. For instance, if the expectation values of the jump-size distributions of two companies have opposite signs, then it is even possible to model a negative correlation of X^i and X^j. However, modeling jumps using common signs always implies a positive correlation of the respective jump components, which agrees with our intuition.

4.2 Default correlation in our model

In the context of credit-risk modeling, the default correlation of two companies is defined below.

Definition 4.1 (Default correlation)
Let the default status of company $i \in \{1, \dots, I\}$ at time t be explained by the indicator variable $D^i = \{D^i_t\}_{t \geq 0}$, where $D^i_t = \mathbf{1}_{\{\tau^i \leq t\}}$. The default correlation of two companies $i \neq j$ up to time $t > 0$ is then defined as

$$\rho^D_t = Corr\left(D^i_t, D^j_t\right) = \frac{\mathbb{E}\left[D^i_t D^j_t\right] - \mathbb{E}[D^i_t]\mathbb{E}[D^j_t]}{\sqrt{Var(D^i_t)\,Var(D^j_t)}}.$$

Being Bernoulli distributed random variables, the expectation and variance of D^i_t can easily be expressed in terms of default probabilities. Without jumps, these default probabilities are even known in closed-form. In the presence of jumps, these default probabilities have to be estimated by means of a Monte Carlo simulation or can be approximated via their Laplace transform if two-sided exponentially distributed jumps are assumed. More delicate is the calculation of $\mathbb{E}[D^i_t D^j_t]$, which is the probability of a joint default of company i and j up to time t. This expectation is difficult to obtain, as for non-degenerated choices of model parameters, the firm-value processes of both companies are dependent random variables and so are their running minimums. However, in a purely continuous model it is possible to express the default correlation of two companies in terms of a double integral of Bessel functions. This result was derived by Zhou (2001a) using several results of Rebholz (1994). In the presence of jumps, we have to rely on a Monte Carlo simulation to estimate these quantities. Before we present our findings, let us remark that such a Monte Carlo simulation requires a large number of runs to produce reliable results, since multiple defaults are rare events. Our Monte Carlo simulation is a simple modification of the first part of Algorithm 5.1, we omit the details of this modification. We controlled the accuracy of our algorithm by reproducing some of the tables of Zhou (2001a) in a pure diffusion case.

Figure 2 exhibits simulated default correlations of two identical companies with parameters $\gamma = 0$, $\sigma = 0.05$, two-sided exponentially distributed jumps with symmetric jumps-size distribution and $\lambda_\oplus = \lambda_\ominus = 20$ as parameter for the exponential distribution, i.e. $\mathbb{P}_Y = $ 2-Exp$\,(20, 20, 50\%)$, and a debt-to-value ratio of $d/v_0 = 85\%$. This experiment is performed in the framework of Section 3.1 with dependent jumps. On the left-hand side, we fix $a = 0.5$ and vary b. To keep the individual default probabilities constant, we fix the individual jump intensity, which is the product $b\lambda$. The figure on the right-hand side is calculated based on a fixed level of $b = 0.5$, $\lambda = 4$ and different levels of a, where we recall that changes in a do not affect individual default probabilities.

An obvious but nevertheless important observation is that our model has two parameters to adjust the default correlation, giving more flexibility compared to pure diffusion models with a single common factor. Moreover, supporting common jumps produces simultaneous defaults already for small maturities. This differs significantly from the situation in a continuous model, where multiple defaults within the first year are extremely rare events. The fact that a continuous model requires much more time to generate a relevant default correlation becomes even more evident in Figure 3. This observation has a massive consequence for the pricing of portfolio-credit derivatives with small maturity, as a continuous model handles the companies almost as if they where

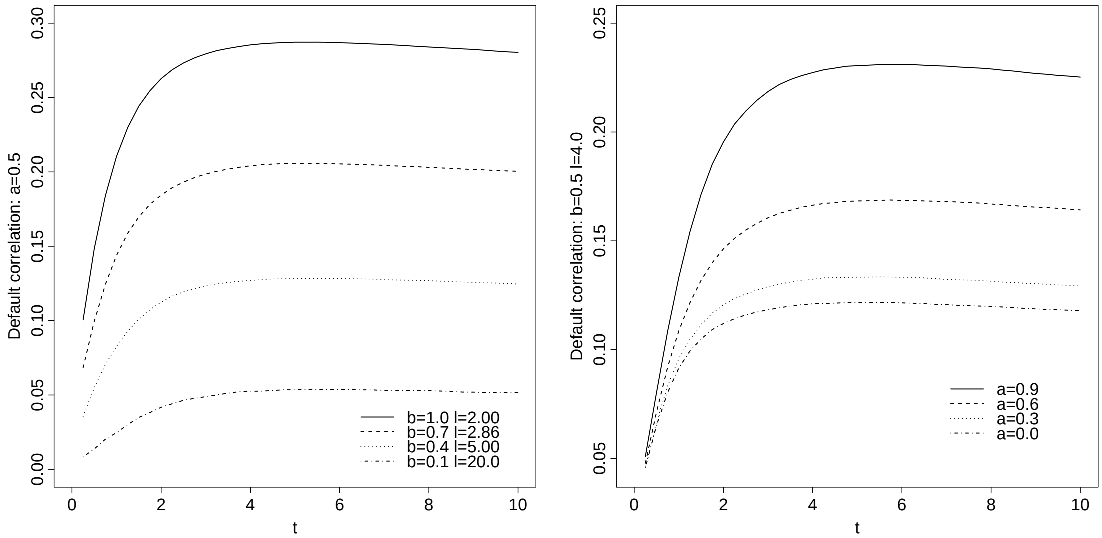

Figure 2: Implied default correlations for fixed a and b, respectively.

