A Note on Life-cycle Funds

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Preprint Series: 2011-10

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This version: 04/08/2011

Abstract

Life-cycle or target-date funds have been gaining tremendous market share and were recently set as default choice of asset allocation in numerous defined contribution schemes or related old age provision products in many countries. Hence, an appropriate assessment of life-cycle funds' risk-return profiles – i.e. the probability distribution of returns – is essential for sustainable financial planning concerning a large group of investors. The core idea of life-cycle funds is to decrease the fund's equity exposure and conversely increase its bond exposure towards the fund's target date. This paper studies the risk-return profile of life-cycle funds in particular compared to standard balanced or lifestyle funds that apply a constant equity portion throughout the fund's term instead.

Assuming a Black-Scholes model, we derive balanced funds that reproduce the risk-return profile of an arbitrary life-cycle fund considering single and regular contributions. We then investigate our results' accuracy applying more sophisticated asset models including stochastic interest rates, stochastic equity volatility and jumps. We further compare our methodology to approximations used so far only taking into account the life-cycle funds' average equity portion throughout the funds' term and conclude that this approximation fails in approximating the life-cycle funds' risk-return profile properly. Our results on the one side facilitate sustainable financial planning and on the other side challenge the very existence of life-cycle funds since appropriately calibrated balanced funds can offer a similar risk-return profile.

1 Introduction

The demographic transition resulting from a continuing increase in life expectancy combined with rather low fertility rates constitutes a severe challenge for government-run pay-as-you-go pension systems in many countries. Therefore, the importance of funded private and/or occupational old age provision has been increasing and will – according to industry surveys – likely continue to increase. Besides state pensions and private old age provision, occupational pension plans provide an essential tier for retirement wealth whose market share is likely to further increase due to planned mandatory 'auto-enrolment' in these plans in many countries[1]. Hence, the underlying pension plan's performance will have a significant impact on retirement wealth and living standard after the active working phase. A large number of these pension plans allocate their contributions to so-called life-cycle or target-date funds[2], since – according to Charlson et al. (2010) – 96% of the large pension plans[3] in the US selected life-cycle funds as their default asset allocation and most of the money allocated in these funds was then kept until retirement. Further, Charlson et al. (2010) state that *"[…] by most measures, target-date funds have been a smashing success"* which is in line with Viceira (2008) finding that assets under management of life-cycle funds in the US have increased from $1bn in 1996 to $120bn in 2006. Summarizing, the assets under management of and further contributions to life-cycle funds are supposed to steadily grow and therefore an appropriate assessment of the life-cycle funds' risk-return profile – i.e. the potential losses or gains – is highly relevant for sustainable financial planning concerning a large group of investors. For assessing the risk-return profile of old age provision products in general, Graf et al. (2010) propose a general methodology and further derive quantitative results for some product types offered in many markets. Among others, they compare an artificial balanced fund[4] – similarly known as lifestyle fund – and an artificial life-cycle fund especially focussing on the effect of different premium modes.

In line with the growing success in terms of assets under management, academia has recently started to investigate life-cycle funds in more detail. Generally, two main streams of research are identified.

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[1] The process of automatically enrolling employees to some pension plan is currently e.g. implemented by the United States or the United Kingdom.

[2] A life-cycle fund invests in risky and riskless assets according to a pre-specified 'glide path' specifying how the asset allocation changes as the target date approaches.

[3] Defined as including more than 5,000 employees.

[4] A balanced fund invests in a constant mix of risky and riskless assets.

The first one is concerned with comparing possible returns of artificial life-cycle funds and artificial balanced funds. Various authors such as Schleef and Eisinger (2007), Spitzer and Singh (2008), Pang and Warschawsky (2008) or Pfau (2010) – to name only few – analyse the results of some life-cycle strategies compared to some balanced funds using different methodologies such as expected utility or some shortfall measures (e.g. the probability of not reaching some investment target at retirement) obtained after simulating returns of the involved risky and riskless asset. Due to the rather artificial choice of the balanced and life-cycle funds under investigation, their conclusions differ from preferring life-cycle funds to balanced funds (e.g. Pfau (2010)) up to concluding that life-cycle funds had to revisit their asset allocation in comparison to balanced funds (e.g. Spitzer and Singh (2008)). In line with the controversy in results and conclusions above, e.g. Pang and Warschawsky (2008) or Schleef and Eisinger (2007) are indifferent of either preferring life-cycle or balanced funds.

