Oscillation and Spectral Theory for for Symplectic Difference Systems with Separated Boundary Conditions

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OSCILLATION AND SPECTRAL THEORY FOR SYMPLECTIC DIFFERENCE SYSTEMS WITH SEPARATED BOUNDARY CONDITIONS

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ABSTRACT. We consider symplectic difference systems involving a spectral parameter together with general separated boundary conditions. We establish the so-called oscillation theorem which relates the number of finite eigenvalues less than or equal to a given number to the number of focal points of a certain conjoined basis of the symplectic system. Then we prove Rayleigh's principle for the variational description of finite eigenvalues and we describe the space of admissible sequences by means of the (orthonormal) system of finite eigenvectors. The principle rôle in our treatment is played by the construction where the original system with general separated boundary conditions is extended to a system on a larger interval with Dirichlet boundary conditions.

1. INTRODUCTION

We consider the (discrete) *symplectic eigenvalue problem*

$$\text{(S)} \qquad x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k, \quad u_{k+1} = \mathcal{C}_k x_k + \mathcal{D}_k u_k - \lambda \mathcal{W}_k x_{k+1}, \quad 0 \le k \le N,$$

together with the general separated boundary conditions

$$\text{(B)} \qquad R_0^* x_0 + R_0 u_0 = 0, \quad R_{N+1}^* x_{N+1} + R_{N+1} u_{N+1}.$$

We assume that $\mathcal{A}_k, \mathcal{B}_k, \mathcal{C}_k, \mathcal{D}_k$ are real $n \times n$ matrices for $0 \le k \le N$ such that the matrices

$$\mathcal{S}_k := \begin{pmatrix} \mathcal{A}_k & \mathcal{B}_k \\ \mathcal{C}_k & \mathcal{D}_k \end{pmatrix}$$

are *symplectic*, i.e., $\mathcal{S}_k^T \mathcal{J} \mathcal{S}_k = \mathcal{J}$, $\mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, I being the $n \times n$ identity matrix. The $n \times n$ matrices $\mathcal{W}_k$ are supposed to be symmetric and nonnegative definite for $0 \le k \le N$. Then the difference system (S) is symplectic for all eigenvalue parameters

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$\lambda \in \mathbb{R}$ and we can write it in the form

$$z_{k+1} = (\mathcal{S}_k - \lambda \hat{\mathcal{S}}_k) z_k, \quad \hat{\mathcal{S}}_k := \begin{pmatrix} 0 & 0 \\ \mathcal{W}_k \mathcal{A}_k & \mathcal{W}_k \mathcal{B}_k \end{pmatrix}.$$

Concerning the matrices in the boundary conditions (B), we suppose that

$$\operatorname{rank}\bigl(R_0^* \; R_0\bigr) = \operatorname{rank}\bigl(R_{N+1}^* \; R_{N+1}\bigr) = n \tag{1}$$

and that

$$R_0^* R_0^T \quad \text{and} \quad R_{N+1}^* R_{N+1}^T \quad \text{are symmetric.} \tag{2}$$

Note that conditions (1) and (2) imply that the eigenvalue problem (S), (B) is *self-adjoint*, we will return to this point later in our paper.

The results given in this paper can be regarded as a continuation of the research initiated in [4, 5, 6]. In [4] we have established an oscillation theorem for (S) together with various boundary conditions. However, when that paper was written, the crucial concept of *multiplicity* of a focal point did not exist (it was established only later in [12]), so the results of [4] are formulated under relatively restrictive assumptions and for parameters λ outside of a certain exceptional set (witch cannot be decribed explicitly). A substantial part of the results of our paper relies on the concepts of *finite eigenvalue* and *finite eigenvector* of (S) which were introduced in [6]. In that paper, the special case of *Dirichlet* boundary conditions $x_0 = 0 = x_{N+1}$ is considered and an oscillation theorem for this eigenvalue problem is proved. As will be seen in the next sections, to extend this statement to general boundary conditions requires a nontrivial construction of a so-called *extended* system. Finally, in [5] we proved, among others, that finite eigenvalues of (S) with Dirichlet boundary conditions can be described via Rayleigh's variational principle, and we proved that the system of finite eigenvectors forms a complete orthonormal basis in the space of the so-called admissible functions. Here we extend these results to general boundary conditions (B), and an important rôle is played again by the construction of the extended eigenvale problem.

Let us finish this introductory section with a brief discussion of our problem in a broader context. Symplectic difference systems are the discrete counterpart of linear Hamiltonian differential systems whose oscillation and spectral theory is deeply developed, and a summary of the results of this theory can be found in [11, 13]. Linear Hamiltonian *difference* systems are a particular case of symplectic difference systems as pointed out in [1], where one can also find the basic theory of these systems. The importance of symplectic difference systems in numerical methods of solving Hamiltonian systems is emphasized in [8]. Symplectic difference systems appear also in discrete calculus of variations and optimal control, and various aspects of this application of symplectic systems, as well as related topics, can be found in [7, 9, 10, 14, 15].

2. Extended system

Throughout the paper, Ker, ind, Im, $\dagger$, and > 0 denote the kernel, index (i.e., the number of negative eigenvalues including their multiplicities), the image, the Moore-Penrose generalized inverse, and positive definiteness of a matrix indicated. Recall that if Q, X are $n \times n$ matrices with Q orthogonal (i.e., $Q^{-1} = Q^T$), then $(QX)^\dagger = X^\dagger Q^{-1}$, see [2] for the definition and properties of generalized inverses.