independent. Figure 3 is produced using the same setup as before, the parameters are $\gamma = 0$, $\sigma = 0.05$, $\mathbb{P}_Y = \text{2-Exp}\,(20, 20, 50\%)$, $d/v_0 = 85\%$, $a = 0.4$, and $b = 0.5$, the three scenarios differ by the influence of jumps. This experiment shows that the largest default correlation was implied across all maturities by the model of Section 3.1 with dependent jumps. Recalling Theorem 4.1, we compute that adding independent jumps with zero expectation to a continuous model actually decreases the asset-value correlation. Still, the model of Section 3.1 with independent jumps implies a larger default correlation than a continuous model for small maturities. For longer maturities, the opposite holds, showing that the asset correlation should not be used alone to measure the dependence among two companies. The phenomenon that continuous models require much more time to produce a relevant default correlation should be kept in mind if portfolio derivatives with short maturities are priced.

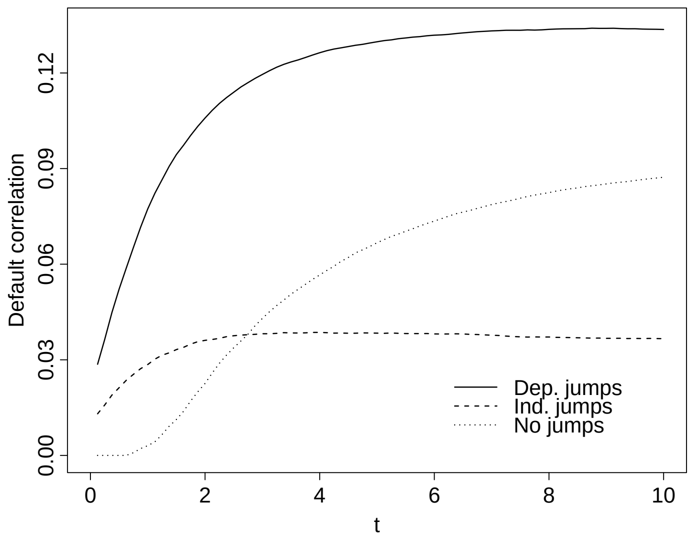

Figure 3: Default correlations with independent, dependent and no jumps.

5 Pricing CDOs via Monte Carlo

CDOs are portfolio derivatives whose payment streams depend on a pool of defaultable assets. The general idea is to pool credit-risky assets and to resell the resulting portfolio in several tranches with different risk profile. Mathematically, the tranches of a CDO can be seen as options on the overall loss distribution of the portfolio. We assume the reader to be familiar with the mechanics of CDOs, a capacious introduction to legal issues, taxation and accounting questions has been published by *JP Morgan*, compare Lukas (2001).

In order to mathematically describe the payment streams we introduce the following abbreviations. Let J be the number of tranches, indexed by j, and the overall portfolio loss process at time t be denoted by L_t. Each tranche $j \in \{1, \ldots, J\}$ is specified by its upper u^j and lower l^j attachment point. For instance, the first tranche usually covers $[0\%, 3\%]$ of the portfolio. Using these attachment points we can express the loss L_t^j in tranche j up to time t as a function of the portfolio loss L_t via

$$L_t^j = \min\left(\max\left(0, L_t - l^j\right), u^j - l^j\right).$$

Todays market standard is defined by the European *iTraxx* portfolio and its American equivalent *DJ CDX*. Table 2 lists their respective segmentation.

$I = 125$ companies		*iTraxx*		*DJ CDX*	
j	Tranche	l^j	u^j	l^j	u^j
1	Equity	0%	3%	0%	3%
2	Junior mezzanine	3%	6%	3%	7%
3	Senior mezzanine	6%	9%	7%	10%
4	Senior	9%	12%	10%	15%
5	Super senior	12%	22%	15%	30%

Table 2: The *iTraxx* and *DJ CDX* segmentation with relative attachment points.