The second strain of literature is concerned with finding 'optimal' life-cycle funds. E.g. Cairns et al. (2006) derive optimal path-dependent life-cycle strategies applying stochastic control techniques to maximize expected utility further taking into account random changes in salary and thus contributions to the life-cycle fund. In line with their results, Basu et al. (2009) or Antolín et al. (2010) conclude path-dependent strategies being superior to 'static' life-cycle funds again using simulation techniques and measuring shortfall. In contrast to determining the optimal path-dependent strategy, e.g. Maurer et al. (2007) and Gomes et al. (2008) derive the optimal static life-cycle strategy, i.e. the optimal pre-specified glide path, by maximizing expected utility and conclude the 'classical' glide path of starting with rather high equity exposure and then reducing the equity exposure over time as being optimal. However, Basu and Drew (2009) prefer a 'contrarian' life-cycle strategy – i.e. starting with moderate equity exposure and increasing the equity exposure towards the target date – to the conventional life-cycle due to their so-called 'portfolio size effect' (cf. Basu and Drew (2009)).

Summarizing, recent research does to our knowledge not provide a sound opinion or common understanding of which investment should be preferred (if any), either balanced or life-cycle funds with some (*path-independent*) glide path. Hence, the difference of life-cycle and balanced funds seems not to be understood completely, yet. However, Poterba et al. (2009) argue in their quantitative comparison that *"the results suggest that the distribution of retirement wealth associated with typical life-cycle investment strategies is similar to that from age-invariant asset allocation strategies that set the equity share of the portfolio equal to the average equity share in the life-cycle strategies",* a 'rule of thumb' that has already been proposed by Lewis (2008).

Further, in the recent period of financial distress, life-cycle funds with rather high equity exposure (obviously) suffered losses in value and many investors were enraged, since they thought of life-cycle funds as rather low-risk investment vehicles. Apparently, investors are not aware of the risk inherent in life-cycle funds and the fact that these funds generally come without any investment guarantee and hence may suffer from financial losses in a similar way as pure equity or balanced funds do. Probably balanced funds are better understood by investors and hence the risk bearing potential of these funds is more easily assessed.

The scope of this paper is therefore to investigate the difference of balanced and life-cycle funds in more detail. However, in contrast to the research so far we are not focusing on comparing artificial balanced and life-cycle strategies, but instead for any pre-specified life-cycle fund we are keen on deriving balanced funds that reasonably well match the life-cycle fund's risk-return profile. This, on the one side gives a better understanding of the 'difference' of life-cycle and balanced funds and on the other side provides investors with an easy risk assessment of their underlying life-cycle investments by just assessing the risk of the corresponding balanced fund (which is essentially determined by its constant equity portion). We start with deriving closed form solutions in a Black-Scholes economy treating single and regular contributions to the funds and further critically investigate the rule of thumb proposed by Lewis (2008). Then, we challenge our approximations' accurateness assuming more sophisticated asset models including stochastic interest rates, stochastic equity volatility and jumps as well and perform numerical analyses considering some life-cycle strategies relying on a Monte-Carlo approach[5].

The remainder of this paper is organized as follows: Section 2 describes our modelling approach of life-cycle and balanced funds whereas Section 3 introduces the financial models under consideration. Section 4 and Section 5 present our results for single and regular contributions respectively. Finally, Section 6 concludes.