First we look at the boundary conditions (B). According to [11, Corollary 3.1.3], there exist matrices $S_0, \hat{S}_0, S_{N+1}, \hat{S}_{N+1}$ such that

$$R_0^* = R_0 S_0 + \hat{S}_0, \quad \hat{S}_0 R_0^T = 0, \quad S_0 \text{ symmetric},$$
$$\operatorname{Ker} R_0 = \operatorname{Im} \hat{S}_0^T, \quad \operatorname{rank}\big(R_0 \ \hat{S}_0\big) = n$$

and similarly

$$R_{N+1}^* = R_{N+1} S_{N+1} + \hat{S}_{N+1}, \quad \hat{S}_{N+1} R_{N+1}^T = 0, \quad S_{N+1} \text{ symmetric},$$
$$\operatorname{Ker} R_{N+1} = \operatorname{Im} \hat{S}_{N+1}^T, \quad \operatorname{rank}\big(R_{N+1} \ \hat{S}_{N+1}\big) = n.$$

We may take

$$S_0 := R_0^\dagger R_0^* R_0^T R_0^{\dagger T}, \quad S_{N+1} := R_{N+1}^\dagger R_{N+1}^* R_{N+1}^T R^{\dagger T}_{N+1}, \tag{3}$$

and $\hat{S}_0 = R_0^* - R_0 S_0 = R_0^*(I - R_0^\dagger R_0)$, $\hat{S}_{N+1} = R_{N+1}^*(I - R_{N+1}^\dagger R_{N+1})$. By [11, Proposition 2.1.2], the conditions (B) are equivalent to

$$\left\{ \begin{array}{l} x_0 \in \operatorname{Im} R_0^T, \quad R_0(u_0 + S_0 x_0) = 0, \\ x_{N+1} \in \operatorname{Im} R_{N+1}^T, \quad R_{N+1}(u_{N+1} + S_{N+1} x_{N+1}) = 0. \end{array} \right. \tag{$\hat{\text{B}}$}$$

Now we construct the *extended system* where we extend the original eigenvalue problem (S), (B) considered for $0 \le k \le N$ to a system for $-1 \le k \le N+1$, where we transform general separated boundary conditions (B) at $k = 0$ and $k = N+1$ to *Dirichlet* boundary condition at $k = -1$ and $k = N+2$. While this construction at the left endpoint follows the ideas from [4], the construction at the right endpoint is new and substantially simplifies some technical computations with respect to that paper. We define

$$\mathcal{S}_{-1} = \begin{pmatrix} \mathcal{A}_{-1} & \mathcal{B}_{-1} \\ \mathcal{C}_{-1} & \mathcal{D}_{-1} \end{pmatrix} := \begin{pmatrix} R_0^{*T} K & -R_0^T \\ R_0^T K & R_0^{*T} \end{pmatrix}, \quad \mathcal{W}_{-1} = 0, \tag{4}$$

where $K := (R_0^* R_0^{*T} + R_0 R_0^T)^{-1}$ is nonsingular in view of (1). Then by a direct computation we see that

$$\mathcal{S}_{-1}^T \mathcal{J} \mathcal{S}_{-1} = \begin{pmatrix} K R_0^* & K R_0 \\ -R_0 & R_0^* \end{pmatrix} \begin{pmatrix} R_0^T K & R_0^{*T} \\ -R_0^{*T} K & R_0^T \end{pmatrix} = \mathcal{J}$$

and $\mathcal{S}_{-1}\binom{0}{I} = \binom{-R_0^T}{R_0^{*T}}$. Hence, the extended system starting at $k = -1$ is also symplectic. Moreover, the so-called *principal solution* at $k = -1$ of the extended system, i.e., the solution given by $X_{-1} = 0$, $U_{-1} = I$, corresponds to the solution of the original system (S) satisfying

$$X_0 = -R_0^T, \quad U_0 = R_0^{*T}. \tag{5}$$

Observe that a substantial rôle palyed the fact that we defined $\mathcal{W}_{-1} = 0$.

Returning to vector solutions of the original and the extended system, $x_{-1} = 0$ implies that $x_0 = \mathcal{A}_{-1}x_{-1} + \mathcal{B}_{-1}u_{-1} = -R_0^T u_{-1} \in \operatorname{Im} R_0^T$. If, in addition, $u_0 = \mathcal{C}_{-1}x_{-1} + \mathcal{D}_{-1}u_{-1} - \lambda\mathcal{W}_{-1}x_0 = R_0^{*T}u_{-1}$, then $R_0(u_0 + S_0x_0) = (R_0R_0^{*T} - R_0S_0R_0^T)u_{-1} = 0$. Hence $R_0^*x_0 + R_0u_0 = 0$ by $(\hat{\mathrm{B}})$, so the boundary conditions (B) at the left endpoint $k = 0$ is satisfied. Conversely, if $x_0 \in \operatorname{Im} R_0^T$, i.e., $x_0 = R_0^T R_0^{\dagger^T} x_0$, then $x_0 = \mathcal{A}_{-1}x_{-1} + \mathcal{B}_{-1}u_{-1}$ for $x_{-1} = 0$ and $u_{-1} = -R_0^{\dagger^T} x_0$. Moreover, if $R_0^*x_0 + R_0u_0 = 0$, and if

$$\begin{pmatrix} x_{-1} \\ u_{-1} \end{pmatrix} = \mathcal{S}_{-1}^{-1}\begin{pmatrix} x_0 \\ u_0 \end{pmatrix} = \begin{pmatrix} \mathcal{D}_{-1}^T & -\mathcal{B}_{-1}^T \\ -\mathcal{C}_{-1}^T & \mathcal{A}_{-1}^T \end{pmatrix}\begin{pmatrix} x_0 \\ u_0 \end{pmatrix},$$

then $x_{-1} = R_0^*x_0 + R_0u_0 = 0$ (and $u_{-1} = -KR_0x_0 + KR_0^*u_0$).