The premium and default legs of a CDO

Specifying a CDO begins with fixing a payment schedule $0 < t_1 < \ldots < t_n = T$, where quarter-yearly payments are a typical convention and assumed in what follows. The protection buyer is committed to pay the product of remaining nominal and spread (relative to the length of the preceding period) of the respective tranche. Hence, the expected discounted premium leg of tranche j is given by (with attachment points as absolute values)

$$EDPL^j = \sum_{k=1}^{n} s^j \Delta t_k e^{-rt_k} \left(u^j - l^j - \mathbb{E}[L_{t_k}^j]\right), \tag{13}$$

where a flat interest rate $r > 0$ is assumed and s^j is the annualized spread of this tranche. In practice, the default (or protection) leg of tranche j allows payments at any time up to maturity. A default payment occurs if some company defaults and the resulting loss affects the specific tranche.

To simplify the computation we defer default payments to the next premium payment date. This assumption allows to conveniently express the expected discounted default leg of tranche j as

$$EDDL^j = \sum_{k=1}^{n} e^{-rt_k} \left(\mathbb{E}[L^j_{t_k}] - \mathbb{E}[L^j_{t_{k-1}}] \right). \tag{14}$$

As for most swap products, CDO tranches (with maturity T) are typically quoted in terms of their annualized spread s^j_T. This spread is chosen such that the expected discounted default leg agrees with the expected discounted premium leg of the same tranche. Let us remark that it has become market practice to modify the premium stream of the equity tranche using a fixed spread of 500 bp, which is usually below the fair spread of this tranche. Therefore, an additional upfront payment is needed which is paid upon settlement. The amount of upfront payment is quoted in percent of the nominal of the equity tranche. The fair upfront payment therefore satisfies

$$(\text{upfront in } \%) \cdot \left(u^1 - l^1\right) \ + \sum_{k=1}^{n} 0.05\, \Delta t_k e^{-rt_k} \left(u^1 - l^1 - \mathbb{E}[L^1_{t_k}]\right) = EDDL^1.$$

An efficient pricing algorithm based on a Monte Carlo simulation is given below, further required notations are the recovery rate R^i and nominal N^i of company i.

Algorithm 5.1 (Monte Carlo estimation of CDO tranche spreads)
Within each simulation run, perform the following steps.

I) Simulate the required random variables

1. *Simulate the number of information arriving until T, i.e. $N_T \sim Poi(\lambda T)$.*
2. *Simulate the location of $0 < \tau_1 < \ldots < \tau_{N_T} < T$. Conditioned on N_T, these random times are distributed as order statistics of $Uni(0,T)$ distributed random variables on $[0,T]$, which allows an efficient implementation.*
3. *Define an equidistant grid on $[0,T]$ with mesh κ which contains all premium payment dates. We propose to choose a monthly grid, that is $\kappa = 1/12$.*
4. *Combine this equidistant grid with $\tau_1 < \ldots < \tau_{N_T}$ to a refined partition of $[0,T]$. Denote the points of this partition by $0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = T$. This partition is not equidistant but contains the jump times of all companies and the times of all premium payments. Note that in between two points of this partition each firm-value process is continuous.*
5. *Simulate a realization of the Brownian motion of the market W^M on the partition above. More precisely, generate a series of independent random variables $x_1, \ldots, x_{n+1}$, where $x_j \sim \mathcal{N}(0, \Delta t_l)$. Then, inductively compute $W^M_{t_l}$ via*

$$W^M_{t_0} = 0, \quad W^M_{t_l} = W^M_{t_{l-1}} + x_l, \quad \forall l \in \{1, \ldots, n+1\}.$$

6. *Simulate the exponent of each firm-value process at the points of the partition. First, determine the jumps of each firm-value process. Simulate I times a series of N_T independent Bernoulli experiments, where the success probabilities in the i^{th} series are given by b_i. The*

outcomes of these experiments are denoted by B_l^i *and specify the jump times of each firm-value process. More precisely, a success in experiment* l *of series* i*, i.e.* $B_l^i = 1$*, corresponds to a jump of* V^i *at position* τ_l*.*

The next step depends on whether we assume independent jump sizes, that corresponds to the model of Section 3.1 with independent jumps, or jumps in the same direction.

(a) *If jumps are assumed to be independent, simulate* I *times* N_T *independent random numbers, denoted by* y_l^i*, where*

$$y_l^i \sim \begin{cases} \mathbb{P}_{Y^i} & : \; B_l^i = 1, \\ \delta_0 & : \; B_l^i = 0. \end{cases}$$

(b) *If dependent jumps are used, perform* N_T *independent Bernoulli experiment with success probability* $p = 50\%$ *to determine the common sign at each* τ_l*. Then, simulate each* y_l^i *as above, conditioned on the respective sign.*

Finally, simulate the increments of the diffusion components in between the points of the grid. This corresponds to simulating I *times* $n+1$ *independent random numbers* x_l^i*, where* $x_l^i \sim \mathcal{N}(0, \Delta t_l)$*, and the construction below.*