2 Modelling life-cycle and balanced funds

Now we introduce how life-cycle and balanced funds are modelled in our analysis. Life-cycle and balanced funds generally invest in a mix of equity and bonds. A balanced fund has a constant portion $x_S \in [0,1]$ of its capital invested in equity and the remaining part in bonds. We further assume the bond investment having a target duration d and therefore model the bond portfolio by a zero-bond investment with time to maturity d. We assume the bond

[5] Cf. Fishman (1996) for a thorough introduction of Monte-Carlo simulation techniques.

portfolio to be permanently reallocated in order to keep the target duration d stable. In contrast to the balanced fund, a life-cycle fund applies a time-dependent (however not path-dependent) asset allocation strategy, where $\left(x_{S,t}\right)_t \in [0,1]$ gives the equity portion at time *t*. Further, fund management fees that reduce the fund's performance are charged on top of the considered funds.

Section 3 introduces the financial models used in our analysis. We start with a simple but tractable Black-Scholes economy and then continue our investigation applying more sophisticated asset models. Whereas we are able to derive closed form solutions for the considered funds in the Black-Scholes setting, we rely on Monte-Carlo simulation techniques when the more sophisticated models are considered. Within the theoretical part in the Black-Scholes model, we assume continuous rebalancing of risky and riskless assets in order to keep the required equity portion stable and further charge the respective management fee on a continuous basis. In contrast, daily rebalancing and charging is applied[6] within the Monte-Carlo framework.

We analyse the risk-return profile – i.e. the probability distribution of returns – of single and regular contributions to the life-cycle and balanced funds given an investment horizon of T years. After determining closed form solutions in the Black-Scholes model that are applicable for an arbitrary life-cycle strategy, we further consider three different life-cycle strategies in our additional numerical analyses. We assume the asset mix to be reset on a yearly basis, i.e. throughout one year the targeted asset allocation is not changed.

- Strategy *A* defines a 'classical' life-cycle strategy starting with a complete investment in equity, i.e. $x_{S,0} = 1$ and each year linearly decreasing its equity exposure over the fund's term, arriving at a complete investment in zero-bond in the last year, i.e. $x_{S,T-1} = 0$[7].

- In contrast, strategy *B* reverses the order of life-cycle and is therefore often referred to as 'contrarian' strategy. In more detail, strategy *B* starts with $x_{S,0} = 0$ and each year linearly increases its exposure to the risky asset finally arriving at $x_{S,T-1} = 1$.

[6] We assume each month consists of 21 trading days.

[7] This approach is obviously only non-degenerated if $T > 1$.

- Finally, for investigating the accurateness of our approximations derived in the theoretical part of Sections 4 and 5, strategy *C* applies a non-standard extreme 'life-cycle' model where the life-cycle fund alternates between a complete investment in the risky and the riskless asset vice versa on a yearly basis, i.e. we set $x_{S,0} = 1, x_{S,1} = 0, x_{S,2} = 1, \ldots$

Basu and Drew (2009) argue that contrarian strategies were superior to the classical life-cycle strategy by comparing expected outcome and various percentiles of the resulting wealth distribution due to the so-called 'portfolio size effect'. Sections 4 and 5 show that contrarian strategies in general offer higher returns on the one side but on the other side increase risk tremendously when compared to classical life-cycle strategies. However note again, the paper's scope is not on comparing different life-cycle strategies but rather on matching them with similar balanced funds.

3 Financial models

Next, we introduce the different financial models under investigation. First we apply a version of the analytically tractable Black-Scholes model (cf. Black and Scholes (1973)) which will be denoted by 'BS' in what follows. Second, we consider a more sophisticated model denominated by 'CIR-SV', additionally allowing for stochastic interest rates using a Cox-Ingersoll-Ross model (cf. Cox et al. (1985)) and further modelling stochastic equity volatility using a version of the Heston-model (cf. Heston (1993)). Finally, we add jumps to the equity process by means of the Merton-model (cf. Merton (1976)). This hybrid stochastic volatility jump diffusion model is then denominated as 'CIR-SVJD' in the following.