In the next part of this section we extend the system (S) at the right endpoint $N+1$. Let $R_{N+1} = QD$ be the polar decomposition of R_{N+1} with Q orthogonal and D symmetric (and nonnegative definite). We define $\mathcal{W}_{N+1} = 0$ and

$$\mathcal{S}_{N+1} = \begin{pmatrix} \mathcal{A}_{N+1} & \mathcal{B}_{N+1} \\ \mathcal{C}_{N+1} & \mathcal{D}_{N+1} \end{pmatrix} := \begin{pmatrix} Q^T R_{N+1}^* & Q^T R_{N+1} \\ -KQ^T R_{N+1} & KQ^T R_{N+1}^* \end{pmatrix}, \tag{6}$$

$K := Q^T(R_{N+1}^* R^{*T}_{N+1} + R_{N+1}R_{N+1}^T)^{-1}Q$. Then (using (2))

$$\mathcal{S}_{N+1}\mathcal{J}\mathcal{S}_{N+1}^T = \begin{pmatrix} Q^T R_{N+1}^* & Q^T R_{N+1} \\ -KQ^T R_{N+1} & KQ^T R_{N+1}^* \end{pmatrix}\begin{pmatrix} R_{N+1}^T Q & R^{*T}_{N+1}QK \\ -R^{*T}_{N+1}Q & R_{N+1}^T QK \end{pmatrix} = \mathcal{J}$$

which is equivalent with $\mathcal{J} = (\mathcal{S}_{N+1}^{-1})^T\mathcal{J}\mathcal{S}_{N+1}^{-1}$ or with $\mathcal{J} = \mathcal{S}_{N+1}^T\mathcal{J}\mathcal{S}_{N+1}$, so that $\mathcal{S}_{N+1}$ is *symplectic*. Moreover (compare [4, formula (14)]), we have that

$$X_{N+2} := \mathcal{A}_{N+1}X_{N+1} + \mathcal{B}_{N+1}U_{N+1} = Q^T\Lambda,$$

where $\Lambda := R_{N+1}^*X_{N+1} + R_{N+1}U_{N+1}$, and

$$X_{N+1}X_{N+2}^\dagger\mathcal{B}_{N+1} = X_{N+1}\Lambda^\dagger R_{N+1},$$

because $X_{N+2}^\dagger = \Lambda^\dagger Q$ (Q is orthogonal) and $\mathcal{B}_{N+1} = Q^T R_{N+1}$.

Next, if $x_{N+1} \in \operatorname{Im} R_{N+1}^T$, i.e., $x_{N+1} = R_{N+1}^T R^{\dagger^T}_{N+1}x_{N+1}$, then

$$x_{N+2} = Q^T(R_{N+1}^*x_{N+1} + R_{N+1}u_{N+1}) = 0 \tag{7}$$

for $u_{N+1} := -R^{*T}_{N+1}R^{\dagger^T}_{N+1}x_{N+1}$. Conversely, if (7) holds, then

$$R_{N+1}^*x_{N+1} + R_{N+1}u_{N+1} = 0,$$

in particular, $x_{N+1} \in \operatorname{Im} R_{N+1}^T$ by $(\hat{\mathrm{B}})$.

The previous computations are summarized in the next statement.

Proposition 1. *A vector* $z = \big(z_k\big)_{k=0}^{N+1}$ *satisfies* (S) *and* (B) *if and only if* $z = \big(z_k\big)_{k=-1}^{N+2}$ *satisfies the extended boundary value problem*

$$z_{k+1} = (\mathcal{S}_k + \lambda\tilde{\mathcal{S}}_k)z_k, \quad -1 \le k \le N+1, \quad x_{-1} = 0 = x_{N+2} \tag{8}$$

with $\mathcal{W}_{-1} = 0 = \mathcal{W}_{N+1}$ *and the matrices* $\mathcal{S}_{-1}, \mathcal{S}_{N+1}$ *given by* (4) *and* (6).

Finally, let $Z = \binom{X_k}{U_k}_{k=0}^{N+1}$ denote the solution of (S) satisfying (5). Then the extended solution $Z_k = \binom{X_k}{U_k}_{k=-1}^{N+2}$ (with $Z_{-1} = \mathcal{S}_{-1}^{-1}Z_0$) is the *principal solution* at $k=-1$ of the extended system with

$$X_{N+2} = Q^T\Lambda \quad \text{and} \quad X_{N+1}X_{N+2}^{\dagger}\mathcal{B}_{N+1} = X_{N+1}\Lambda^{\dagger}R_{N+1}.$$

Observe also that if $z = \big(z_k\big)_{k=-1}^{N+2} = \binom{x_k}{u_k}_{k=-1}^{N+2}$ solves the extended problem (8) with $\big(\mathcal{W}_k x_{k+1}\big)_{k=-1}^{N+1} \ne 0$, then also $\big(\mathcal{W}_k x_{k+1}\big)_{k=0}^{N} \ne 0$ because $\mathcal{W}_{-1} = 0$ and $x_{N+2} = 0$.

3. Quadratic functionals

We say that a sequence $z = \binom{x_k}{u_k}_{k=0}^{N+1}$ is *admissible*, if

$$x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k, \quad 0 \le k \le N, \quad x_0 \in \operatorname{Im} R_0^T,\ x_{N+1} \in \operatorname{Im} R_{N+1}^T, \tag{9}$$

and we say that $z = \binom{x_k}{u_k}_{k=-1}^{N+2}$ is *admissible for the extended system*, if

$$x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k, \quad -1 \le k \le N+1, \quad x_{-1} = 0 = x_{N+2}, \tag{10}$$

where $\mathcal{A}_{-1}, \mathcal{B}_{-1}, \mathcal{A}_{N+1}, \mathcal{B}_{N+1}$ are given by (4) and (6).