For each company, inductively compute $X_{t_0}^i, X_{t_1-}^i, X_{t_1}^i, \ldots, X_{t_{n+1}}^i$ *by* $X_{t_0}^i = 0$ *and*

$$\begin{aligned} X_{t_l-}^i &= X_{t_{l-1}}^i + \gamma_i \Delta t_l + \sigma_i \big(a_i \Delta W_{t_l}^M + \sqrt{1 - a_i^2} x_l^i \big), \quad \forall\, l \in \{1, \ldots, n+1\}, \\ X_{t_l}^i &= \begin{cases} X_{t_l-}^i + y_l^i & : \; \exists\, j \in \{1, \ldots, N_T\} \,:\, t_l = \tau_j, \\ X_{t_l-}^i & : \; t_l \neq \tau_j, \, \forall\, j \in \{1, \ldots, N_T\}, \end{cases} \\ & \qquad \forall\, l \in \{1, \ldots, n+1\}. \end{aligned}$$

7. *Define* $\mathcal{F}^*$ *as the information of all firm-value processes on the grid, i.e.*

$$\mathcal{F}^* = \sigma \left\{ X_{t_l-}^i, X_{t_l}^i, t_l : \, l \in \{0, \ldots, n+1\}, \, i \in \{1, \ldots, I\} \right\}.$$

8. *Calculate survival probabilities of each company up to each point of the grid, conditional on* $\mathcal{F}^*$*. These are found using the distribution of the running minimum of a Brownian bridge, compare Borodin and Salminen (1996), page 61, or Karatzas and Shreve (1997), page 265. Let*

$$\mathbb{P}\mathbb{P}_l^i = \mathbb{P}\left(\tau^i \geq t_l | \mathcal{F}^* \right) = \mathbb{P}\left(\inf_{0 \leq s < t_l} \left\{ X_s^i \right\} > \log\left(d^i / v_0^i \right) \Big| \mathcal{F}^* \right).$$

Moreover, denote the conditional probability of each firm not to default within $[t_{l-1}, t_l)$ *by*

$$\mathbb{P}_l^i = \mathbb{P}\left(\tau^i \notin [t_{l-1}, t_l) | \mathcal{F}^* \right) = \mathbb{P}\left(\inf_{t_{l-1} \leq s < t_l} \left\{ X_s^i \right\} > \log\left(d^i / v_0^i \right) \Big) \Big| \mathcal{F}^* \right),$$

which is found using the distribution of the minimum of a Brownian motion.

II) Estimate the expected loss at each t_l

1. *Initialize* $\left(L_{t_0}, \ldots, L_{t_{n+1}} \right)$ *by* $(0, \ldots, 0)$*.*

2. *For each company* $i \in \{1, \ldots, I\}$ *let* $l \in \{0, \ldots, n\}$ *loop through each point of the grid and consider the following cases.*

(a) $X^i_{t_{l+1}-} > \log(d^i/v^i_0)$ *and* $X^i_{t_{l+1}} > \log(d^i/v^i_0)$.

These conditions prevent company i from defaulting by jump at t_{l+1}. Nevertheless, even if the firm-value process at time t_{l+1} is above d^i, the probability of its running minimum to touch this level is given by $1 - \mathbb{P}^i_{l+1}$. We consider this probability of overseeing such a default by increasing $L_{t_{l+1}}$ by

$$\mathbb{P}\mathbb{P}^i_l \left(1 - \mathbb{P}^i_{l+1}\right) N^i(1 - R^i).$$

The factor $\mathbb{P}\mathbb{P}^i_l \left(1 - \mathbb{P}^i_{l+1}\right)$ is the conditional probability given $\mathcal{F}^*$ of company i to survive up to t_l- and to default in $[t_l, t_{l+1})$.

(b) $X^i_{t_{l+1}-} \leq \log(d^i/v^i_0)$ *or* $X^i_{t_{l+1}} \leq \log(d^i/v^i_0)$.

In this case, company i defaults within $(t_l, t_{l+1}]$. We increase $L_{t_{l+1}}$ by

$$\mathbb{P}\mathbb{P}^i_l N^i(1 - R^i).$$

3. *So far, we calculated the losses which occurred in each interval of the partition. These losses are now aggregated to obtain the cumulative loss at each t_l. For $l = 0, \dots, n-1$, increase $L_{t_{l+1}}$ by L_{t_l}.*

III) Estimate the expected discounted premium leg of each tranche

1. *Initialize the estimate $EDPL^j$ by zero for all tranches $j \in \{1, \dots, J\}$.*

2. *For each tranche $j \in \{1, \dots, J\}$ loop through all premium payment dates $\{t^p_1, \dots, t^p_n\}$. At each payment date, increase $EDPL^j$ by the estimated expected discounted premium payment given $\mathcal{F}^*$, which is given by*

$$e^{-rt^p_l} \cdot \Delta t^p_l \cdot \left(u^j - l^j - \min\left\{\max\left\{0, L_{t^p_l} - l^j\right\}, u^j - l^j\right\}\right), \quad t^p_l \in \{t^p_1, \dots, t^p_n\}.$$

3. *If accrued interest has to be considered, we proceed as follows. Given the payment dates t^p_{l-1} and t^p_l and the corresponding losses, we have to assess where these losses occurred. To approximate this, let t_k loop from the point after t^p_{l-1} to t^p_l. For each t_k, increase $EDPL^j$ by*

$$e^{-rt^p_l} \left((t_k + t_{k-1})/2 - t^p_{l-1}\right) \left(L^j_{t_k} - L^j_{t_{k-1}}\right).$$

This product is the discount factor of the payment date t^p_l, the distance between the midpoint of t_{k-1} and t_k to the previous payment date t^p_{l-1} and the loss of tranche j in $(t_{k-1}, t_k]$.