Let $r(t)$ denote the short-rate and $S(t)$ denote the equity's spot price at time t. The (BS) model is then entirely described by defining the *real-world*[8] dynamics of the underlying equity process as

$$dS(t) = S(t)\big((r(t) + \lambda_S)dt + \sigma_S dW_1(t)\big) \text{ and } r(t) = r, \forall t$$

where λ_S denotes the equity risk premium, $\sigma_S > 0$ is the annualized volatility of the equity process and r gives the non-random constant risk-less[9] interest rate. Further $W_1(t)$ is a Brownian Motion under the considered probability measure.

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[8] In our setting, the risk-return profiles are obviously assessed under the objective probability measure.

[9] We are not considering default risk within this paper.

The (CIR-SV) model adds stochastic interest rates and stochastic equity volatility to the (BS) model and is summarized by

$$\begin{aligned} dS(t) &= S(t)\left(\left(r(t)+\lambda_S\right)dt + \sqrt{V(t)}dW_1(t)\right) \\ dV(t) &= \kappa_V\left(\theta_V - V(t)\right)dt + \sigma_V\sqrt{V(t)}dW_2(t) \\ dr(t) &= \kappa_r\left(\theta_r - r(t)\right)dt + \sigma_r\sqrt{r(t)}dW_3(t) \end{aligned}$$

where $W_2(t)$ and $W_3(t)$ are Brownian Motions under the probability measure. We further assume $dW_1(t)dW_3(t) = dW_2(t)dW_3(t) = 0$ and $dW_1(t)dW_2(t) = \rho dt$ with $\rho \in [-1,1]$ driving the correlation between the spot price and its instantaneous variance $V(t)$.

The (CIR-SVJD) model finally extends the (CIR-SV) model by further allowing for the occurrence of jumps in the equity process and is given by

$$\begin{aligned} dS(t) &= S(t)\left(\left(r(t)+\lambda_S\right)dt + \sqrt{V(t)}dW_1(t) + dJ(t)\right) \\ dV(t) &= \kappa_V\left(\theta_V - V(t)\right)dt + \sigma_V\sqrt{V(t)}dW_2(t) \\ dr(t) &= \kappa_r\left(\theta_r - r(t)\right)dt + \sigma_r\sqrt{r(t)}dW_3(t) \end{aligned}$$

where $dJ(t) = Z(t)dN(t)$ gives the jump's dynamics. Within this setting $N(t)$ is a Poisson counter with intensity λ indicating the occurrence of a jump and $Z(t)$ gives the random jump size. In the spirit of Merton (1976), we assume the jump sizes being normally distributed and hence set $Z(t) \sim N\left(\mu_Z, \sigma_Z^2\right)$.

The term structure of interest rates – i.e. zero-bond prices with different maturities – is driven by the short-rate $r(t)$ in the considered models. Regarding the models with stochastic interest rates, standard no-arbitrage arguments[10] give zero-bond prices at time *t* with time-to-maturity *d* as function of the short rate by $P(t,d) = A(d)\exp\left(-B(d)r(t)\right)$ with

$$A(d) = \left[\frac{2\cdot h\cdot \exp\left(\left(\tilde{\kappa}_r + h\right)\cdot d/2\right)}{\left(\tilde{\kappa}_r + h\right)\cdot\left(\exp\left(h\cdot d\right)-1\right)+2\cdot h}\right]^{\frac{2\tilde{\kappa}_r\tilde{\theta}_r}{\sigma_r^2}} \quad \text{and} \quad B(d) = \frac{2\cdot\left(\exp\left(h\cdot d\right)-1\right)}{\left(\tilde{\kappa}_r + h\right)\left(\exp(h\cdot d)-1\right)+2\cdot h}$$

where $h = \sqrt{\tilde{\kappa}_r^{\,2} + 2\cdot\sigma_r^{\,2}}$, $\tilde{\kappa}_r = \kappa_r + \lambda_r\sigma_r$, $\tilde{\theta}_r = \dfrac{\kappa_r\theta_r}{\kappa_r + \lambda_r\sigma_r}$ and λ_r denoting the market price of interest rate risk.

[10] Cf. e.g. Bingham and Kiesel (2004).