In the previous section we have shown that, with the corresponding setting, z is admissible if and only if it is admissible for the extended problem. Moreover, the bilinear forms

$$\langle z, \tilde z\rangle := \sum_{k=0}^{N} x_{k+1}^T\mathcal{W}_k\tilde x_{k+1}, \quad \langle z, \tilde z\rangle_e := \sum_{k=-1}^{N+1} x_{k+1}^T\mathcal{W}_k\tilde x_{k+1} \tag{11}$$

are *the same* because $\mathcal{W}_{-1} = 0 = \mathcal{W}_{N+1}$.

Symplectic difference systems are closely related to quadratic functionals which we consider in the remaining part of this section. Associated with the extended problem is the bilinear form (and the quadratic functional $\mathcal{F}_e(z) := \mathcal{F}_e(z,z)$)

$$\begin{aligned}\mathcal{F}_e(z,\tilde z) &:= \sum_{k=-1}^{N+1}\{x_k^T\mathcal{A}_k^T\mathcal{C}_k\tilde x_k + x_k^T\mathcal{C}_k^T\mathcal{B}_k\tilde u_k + u_k^T\mathcal{B}_k^T\mathcal{C}_k\tilde x_k + u_k^T\mathcal{B}_k^T\mathcal{D}_k\tilde u_k\} \\ &= \mathcal{F}_0(z,\tilde z) + \Delta_{-1} + \Delta_{N+1},\end{aligned} \tag{12}$$

where

$$\mathcal{F}_0(z,\tilde z) := \sum_{k=0}^{N}\{x_k^T\mathcal{A}_k^T\mathcal{C}_k\tilde x_k + x_k^T\mathcal{C}_k^T\mathcal{B}_k\tilde u_k + u_k^T\mathcal{B}_k^T\mathcal{C}_k\tilde x_k + u_k^T\mathcal{B}_k^T\mathcal{D}_k\tilde u_k\} \tag{13}$$

and Δ_{-1}, Δ_{N+1} are given by

$$\begin{aligned}\Delta_{-1} =& x_{-1}^T\mathcal{A}_{-1}^T\mathcal{C}_{-1}\tilde{x}_{-1} + x_{-1}^T\mathcal{C}_{-1}^T\mathcal{B}_{-1}\tilde{u}_{-1} + u_{-1}^T\mathcal{B}_{-1}^T\mathcal{C}_{-1}\tilde{x}_{-1} + u_{-1}^T\mathcal{B}_{-1}^T\mathcal{D}_{-1}\tilde{u}_{-1}\\ =& u_{-1}^T\mathcal{B}_{-1}^T\mathcal{D}_{-1}\tilde{u}_{-1} = -u_{-1}^T R_0 R_0^{*T}\tilde{u}_{-1} = -u_{-1}^T\mathcal{B}_{-1}^T S_0\mathcal{B}_{-1}\tilde{u}_{-1} = -x_0^T S_0\tilde{x}_0,\end{aligned}$$

(here we have used that $x_{-1} = 0 = \tilde{x}_{-1}$ and that $\mathcal{B}_{-1} = -R_0^T$, $\mathcal{D}_{-1} = R_0^{*T}$), and

$$\begin{aligned}\Delta_{N+1} :=& x_{N+1}^T\mathcal{A}_{N+1}^T\mathcal{C}_{N+1}\tilde{x}_{N+1} + x_{N+1}^T\mathcal{C}_{N+1}^T\mathcal{B}_{N+1}\tilde{u}_{N+1}\\ &+ u_{N+1}^T\mathcal{B}_{N+1}^T\mathcal{C}_{N+1}\tilde{x}_{N+1} + u_{N+1}^T\mathcal{B}_{N+1}^T\mathcal{D}_{N+1}\tilde{u}_{N+1}.\end{aligned}$$

In computing Δ_{N+1} we proceed as follows. Denote $\mathcal{K} := \begin{pmatrix} 0 & 0\\ I & 0\end{pmatrix}$. Then using the block structure of $\mathcal{S}$ we have

$$\Delta_{N+1} = z_{N+1}^T\{\mathcal{S}_{N+1}^T\mathcal{K}\mathcal{S}_{N+1} - \mathcal{K}\}\tilde{z}_{N+1} = -z_{N+1}^T\mathcal{K}z_{N+1} = -x_{N+1}^T\tilde{u}_{N+1},$$

where we have used the fact that $z_{N+1}^T\mathcal{S}_{N+1}\mathcal{K}\mathcal{S}\tilde{z}_{N+1} = 0$ since $\mathcal{K}\mathcal{S}_{N+1}z_{N+1} = 0 = \mathcal{K}\mathcal{S}_{N+1}\tilde{z}_{N+1}$ because of admissimility of $z, \tilde{z}$ and $x_{N+2} = 0 = \tilde{x}_{N+2}$. Further,

$$0 = x_{N+2} = Q^T(R_{N+1}^* x_{N+1} + R_{N+1}u_{N+1}),$$

and we have the same equation for $\tilde{x}_{N+2}, \tilde{x}_{N+1}, \tilde{u}_{N+1}$. By $(\hat{\mathrm{B}})$ there exist $c, \tilde{c} \in \mathbb{R}^n$ such that $x_{N+1} = R_{N+1}^T c$, $\tilde{x}_{N+1} = R_{N+1}^T\tilde{c}$. Substituting this into Δ_{N+1}, we obtain

$$\begin{aligned}\Delta_{N+1} =& - x_{N+1}^T\tilde{u}_{N+1} = -c^T R_{N+1}\tilde{u}_{N+1} = c^T R_{N+1}^*\tilde{x}_{N+1} = c^T R_{N+1}^* R_{N+1}^T\tilde{c}\\ =& c^T R_{N+1}S_{N+1}R_{N+1}^T\tilde{c} = x_{N+1}^T S_{N+1}\tilde{x}_{N+1}.\end{aligned}$$