IV) Estimate the expected discounted default leg of each tranche

1. *Initialize the estimate $EDDL^j$ by zero for all tranches $j \in \{1, \dots, J\}$.*

2. *For each tranche $j \in \{1, \dots, J\}$ loop through all premium payment dates $\{t^p_1, \dots, t^p_n\}$. At each payment date, increase $EDDL^j$ by the expected discounted loss within the preceding period given $\mathcal{F}^*$, which is given by*

$$e^{-rt^p_l} \cdot \left(L^j_{t^p_l} - L^j_{t^p_{l-1}}\right), \quad t^p_l \in \{t^p_1, \dots, t^p_n\},$$

where $L^j_{t^p_l} = \min\left\{\max\left\{0, L_{t^p_l} - l^j\right\}, u^j - l^j\right\}$.

V) Summarizing each Monte Carlo run

1. *After each Monte Carlo run, store the expected discounted premium and default leg of all tranches $j \in \{1, \ldots, J\}$, indexed by the number of the current run.*
2. *Reinitialize all variables and proceed with the next run.*
3. *After the final run, calculate the average of all expected discounted premium and default legs of each tranche, as simulated in the different runs. These quantities are then used to estimate the expected fair spread of the corresponding tranche or to assess the market value of the contract, whichever is of interest.*

5.1 Discussion of the pricing algorithm

It is typical for numerical routines to face a tradeoff between speed and accuracy. In what follows, we explain how the algorithm is efficiently implemented. Also, we address some numerical pitfalls which have to be considered. A common configuration of CDO portfolios (*iTraxx* convention) sets the number of companies to $I = 125$, which emphasizes the importance of a fast implementation of this high-dimensional problem.

1. To begin with, let us discuss on how many random numbers the algorithm requires in each Monte Carlo run. In total, we have to simulate I firm-value processes on a grid. On average, this grid consists of $\mathbb{E}[N_T] = \lambda T$ random times and $T\kappa^{-1} + 1$ systematic points. For each increment Δt_l, a realization of a normally distributed random variable is required for each company and the Brownian motion of the market. At each τ_l it is further required to check for jumps via a Bernoulli experiment and to simulate a jump size if the respective experiment succeeds. In total, about $\mathcal{O}(T(3\lambda + \kappa^{-1})I)$ simulations of random numbers are required[4]. In our implementation, we use the random-number generators of the NAG-software library.
2. For each company, it is further required to compute the conditional survival probabilities $\mathbb{P}\mathbb{P}_l^i$ and $\mathbb{P}_l^i$. To do so, the iteration $\mathbb{P}\mathbb{P}_{l+1}^i = \mathbb{P}\mathbb{P}_l^i \cdot \mathbb{P}_{l+1}^i$ is useful. Moreover, as soon as $\mathbb{P}\mathbb{P}_l^i = 0$ for some l, this iteration is stopped and all following $\mathbb{P}\mathbb{P}_k^i$ are set to zero. The remaining $\mathbb{P}_k^i$ are not required any more.
3. An important concern is the mesh κ of the systematic grid of $[0, T]$, as our algorithm implies a small discretization bias with respect to the common factor W^M. In between two points of the grid, we take into account the individual probability of a company to default unobserved. What we do not control is the evolution of the common factor W^M, which is possibly responsible for multiple defaults. Therefore, we slightly underestimate the default correlation of the model. However, several numerical experiments (with a pure-diffusion model as worst case scenario for the discretisation bias) have shown that a monthly grid is fine enough for pricing CDOs. In our experiments, a finer grid did not change spreads below the noise of the Monte Carlo simulation.
4. If a CDO contract is already on the run, the time until the first premium payment does not agree with the time between every other two premium payments. In this case, an equidistant

[4] For a typical 5-year contract in an *iTraxx* framework, this number is around 20,000.

grid typically does not contain all premium payment dates. Therefore, one needs to insert one additional small interval, for instance, $\Delta t_1 < \kappa$ can be chosen appropriately.

6 Calibration

A calibration of our model is ambitious, since the model contains a large number of parameters. However, we designed the model to simultaneously explain single and multi-name derivatives, which allows the use of a vast quantity of market quotes as input variables. The key observation for fitting the model is Lemma 3.1. This result shows that the parameter a_i does not affect the distribution of τ^i. Also, as long as the product $b_i\lambda$ is kept constant, one can adjust b_i without altering the term structure of default probabilities of company i. Summarized, this suggests to initially fit the individual firm-value parameters, followed by a calibration of the parameters of the dependence structure in which the marginal default probabilities remain unchanged.