In the numerical analyses in Sections 4 and 5, we adopt capital market parameters from Graf et al. (2010) by assuming $\lambda_S = 3\%$ and further use interest rate model parameters as given in Table 1.

κ_r	θ_r	σ_r	λ_r
20%	4.5%	7.5%	0%

Table 1: Interest rate model parameters

For reasons of consistency with the (BS) model, we further assume the initial short rate being equal to its long-term expectation and consequently set $r(0) = 4.5\%$[11].

Stochastic volatility (i.e. stochastic variance) and the jump component's parameters are taken from Eraker (2004) who estimates these parameters by combining historical time series and option market analyses. However, in contrast to Eraker (2004) we give the estimated parameters in annualized form in Table 2, which is e.g. done in Paulsen et al. (2009).

κ_V	θ_V	σ_V	ρ	$V(0)$
475%	$(22\%)^2$	55%	-57%	$(22\%)^2$

Table 2: Volatility model parameters

Finally, the jump parameters are summarized in Table 3

λ	μ_Z	σ_Z
0.5	- 0.4%	6.6%

Table 3: Jump model parameters

The Black-Scholes model is then parameterized accordingly by setting $r = r(0)$ and $\sigma_S = V(0)$. Further, zero-bond prices in the (BS) model are given by $P(t,d) = \exp(-rd)$.

[11] Note, we could also assume some different short-rate at $t = 0$ which then implies a non-constant but time-dependent drift term in the (BS) model and thus changes the theoretical results in Sections 4 and 5 slightly.

4 Single contribution

At first, we analyse a single contribution to an arbitrary life-cycle fund as described in Section 2. We start with deriving some closed form solutions within the (BS) model in Section 4.1. Section 4.2 then analyses the risk of parameter misspecification in the (BS) model in more detail whereas Section 4.3 further challenges the approximations applying more sophisticated financial models.

4.1 Calibration in the (BS) model

In the following, let $V_{LF}(t)$ denote the spot price of a life-cycle fund at time t with time-dependent equity allocation $(x_{S,t})_t$, target-duration d and management fee c_{LF}. In a Black-Scholes framework, the dynamics of $V_{LF}(t)$ are then written as

$$dV_{LF}(t) = V_{LF}(t)\left(\left(x_{S,t}(r+\lambda_S) + (1-x_{S,t})r - c_{LF}\right)dt + x_{S,t}\sigma_S dW(t)\right)$$

which is solved as

$$V_{LF}(t) = \exp\left(\int_0^t \left(x_{S,u}(r+\lambda_S) + (1-x_{S,u})r - c_{LF} - \frac{1}{2}(x_{S,u}\sigma_S)^2\right)du + \int_0^t (x_{S,u}\sigma_S)dW_u\right)$$

Further, let $V_{BF}(t)$ denote the spot price of a balanced fund at time t with fixed equity allocation x_S, the same target-duration d and management fee c_{BF}. Hence, the considered balanced fund only differs from the life-cycle fund by the constant equity portion and the different management fee.

$V_{BF}(t)$ is similarly written as

$$V_{BF}(t) = \exp\left(\left(x_S(r+\lambda_S) + (1-x_S)r - c_{BF} - \frac{1}{2}(x_S\sigma_S)^2\right)t + \int_0^t (x_S\sigma_S)dW_u\right)$$

Considering an investment horizon of T years, both $V_{LF}(T)$ and $V_{BF}(T)$ follow a lognormal distribution and their distributions hence coincide if and only if $\mathbf{E_P}V_{LF}(T) = \mathbf{E_P}V_{BF}(T)$ and $\mathbf{Var_P}V_{LF}(T) = \mathbf{Var_P}V_{BF}(T)$ or equivalently if and only if

$$\int_0^T (x_S \sigma_S)^2 du \overset{!}{=} \int_0^T (x_{S,u} \sigma_S)^2 du \Rightarrow x_S = \sqrt{\frac{1}{T}\int_0^T x_{S,u}^2 du} \tag{1}$$

and

$$\begin{aligned} T\left(x_S(r+\lambda_S) + (1-x_S)r - c_{BF}\right) &\overset{!}{=} \int_0^T \left(x_{S,u}(r+\lambda_S) + (1-x_{S,u})r - c_{LF}\right) du \\ \Rightarrow c_{BF} &= c_{LF} + \lambda_S \left(x_S - \frac{1}{T}\left(\int_0^T x_{S,u} du\right)\right) \end{aligned} \tag{2}$$

Therefore, using the equations above, the balanced fund's equity portion x_S and its management fee c_{BF} can be derived such that the distribution of the life-cycle and balanced fund *at the end of the investment horizon* coincide.