Therefore

$$\mathcal{F}_e(z, \tilde{z}) = \mathcal{F}_0(z, \tilde{z}) - x_0^T S_0\tilde{x}_0 + x_{N+1}^T S_{N+1}\tilde{x}_{N+1}.$$

In the remaining part of this section we look for conditions which guarantee that

$$\mathcal{F}_e(z, z) - \lambda\langle z, z\rangle > 0$$

for any nontrivial admissible z if λ is sufficiently negative. Directly one may verify that the matrix $\mathcal{E}_k := \mathcal{B}_k\mathcal{B}_k^\dagger\mathcal{D}_k\mathcal{B}_k^\dagger$ is symmetric and $\mathcal{B}_k^T\mathcal{E}_k\mathcal{B}_k = \mathcal{B}_k^T\mathcal{D}_k$. Then the particular summands in $\mathcal{F}_0$ (with $z = \tilde{z}$) can be expressed as follows:

$$\begin{aligned}\Delta_k :=& x_k^T\mathcal{A}_k^T\mathcal{C}_k x_k + 2x_k^T\mathcal{C}_k^T\mathcal{B}_k u_k + u_k^T\mathcal{B}_k^T\mathcal{D}_k u_k\\ =& x_k^T\mathcal{A}_k^T C_k x_k + 2x_k^T\mathcal{C}_k^T(x_{k+1} - \mathcal{A}_k x_k) + (x_{k+1} - \mathcal{A}_k x_k)^T\mathcal{E}_k(x_{k+1} - \mathcal{A}_k x_k).\end{aligned}$$

Hence, there exists a constant $c > 0$ such that

$$|\Delta_k|^2 \le c(|x_k|^2 + |x_{k+1}|^2), \quad 1 \le k \le N,$$

where $|\cdot|$ is any norm in $\mathbb{R}^n$. Because x_0 does not appear in the product $\langle z, z\rangle$, we treat Δ_{-1} and Δ_0 separately. We have (again with $\tilde{z} = z$)

$$\begin{aligned}\Delta_{-1} + \Delta_0 = &- x_0^T S_0 x_0 + x_0^T(\mathcal{A}_0^T\mathcal{C}_0 - 2\mathcal{A}_0^T\mathcal{C}_0 + \mathcal{A}_0^T\mathcal{E}_0\mathcal{A}_0)x_0\\ &+ 2x_0^T(\mathcal{C}_0^T - \mathcal{A}_0^T\mathcal{E}_0)x_1 + x_1^T\mathcal{E}_0 x_1\\ &= x_0^T E_0 x_0 + 2x_0^T E_1 x_1 + x_1^T \mathcal{E}_0 x_1,\end{aligned}$$

where

$$E_0 := -S_0 - \mathcal{A}_0^T\mathcal{C}_0 + \mathcal{A}_0^T\mathcal{E}_0\mathcal{A}_0 \quad \text{and} \quad E_1 := \mathcal{C}_0^T - \mathcal{A}_0^T\mathcal{E}_0.$$

Now assume that $x_0^T E_0 x_0 > 0$ on $\operatorname{Im} R_0^T \setminus \{0\}$, i.e., $c^T R_0 E_0 R_0^T c > 0$ for every $c \in \mathbb{R}^n$ such that $R_0^T c \neq 0$. Hence, there exists $\varepsilon > 0$ such that

$$x_0^T E_0 x_0 \geq \varepsilon |x_0|^2 \quad \text{for all } x_0 \in \operatorname{Im} R_0^T \setminus \{0\}.$$

Note that this assumption is "empty" if $R_0 = 0$, i.e., for the Dirichlet boundary conditions. Since $||E_1|| \leq \delta_0$ for some δ_0 ($||\cdot||$ denotes the matric norm in $\mathbb{R}^n$ associated with $|\cdot|$), we obtain for $x_0 \in \operatorname{Im} R_0^T$ and some $\delta_1 > 0$:

$$\begin{aligned}\Delta_{-1} + \Delta_0 \geq &\,\varepsilon |x_0|^2 - 2\delta_0 |x_0||x_1| - \delta_1 |x_1|^2\\ = &\,\varepsilon |x_0|^2 + \left(\sqrt{\varepsilon}|x_0| - \frac{\delta_0}{\sqrt{\varepsilon}}|x_1|\right)^2 - \left(\delta_1 + \frac{\delta_0^2}{\varepsilon}\right)|x_1|^2\\ \geq &\,\varepsilon |x_0|^2 - \left(\delta_1 + \frac{\delta_0^2}{\varepsilon}\right)|x_1|^2.\end{aligned}$$

The previous considerations are summarized in the next statement.