6.1 Calibration of each firm to individual CDS quotes

We used CDS quotes of the seventh European *iTraxx* series to fit the individual firm-value processes. The *iTraxx* portfolio contains 125 companies from six business sectors with a large spectrum of ratings, emphasizing the inadequacy of a homogeneous portfolio assumption. Spreads are quoted for contracts maturing in one, three, five, seven and ten years, respectively. All contracts are computed based on a recovery rate of 40%. Missing values where extrapolated or interpolated using the average slope of all available companies, attached to the listed CDS contract with the closest maturity of the same company. Additionally, *iTraxx* provides portfolio CDS spreads and spreads for the different tranches of the CDO. Matching these prices was our criteria for a fit of the dependence structure. The term structure of default-free interest rates in the Eurozone was obtained from *Bloomberg*. These deterministic interest rates were used to replace the flat interest rate $r > 0$ in all pricing formulas.

The calibration algorithm

We assume two-sided exponentially distributed jumps, which allows the use of our numerical routine for evaluating CDS spreads. To decrease the dimension of the parameter space we fix $p^i = 50\%$ (the probability for upward jumps) and $\lambda^i_\oplus = \lambda^i_\ominus$ (the parameter of the exponential distribution). This simplification significantly accelerates the convergence of the minimization and accounts for the fact that our set of data only contains five data points per company, which makes a calibration with more parameters too unstable. The objective function that is minimized is the sum of relative differences

$$\left(\hat{\gamma}^i,\, \hat{\sigma}^i,\, \hat{\lambda}^i,\, \hat{\lambda}^i_\oplus = \hat{\lambda}^i_\ominus\right) = \operatorname{argmin} \sum_{t\in\{1,3,5,7,10\}} \left| \frac{c_t^{R,i} - c_t^{M,i}}{c_t^{R,i}} \right|, \tag{15}$$

where $c_t^{M,i}$ is the model spread of company i, depending on the set of parameters, and $c_t^{R,i}$ is the market quote for the respective firm and maturity. This construction guarantees that all maturities are considered to equal parts in the minimization. The initial leverage ratio was obtained from the last available balance sheet of each company, as reported by *Bloomberg*. The level d was chosen as the sum of all short term liabilities and 50% of all long term debt.

Our implementation starts with a search for parameters on a coarse grid over the parameter space. Then, the minimization routine `nag_opt_bounds_no_deriv` is started at the best grid point of the previous step. It turned out that this minimization routine converged extremely fast from this initial position. For the multidimensional model, the additional constraint $\lambda \geq \max_{i\in\{1,\dots,I\}} \hat{\lambda}^i$ is required, as jumps are triggered by the common ticker process N with intensity λ. Therefore, we restrict each parameter λ^i by some artificial upper bound λ^{max} in Equation (15). In our set of data, all companies had an implied jump intensity of less than two, so $\lambda^{max} = 2$ was a condition which did not decrease the fitting capability of the model. For the latter calibration of the dependence parameters, the initial value for b_i is set to $b_i = \hat{\lambda}^i/\lambda^{max}$, the initial intensity λ of the ticker process N is set to λ^{max}.

The calibration to CDO spreads

At first, we have to specify the measure of distance of model to market quotes. Choosing such a measure is not obvious, as the first tranche is quoted in terms of an upfront payment, while the other tranches are quoted in basis points. In what follows, we choose the parameters of the dependence structure such that the quoted equity tranche is matched by the model, i.e. the deviation of model to market upfront is below bid-ask spreads (for which we assumed 0.08%). Then, the sum of absolute distances of model to market spreads over all remaining tranches is used as a measure of the fitting capability of the model. Our CDO pricing algorithm is a Monte Carlo simulation, which complicates the use of sophisticated search routines. Therefore, we implement a naïve search on a grid over the dependence parameter of the model. To make this approach numerically tractable, we have to reduce the dimension of the problem. To do so, we assume a homogeneous correlation of all firms to the market factor, i.e. $a = a_i$ for all $i \in \{1, \dots, I\}$. Adjusting the parameters b_i is done conditional on the constraint $\hat{\lambda}^i = b_i\lambda$ for all $i \in \{1, \dots, I\}$, which is required for preserving the previously calibrated individual default probabilities. Therefore, we gradually increase λ and adjust each b_i appropriately. The implied dependence is obviously decreasing in λ. More precisely, with λ^{max} as described above, we define

$$\lambda(x) = \lambda^{max}/x, \quad b_i(x) = \hat{\lambda}^i x/\lambda^{max}, \quad x \in (0,1]. \tag{16}$$

This construction guarantees a constant jump intensity of $b_i(x)\lambda(x) \equiv \hat{\lambda}^i$ for each firm and all $x \in (0,1]$. Given this construction, we proceed as follows.

1. Define a grid on $[0,1) \times (0,1]$. We used 30×30 equidistant points, where more points would obviously improve the fitting capability of the model for the costs of more computations.