However, it is not yet clear if the calibrated equity portion and management fee actually define a 'valid' balanced fund, i.e. a balanced fund with an equity portion between 0% and 100% and non-negative management fee. First, we obtain $x_S \in [0,1]$ by construction and second, Cauchy-Schwartz inequality yields $\int_0^T \frac{1}{T} x_{S,u} du \leq \sqrt{\int_0^T \frac{1}{T^2} du \int_0^T x_{S,u}^2 du} = \sqrt{\frac{1}{T}\int_0^T x_{S,u}^2 du} = x_S$. Hence, if the risk premium λ_S is non-negative (as it is expected to be), we conclude $c_{BF} \geq c_{LF}$ from equation (2). In consequence, the derived balanced fund is indeed valid. Therefore on the one side, regarding single contributions – neglecting parameter and model risk for a second – the very existence of life-cycle fund investment vehicles seems challenged since appropriately calibrated balanced funds deliver exactly the same risk-return profile at maturity. On the other side, investors may easily assess the life-cycle fund's upside potential and downside risk by solely investigating the appropriately calibrated balanced fund.

Applying the capital market parameter set as defined in Section 3, Table 4 gives equity portion and management fee of the balanced funds calibrated to the sample life-cycle strategies A, B and C as introduced in Section 2. We set the life-cycle funds' management fee to $c_{LF} = 1.3\%$ p.a. and assume an investment horizon of $T = 12$ years.

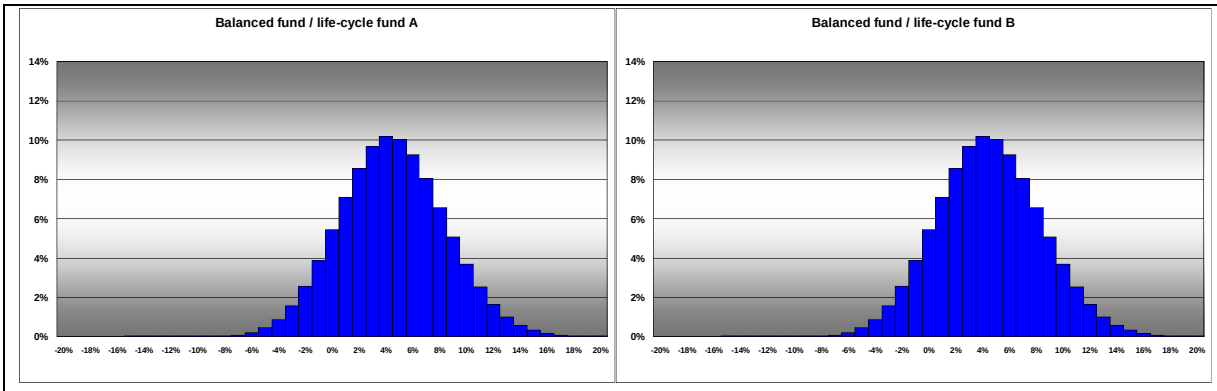
Strategy	Equity portion x_S	Management fee c_{BF}
A	59.04%	1.57% p.a.
B	59.04%	1.57% p.a.
C	70.71%	1.91% p.a.

Table 4: Balanced funds' calibration – single contribution

At first note that in case of a single contribution and a Markovian capital market model – e.g. the considered (BS) model – strategies A and B yield the same risk-return profile and therefore obviously correspond to the same balanced fund. Further, strategy C results in the riskiest balanced fund in terms of equity portion and the most 'expensive' fund in terms of management fee. As shown above, all balanced funds require management fees greater than the life-cycle funds' management fee.