Proposition 2. *Suppose that*

$$\mathcal{W}_k > 0 \quad \textit{for} \quad 0 \leq k \leq N \tag{14}$$

and that

$$E_0 := -S_0 - \mathcal{A}_0^T\mathcal{C}_0 + \mathcal{A}_0^T\mathcal{B}_0\mathcal{B}_0^{\dagger}\mathcal{D}_0\mathcal{B}_0^{\dagger}\mathcal{A}_0 \ \textit{is positive definite on} \ \operatorname{Im} R_0^T \tag{15}$$

hold. Then

$$\begin{cases} \textit{there exists } \lambda_0 < 0 \textit{ such that}\\ \mathcal{F}_0(z,z) - x_0^T S_0 x_0 + x_{N+1}^T S_{N+1} x_{N+1} - \lambda \sum_{k=0}^N x_{k+1}^T \mathcal{W}_k x_{k+1} > 0\\ \textit{for } \lambda \leq \lambda_0 \textit{ and all admissible } z = \binom{x_k}{u_k}_{k=0}^N \textit{ with } \left(\mathcal{W}_k x_{k+1}\right)_{k=0}^N \neq 0. \end{cases} \tag{16}$$

4. Oscillation and spectral theory

In this section we formulate the main results of our paper: oscillation and spectral theorems for the symplectic eigenvalue problem (S) with general separated boundary conditions (B). Recall first some necessary concepts and results.

A $2n \times n$ matrix solution $\binom{X_k(\lambda)}{U_k(\lambda)}$ is said to be a *conjoined basis* of (S) if

$$\begin{cases} X_k^T(\lambda)U_k(\lambda) = U_k^T(\lambda)X_k(\lambda) \quad \text{and} \quad \operatorname{rank}(X_k^T(\lambda) \;\; U_k^T(\lambda)) = n \\ \text{for all } 0 \le k \le N \text{ and } \lambda \in \mathbb{R}. \end{cases}$$

According to [12], we define the focal points including their multiplicities of $Z := \binom{X_k(\lambda)}{U_k(\lambda)}$, using the notation

$$\begin{cases} M_k(\lambda) := (I - X_{k+1}(\lambda)X_{k+1}^\dagger(\lambda))\mathcal{B}_k, \\ T_k(\lambda) := I - M_k^\dagger(\lambda)M_k(\lambda), \\ D_k(\lambda) := T_k(\lambda)X_k(\lambda)X_{k+1}^\dagger(\lambda)\mathcal{B}_kT_k(\lambda). \end{cases} \tag{17}$$

The *number of focal points of* Z in the interval $(k, k+1]$ is defined by

$$m(k,\lambda) := m_1(k,\lambda) + m_2(k,\lambda), \tag{18}$$

where $m_1(k,\lambda) = \operatorname{rank} M_k(\lambda)$ is the multiplicity in the point $k+1$ and $m_2(k,\lambda) = \operatorname{ind} D_k(\lambda)$ is the number of focal points in the open interval $(k, k+1)$.

The central rôle in our treatment is played by the concepts of a finite eigenvalue and a finite eigenvector. These concepts are introduced in [6] for symplectic eigenvalue problems with Dirichlet boundary conditions. We modify their definition as follows.

Definition 1. A number λ is called a *finite eigenvalue* of (S), (B) if

$$\operatorname{rank}\Lambda(\lambda) < r := \max_{\mu\in\mathbb{R}} \operatorname{rank}\Lambda(\mu),$$

and this is equivalent with the existence of a corresponding *finite eigenvector* z, i.e. $z = \binom{x_k}{u_k}_{k=0}^{N+1}$ satisfies (S) and (B) such that $\left(\mathcal{W}_k x_{k+1}\right)_{k=0}^{N} \ne 0$.

Note that the concepts of finite eigenvalue and finite eigenvector of the symplectic eigenvalue problem (S) with Dirichlet boundary conditions (i.e., $R_{N+1}^* = I$, $R_{N+1} = 0$ in (B)) were introduced in [6] and reflected the fact that, in contrast to [4], the eigenvalue problem may be *singular*, i.e., $\det X_{N+1}(\lambda) = 0$ for all $\lambda \in \mathbb{R}$. Also here we admit the singular case, i.e., we *do not exclude* the case that $\det\Lambda(\lambda) = 0$ for every $\lambda \in \mathbb{R}$. Let us also point out that the assumptions (1) and (2) imply that the boundary conditions (B) are *self-adjoint* (see [11, Definition 2.1.2]). It means that $\mathcal{F}(z,\tilde z) = \mathcal{F}(\tilde z, z)$ for every admissible $z = \binom{x}{u}$, $\tilde z = \binom{\tilde x}{\tilde u}$, where

$$\mathcal{F}(z,\tilde z) := \mathcal{F}_0(z,\tilde z) + x_0^T S_0 \tilde x_0 + x_{N+1}^T S_{N+1}\tilde x_{N+1}$$

with $\mathcal{F}_0$ given by (13). It also implies that finite eigenvalues of (S), (B) are *real* and finite eigenvectors corresponding to different finite eigenvalues are orthogonal, see [4, Remark 2(iii)] and [6, Proposition 2].

We will use in our proofs the main results of [6] which we now briefly recall.

Proposition 3. *Let* $Z = \binom{X_k(\lambda)}{U_k(\lambda)}$ *be the principal solution of* (S) *at* $k = 0$, *i.e.,* $X_0 = 0$, $U_0(0) = I$, *and denote by* $n_1(\lambda)$ *the number of its focal points in the interval* $(0, N+1]$. *Further, denote by* $n_2(\lambda)$ *the number of finite eigenvalues of* (S) *together* *with Dirichlet boundary conditions, (i.e.,* $R_0^* = I = R_{N+1}^*$*,* $R_0 = 0 = R_{N+1}$ *in* (S)*) which are less than or equal to* λ*. Then there exists* $\ell \in \mathbb{N} \cup \{0\}$*, such that*

$$n_1(\lambda) = \ell + n_2(\lambda) \quad \textit{for all } \lambda \in \mathbb{R}. \tag{19}$$