2. Derive CDO spreads using Algorithm 5.1 for each point of the grid.

3. Compute the required measure of distance for each point of the grid.

4. Find the minimal distance of model to market prices on the grid.

5. Use Equation (16) to retrieve $\hat{b}_i$ from $\hat{x}$, λ^{max} and $\hat{\lambda}^i$.

Adjusting d/v_0 to match index CDS spreads

Up to this point, we only used individual CDS and CDO tranche quotes to fit the model. Additionally, *iTraxx* provides index CDS (or portfolio CDS) quotes of the respective portfolio. One can use index spreads to check whether the combined individual default probabilities agree with the market's expectation for the portfolio, which typically differs due to liquidity effects. In our calibration, we observed that these prices are matched relatively well. Typically, a deviation of less than two basis points was observed for the five year spread. To also incorporate these spreads, it is possible to adjust the individual default probabilities of all companies until a better fit to observed portfolio CDS spreads is achieved. To do so, we suggest multiplying each leverage ratio d^i/v_0^i by an appropriate correction term, which is determined prior to the calibration to CDO quotes. In our calibration of the jump-diffusion model, a correction of less than 1.5% was usually sufficient, while for the pure diffusion model a factor of about 16% was required. Of course, adjusting the individual default probabilities goes along with sacrificing precision in matching individual CDS contracts. Therefore, this adjustment should not be used if individual and portfolio derivatives are priced simultaneously.

6.2 Results of the individual calibration to CDS spreads

We calibrate each of the 125 firms to CDS data of five days - giving 625 individual calibrations. The same calibrations are performed within a pure diffusion setup (PD) - corresponding to the model of Black and Cox (1976). Reported are pricing errors in bp, averaged over all 125 companies, for each day and maturity in Table 3. We observe that the pure diffusion model has massive problems in matching one- and ten-year spreads. This mispricing is the result of the unrealistic hump-size structure of spreads, with zero limit at the short end. Therefore, one- and ten-year spreads are typically underestimated, while three-, five- and seven-year spreads are overestimated, instead. In contrast, the jump-diffusion model (JD) turns out to be flexible enough to fit every term-structure of CDS spreads of the *iTraxx* portfolio with high precision.

Maturity	1 year		3 years		5 years		7 years		10 years	
Day	JD	PD	JD	PD	JD	PD	JD	PD	JD	PD
6.12	1.01	4.85	0.40	0.74	0.66	2.52	0.61	1.56	1.97	9.98
6.14	1.10	5.08	0.55	0.59	0.62	2.55	0.23	1.46	2.10	10.05
6.19	0.95	4.86	0.57	0.75	0.62	2.55	0.29	1.57	2.11	10.51
6.21	1.26	5.54	0.60	0.75	0.70	2.60	0.60	1.78	2.36	10.37
6.26	0.96	5.62	0.61	0.98	0.94	2.97	0.52	2.15	2.54	11.37

Table 3: Average absolute pricing error in bp.

6.3 Results of the calibration to CDO spreads

The results of our CDO calibration are computed with 250,000 Monte Carlo runs and presented in Table 4. If we use the firm-value parameters as obtained from the CDS calibration (to simultaneously describe CDS and CDO spreads), we observe that index CDS quotes are matched relatively well by the jump-diffusion model but not at all by the pure diffusion model, whose results are so far from market quotes that we did not report them. Reasons for not perfectly matching the index CDS are mispricings in the individual calibration (especially for the pure diffusion case) and also a liquidity premium which raises individual spreads. Still, we obtained realistic quotes for CDO tranches with this setup. The fitting quality of the individual tranches is further improved if an adjustment, as described in Section 6.1, is used. For the pure diffusion case (aPD), an adjustment of about 16% was required, the required adjustment in the jump-diffusion framework (aJD) was about 1.5%. Focusing on the fitting capability of the jump-diffusion model, we notice that the fit of the second and fifth tranche is not satisfying, the other tranches are priced very accuracy. As long as a perfect fit to the equity tranche is required (which comprises by far the most spread), the second tranche is overpriced by the model, the fifth tranche is underpriced. This phenomenon[5] is reduced, but not completely eliminated by presence of jumps.