Further, it is worthwhile noting that the average equity portion of the considered life-cycle strategies A, B and C over time equals 50%. Hence, the rule of thumb as proposed by Lewis (2008) is not able to detect any difference in the life-cycle funds considered although differences are quite substantial (cf. Table 4 and Figure 1).

Figure 1 provides an illustration of the probability distribution of internal rates of return assuming the (BS) model and an investment horizon of $T = 12$ years. We consider the probability distributions of the different life-cycle strategies (or the corresponding balanced funds as given in Table 4) and a balanced fund equipped with a constant equity portion of 50% and a management fee of 1.3% p.a. applying the rule of thumb calibration technique.



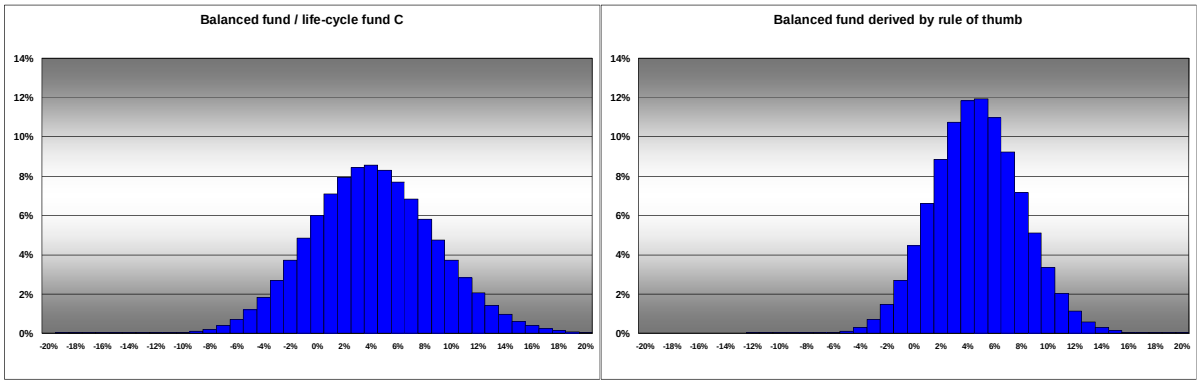


Figure 1: Histogram of returns: Life-cycle and balanced fund – single contribution (BS)

As already summarized in Table 4, strategy A and B give exactly the same risk-return profile whereas strategy C allows for higher returns at the cost of more volatility and hence potentially larger losses. This effect is mainly driven by the higher equity portion of approximately 70% as compared to approximately 60% in strategies A and B. In addition, even in the 'simple' single contribution case, the rule of thumb fails to approximate the risk-return profiles of the considered life-cycle funds and is therefore of only limiting explanatory value. This misspecification becomes even more apparent when regular contributions are considered (cf. Section 5).

Finally, Table 5 summarizes some percentiles of the underlying probability distributions of returns, again stressing the weaknesses of the rule of thumb approximation.

Strategy	5%	25%	Median	75%	95%
Life-cycle A	-2.29%	1.33%	3.92%	6.58%	10.53%
Life-cycle B	-2.29%	1.33%	3.92%	6.58%	10.53%
Life-cycle C	-3.83%	0.45%	3.54%	6.73%	11.48%
Rule Of Thumb	-1.13%	1.96%	4.17%	6.43%	9.76%

Table 5: Characteristic figures – single contribution (BS)

Sections 4.2 and 4.3 are now concerned with the balanced fund's calibration as derived from equations (1) and (2) focussing on parameter risk in the (BS) model on the one side and model risk on the other side.