Based on this statement, we can now formulate the oscillation theorem for symplectic eigenvalue problems with general separated boundary conditions. In its formulation we use the following notation. Let $Z = \binom{X_k(\lambda)}{U_k(\lambda)}$ be the solution of (S) given by the initial condition (5) (sometimes, such a solution is called *natural conjoined basis* with respect to (B)). We denote by

$$n_1(\lambda) := \sum_{k=0}^{N} m(k, \lambda)$$

the number of its focal points in the interval $(-1, N+1]$, where $m(k, \lambda)$ is given by (18). Further, let

$$\begin{aligned} &\Lambda(\lambda) := R_{N+1}^* X_{N+1}(\lambda) + R_{N+1} U_{N+1}(\lambda), \quad M(\lambda) := [I - \Lambda(\lambda)\Lambda^\dagger(\lambda)] R_{N+1}, \\ &T(\lambda) := I - M^\dagger(\lambda) M(\lambda), \quad D(\lambda) := T(\lambda) X_{N+1}(\lambda) \Lambda^\dagger(\lambda) R_{N+1} T(\lambda), \end{aligned} \tag{20}$$

and

$$m(\lambda) := \operatorname{rank} M(\lambda) + \operatorname{ind} D(\lambda). \tag{21}$$

Finally, we denote by $n_2(\lambda)$ the number of finite eigenvalues of (S), (B) which are less that or equal to λ.

Theorem 1 (Oscillation Theorem)**.** *Suppose that* (16) *holds. Then*

$$n_1(\lambda) + m(\lambda) = n_2(\lambda) \quad \textit{for all } \lambda \in \mathbb{R}.$$

Proof. We transfrom the original eigenvalue problem (S), (B) to the extended eigenvalue problem with Dirichlet boundary conditions (8), where the extended matrices $\mathcal{S}_{-1}$, $\mathcal{S}_{N+1}$ are given by (4), (6), respectively, and $\mathcal{W}_{-1} = 0 = \mathcal{W}_{N+1}$. Then the solution $Z = \binom{X}{U}$ given by (5) is the principal solution of the extended system at $k = -1$. Since

$$M_{-1} = (I - X_0 X_0^\dagger)\mathcal{B}_{-1} = 0, \quad D_{-1} = T_{-1} X_{-1} X_0^\dagger \mathcal{B}_{-1} T_{-1} = 0,$$

no additional focal point appears in $(-1, 0]$ in the extension at -1. Hence the number of focal points of $\binom{X}{U}$ as a solution of the original system in $(0, N+1]$ and as a solution of the extended system in $(-1, N+1]$ is the same. Concerning the interval $(N+1, N+2]$ for the extended system,

$$\begin{aligned} X_{N+2}(\lambda) &= Q^T \Lambda(\lambda), \\ M_{N+1}(\lambda) &= (I - X_{N+2}(\lambda) X_{N+2}^\dagger(\lambda))\mathcal{B}_{N+1}(\lambda) = (I - Q^T \Lambda(\lambda)\Lambda^\dagger(\lambda) Q) Q^T R_{N+1} \\ &= Q^T (I - \Lambda(\lambda)\Lambda^\dagger(\lambda)) R_{N+1} = Q M(\lambda), \end{aligned}$$

where $M(\lambda)$ and $\Lambda(\lambda)$ are given by (20). So $\binom{X}{U}$ has $m(\lambda)$ focal points in $(N+1, N+2]$, including multiplicities, where $m(\lambda)$ is given by (21). By (16), $\mathcal{F}_e(z,z) - \lambda\langle z,z\rangle > 0$ for every admissible $z = \binom{x}{u}$ with $\langle z,z\rangle > 0$ for $\lambda < \lambda_0$, which means (see, e.g. [3, Theorem 1]) that the principal solution at $k = -1$ of the extended system has no focal point in $(-1, N+2]$ if $\lambda < \lambda_0$. Moreover, since $n_2(\lambda) = 0$ for λ sufficiently small, (see [6, Proposition 2]), we have $\ell = 0$ in (19). Now, Proposition 1 applied to the extended system yields the required result. □

Next we recall the main results of [5], which concern the variational description of finite eigenvalues and completeness of the system of finite eigenvectors in the space of admissible sequences.

Consider again the eigenvalue problem (S) for $0 \le k \le N$ with *Dirichlet* boundary conditions $x_0 = 0 = x_{N+1}$. Note that in [5] we supposed $\mathcal{W}_k = I$, $0 \le k \le N$. However, all results of that paper extend literally to nonnegative definite weight matrices $\mathcal{W}_k$ and admissible sequences $z = \binom{x}{u}$ with $\left(\mathcal{W}_k x_{k+1}\right)_{k=0}^{N-1} \neq 0$, i.e. $\langle z,z\rangle > 0$.

Denote by r the number of its finite eigenvalues ($r \le nN$ by [6, Proposition 2]) of (S) with the Dirichlet boundary conditions and suppose that the finite eigenvalues are ordered by their size $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_r$. The finite eigenvectors corresponding to different finite eigenvalues are *orthogonal*, so we may assume that they are orthonormal, and we denote them by $z^{(\mu)} = \binom{x_k^{(\mu)}}{u_k^{(\mu)}}_{k=0}^{N+1}$, $1 \le \mu \le r$. Moreover, finite eigenvectors form a *complete* orthonormal system in the space of admissible sequences, i.e., in the space of $z = \binom{x_k}{u_k}_{k=0}^{N+1}$ satisfying

$$x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k,\ 0 \le k \le N, \quad x_0 = 0 = x_{N+1}.$$

This means that the first component of any admissible $z = \binom{x}{u}$ can be expressed in the form

$$\| \, x - \sum_{\mu=1}^{r} c_\mu x^{(\mu)} \, \|_{\mathcal{W}} = 0 \quad \text{with} \quad c_\mu = \langle z^{(\mu)}, z\rangle, \quad \text{where} \quad \| \cdot \|_{\mathcal{W}} := \sqrt{\langle \cdot, \cdot \rangle}. \tag{22}$$