Day	Mod.	up	s^2	s^3	s^4	s^5	iCDS	a	x	d_1	d_{2-5}
6.12	real	7.40	45.12	11.84	5.18	2.14	21.48	-	-	-	-
	aJD	7.46	76.21	14.90	3.60	0.38	21.37	0.93	0.53	0.06	37.49
	JD	7.48	105.02	26.79	7.97	0.94	22.75	0.77	0.87	0.08	78.84
	aPD	7.92	72.19	11.69	2.25	0.14	21.65	0.40	-	0.52	32.14
6.14	real	7.36	45.75	12.00	5.11	2.19	21.10	-	-	-	-
	aJD	7.35	65.76	11.02	2.36	0.24	20.84	0.90	0.47	0.01	25.69
	JD	7.39	106.39	27.91	9.01	1.50	22.85	0.97	0.77	0.03	81.14
	aPD	7.44	69.77	11.17	2.12	0.13	21.27	0.40		0.08	29.91
6.19	real	7.13	47.28	12.344	5.77	2.12	21.01	-	-	-	-
	aJD	7.11	70.53	12.55	2.78	0.25	20.88	0.93	0.50	0.01	28.32
	JD	7.20	108.12	28.92	8.90	1.35	22.79	0.87	0.87	0.08	81.32
	aPD	7.34	69.03	11.00	2.07	0.13	21.18	0.40	-	0.21	28.79
6.21	real	8.48	50.47	13.28	6.11	2.43	22.78	-	-	-	-
	aJD	8.51	89.59	18.74	5.01	0.58	22.58	0.97	0.57	0.01	47.52
	JD	8.48	122.69	34.43	10.80	1.70	24.26	0.80	0.93	0.00	98.79
	aPD	7.98	88.81	18.68	4.73	0.50	22.47	0.43	-	0.50	47.04
6.26	real	11.87	63.70	16.26	7.35	3.18	24.12	-	-	-	-
	aJD	11.84	77.69	11.78	2.41	0.25	23.99	0.90	0.37	0.03	26.34
	JD	11.87	130.57	33.05	9.53	1.34	26.45	0.90	0.70	0.00	87.68
	aPD	11.43	75.26	9.75	1.52	0.09	23.83	0.37	-	0.47	27.00

Table 4: CDO prices (5 years): JD model with dependent jumps, w/w.o adjusted d/v_0.

7 Conclusion

In this paper we presented a tractable structural model for analysing the term structure of portfolio loss distributions. Starting from the single firm-value processes we are able to couple these univariate processes to a multidimensional model based on a firm economic interpretation. An

[5]This phenomenon is well known for pure diffusion models and also present in Willemann's model.

important feature of our approach is that the distribution of the individual firm-value processes remains unchanged. Hence, the term structure of marginal default probabilities is also retained and can therefore be fitted individually. In addition, we are able to accurately model the dependence among the companies. This is possible by tacking account for dependent jumps in the firm value processes. To demonstrate the applicability of our approach we present a Monte Carlo simulation for the pricing of CDOs which takes advantage of the specific structure of our jump-diffusion model.

References

Albrecher, H., S. Ladoucette, and W. Schoutens, 2007, *A generic one-factor Levy model for pricing synthetic CDOs*Birkhäuser Verlag pp. 259–278.

Black, F., and C. Cox, 1976, Valuing corporate securities: some effects of bond indenture provisions, *Journal of Finance* 31, 351–367.

Borodin, A., and P. Salminen, 1996, *Handbook of Brownian motion - facts and formulae.* (Birkhäuser).

Hull, J., M. Pedrescu, and A. White, 2006, The valuation of correlation-dependent credit derivatives using a structural model, working paper, retrieved from http://www.rotman.utoronto.ca/ hull/DownloadablePublications/StructuralModel.pdf.

Hull, J., and A. White, 2004, Valuation of a CDO and an n-th to default CDS without a Monte Carlo simulation, *Journal of Derivatives* 12, 8–23.

Karatzas, I., and S. Shreve, 1997, *Brownian motion and stochastic calculus.* (Springer) second edition edn.

Kou, S., and H. Wang, 2003, First passage times of a jump diffusion process, *Advances in Applied Probability* 35, 504–531.

Laurent, J-P., and J. Gregory, 2003, Basket Default Swaps, CDO's and Factor Copula, Preprint ISFA Actuarial School, University of Lyon & BNP Paribas.

Lukas, D., 2001, CDO handbook, working paper of *JP Morgan.*

Merton, R., 1974, On the pricing of corporate debt: the risk structure of interest rates, *Journal of Finance* 29, 449–470 reprinted as Chapter 12 in Merton, R. (1990), Continuous-time finance, Blackwell.

Metwally, S., and A. Atiya, 2002, Using Brownian bridge for fast simulation of jump-diffusion processes and barrier options, *The Journal of Derivatives* 10, 43–54.

Rebholz, J., 1994, Planar diffusions with applications to mathematical finance, Ph.D. thesis University of California, Berkeley.

Scherer, M., 2007, Tractable multi-firm default models based on discontinuous processes, Ph.D. thesis Universität Ulm.

Schönbucher, P., 2006, Portfolio losses and the term structure of loss transition rates: A new methodology for the pricing of portfolio credit derivatives, Preprint, ETH Zürich.

Sidenius, J., v. Piterbag, and L. Andersen, 2005, A new framework for dynamik credit portfolio loss modelling, Working paper, Royal Bank of Scotland and Barclay's Capital and Bank of America.

Vasicek, O., 1987, Probability of loss on loan portfolio, KMV Corporation.

Willemann, S., 2007, Fitting the CDO Correlation Skew: A Tractable Structural Jump-Diffusion Model., *Journal of Credit Risk* 3.

Zhou, C., 2001a, An analysis of default correlations and multiple defaults, *Review of Financial Studies* 14, 555–576.

Zhou, C., 2001b, The term structure of credit spreads with jump risk, *Journal of Banking and Finance* 25, 2015–2040.