Finite eigenvalues λ_μ also admit the variational description via Rayleigh's principle

$$\lambda_{m+1} = \min \left\{ \frac{\mathcal{F}_0(z)}{\langle z,z\rangle} : \quad \begin{array}{l} z = (x,u) \text{ is admissible with } x_0 = x_{N+1} = 0, \\ z \perp z^{(\mu)} \text{ for all } 1 \le \mu \le m, \text{ and } \left(\mathcal{W}_k x_{k+1}\right)_{k=1}^{N-1} \neq 0 \end{array} \right\}. \tag{23}$$

Finally, if the minimum in (23) is attained by some $\hat{z} = \binom{\hat{x}}{\hat{u}}$, i.e.,

$$\mathcal{F}_0(\hat{z}) = \lambda_{m+1} \langle \hat{z}, \hat{z}\rangle \quad \begin{array}{l} \text{for some admissible } \hat{z} \text{ with } \hat{x}_0 = \hat{x}_{N+1} = 0 \\ \text{and } \hat{z} \perp z^{(\mu)} \text{ for all } 1 \le \mu \le m, \end{array} \tag{24}$$

then $\hat{z}$ satisfies not only the admissibility equation $\hat{x}_{k+1} = \mathcal{A}_k \hat{x}_k + \mathcal{B}_k \hat{u}_k$ but also the second equation in (S), the so called *Euler equation*:

$$\begin{cases} \tilde{u}_{k+1} = \mathcal{C}_k \hat{x}_k + \mathcal{D}_k \tilde{u}_k - \lambda_{m+1} \mathcal{W}_k \hat{x}_{k+1} \text{ for suitable vectors } \tilde{u}_k \text{ with} \\ \mathcal{B}_k \tilde{u}_k = \mathcal{B}_k \hat{u}_k \text{ for } 0 \le k \le N. \end{cases} \tag{25}$$

Using constructions from Sections 2 and 3 we may now extend these results to the symplectic eigenvalue problem (S), (B). Recall that due to the fact that $\mathcal{W}_{-1} = 0 = \mathcal{W}_{N+1}$, the inner products for the original and extended problems defined by the bilinear forms (11) coincide, so we do not need to distinguish between the products $\langle , \rangle$, and $\langle , \rangle_e$. Moreover, the functional corresponding to the extended problem (12) and the functional

$$\mathcal{F}(z) := -x_0^T S_0 x_0 + x_{N+1}^T S_{N+1} x_{N+1} + \sum_{k=0}^{N} \{ x_k^T \mathcal{A}_k^T \mathcal{C}_k x_k + 2 x_k^T \mathcal{C}_k^T \mathcal{B}_k u_k + u_k^T \mathcal{B}_k^T \mathcal{D}_k u_k \}. \tag{26}$$

are the same, hence the statements of the next theorems follow directly from (22), (23), (24), and (25) applied to the extended system on $[-1, N+2]$. Note that $\mathcal{W}_{-1} = 0$ and use [6, Proposition 2] to see that $r \le n(N+1)$ below.

Theorem 2 (Rayleigh's principle). *Suppose that* (16) *holds. Then the eigenvalue problem* (S), (B) *has at most $n(N+1)$ finite eigenvalues, including multiplicities. Let*

$$\lambda_1 \le \cdots \le \lambda_r, \quad r \le n(N+1),$$

be these eigenvalues ordered by their size, with corresponding orthonormal finite eigenvectors $z^{(1)}, \dots, z^{(r)}$. Then for $0 \le m \le r-1$

$$\lambda_{m+1} = \min \left\{ \frac{\mathcal{F}(z)}{\langle z, z \rangle} : \quad z = \binom{x_k}{u_k}_{k=0}^{N+1} \text{ is admissible, } z \perp z^{(\mu)} \text{ for all } 1 \le \mu \le m, \right. \\ \left. \text{and } \left(\mathcal{W}_k x_{k+1} \right)_{k=0}^{N} \ne 0 \right\},$$

where the quadratic functional $\mathcal{F}$ is given by (26).

Theorem 3 (Euler Equation). *Suppose that* (16) *holds and that λ_μ, $z^{(\mu)}$, $\mu = 1, \dots, r$, are the same as in the previous theorem. If*

$$\mathcal{F}(\hat{z}) = \lambda_{m+1} \langle \hat{z}, \hat{z} \rangle$$

for some admissible $\hat{z} = \binom{\hat{x}}{\hat{u}} \perp z^{(\mu)}$, $\mu = 1, \dots, m$, then (25) *holds.*

Theorem 4 (Expansion Theorem). *Suppose that* (16) *holds and that λ_μ, $z^{(\mu)}$, $\mu = 1, \dots, r$, are the same as in the previous two theorems. Then the finite eigenvectors $z^{(\mu)} = \binom{x^{(\mu)}}{u^{(\mu)}}$, $1 \le \mu \le r$, form an orthonormal basis in the space of admissible sequences given by* (9) *with respect to the norm on* $\operatorname{Im} \mathcal{W}$ *induced by $\mathcal{W}$, i.e., the first component x of any admissible $z = \binom{x}{u}$ satisfies*

$$\| \, x - \sum_{\mu=1}^{r} c_\mu x^{(\mu)} \, \|_{\mathcal{W}} = 0 \quad \textit{with} \quad c_\mu = \langle z^{(\mu)}, z \rangle, \quad \textit{where} \quad \| \cdot \|_{\mathcal{W}} := \sqrt{\langle \cdot, \cdot \rangle}.$$

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