4.2 Parameter risk and stochastic dominance

Even if we did not face any model risk – i.e. if the (BS) model was the 'right' model – we might still face the risk of inappropriate parameter estimates, i.e. mis-specified risk-free rate, equity risk premium or equity volatility. According to equations (1) and (2), the calibrated balanced fund's equity portion and its management fee are a function of the life-cycle's glide path, its management fee and the equity risk premium only and are hence independent of equity volatility or the risk-free rate. Hence, only if the assumed or estimated risk premium differs from the 'real' risk premium above derivations may fail. However, if the 'real' risk-premium is non-negative (as it is supposed to be), $c_{BF} \geq c_{LF}$ follows from equation (2). Hence, a balanced fund with management $c_{BF} = c_{LF}$ and an equity portion derived according to equation (1) is independent of capital market parameters (and thus not affected by parameter risk). Further, this balanced fund yields a distribution with volatility equal to the life-cycle fund's volatility and has a higher mean if the 'real' risk premium of the (BS) economy is actually greater than zero. Therefore, within the (BS) model the corresponding balanced fund dominates any given life-cycle fund by means of first order stochastic dominance[12] and hence there is no need for life-cycle fund investments at all[13].

This result is in line with recent results from Bernard and Boyle (2010). In a more general set-up, they investigate 'cost-efficient' investment strategies to achieve a pre-specified *real-world* probability distribution at maturity, i.e. strategies that can be set up with minimal initial cost (using a *risk-neutral* pricing measure). Applying a Black-Scholes economy, they identify any financial derivative whose payout is written as a non-decreasing function of the risky asset as 'cost-efficient', i.e. there is no cheaper way of generating the same real-world return distribution at maturity.

[12] Note, when log-normal distributions are considered, mean and volatility are sufficient to determine first order stochastic dominance.

[13] If solely the returns at maturity are considered and the returns during the investment phase are neglected.

In our setting, we have

$$S(T) = \exp\left(\left(r + \lambda_S - \frac{1}{2}\sigma_S^2\right)T + \sigma_S\left(W(T) - W(0)\right)\right)$$

$$\Leftrightarrow$$

$$\sigma_S\left(W(T) - W(0)\right) = \log\left(S(T)\right) - \left(r + \lambda_S - \frac{1}{2}\sigma_S^2\right)T$$

$$=: h\left(S(T)\right)$$

where $h\left(S(T)\right)$ is a non-decreasing function of the risky asset. We further obtain

$$V_{BF}(T) = \exp\left(\left(x_S\left(r + \lambda_S\right) + \left(1 - x_S\right)r - c_{BF} - \frac{1}{2}\left(x_S\sigma_S\right)^2\right)T + x_S h\left(S(T)\right)\right)$$

and therefore the balanced fund's spot price at time T is given as a non-decreasing function of the risky asset. Hence following Bernard and Boyle (2010), the cost-efficient strategy (in the sense of Bernard and Boyle (2010)) of generating the return distribution of a life-cycle fund is given by the balanced fund derived from equations (1) and (2).

4.3 Model risk

Section 4.2 treated the potential mis-specification of parameters in the (BS) economy and it was shown that for any given life-cycle fund, there exists a balanced fund independent of capital market parameters and thus independent of parameter risk which stochastically dominates the life-cycle fund. This section now analyses the impact of model risk by allowing for the more sophisticated financial models introduced in Section 3. We compare the life-cycle funds' and balanced funds' risk-return profiles by means of Monte-Carlo simulation techniques. The balanced funds under consideration are given as summarized in Table 4. Hence we still assume a (BS) model for deriving the balanced funds corresponding to a given life-cycle fund and now analyze whether these funds still constitute good approximations for the life-cycle fund when more randomness is introduced in the financial models

Note that the (CIR-SV) model adds stochastic interest rates and stochastic volatility to the (BS) model. Since in this setting, equity returns are modelled using a 'spread approach' (cf. Section 3), modelling interest rates stochastically has an immediate impact on equity returns and volatility as well. By additionally introducing 'fat tails' of the risky assets due to modelling of stochastic volatility, the effect on the probability distributions of returns due to changing models from (BS) to (CIR-SV) is not obvious. Therefore, Figure 2 illustrates the empirical

returns of the underlying life-cycle and balanced funds assuming the (CIR-SV) model using 20,000 Monte-Carlo trajectories for deriving the estimates.

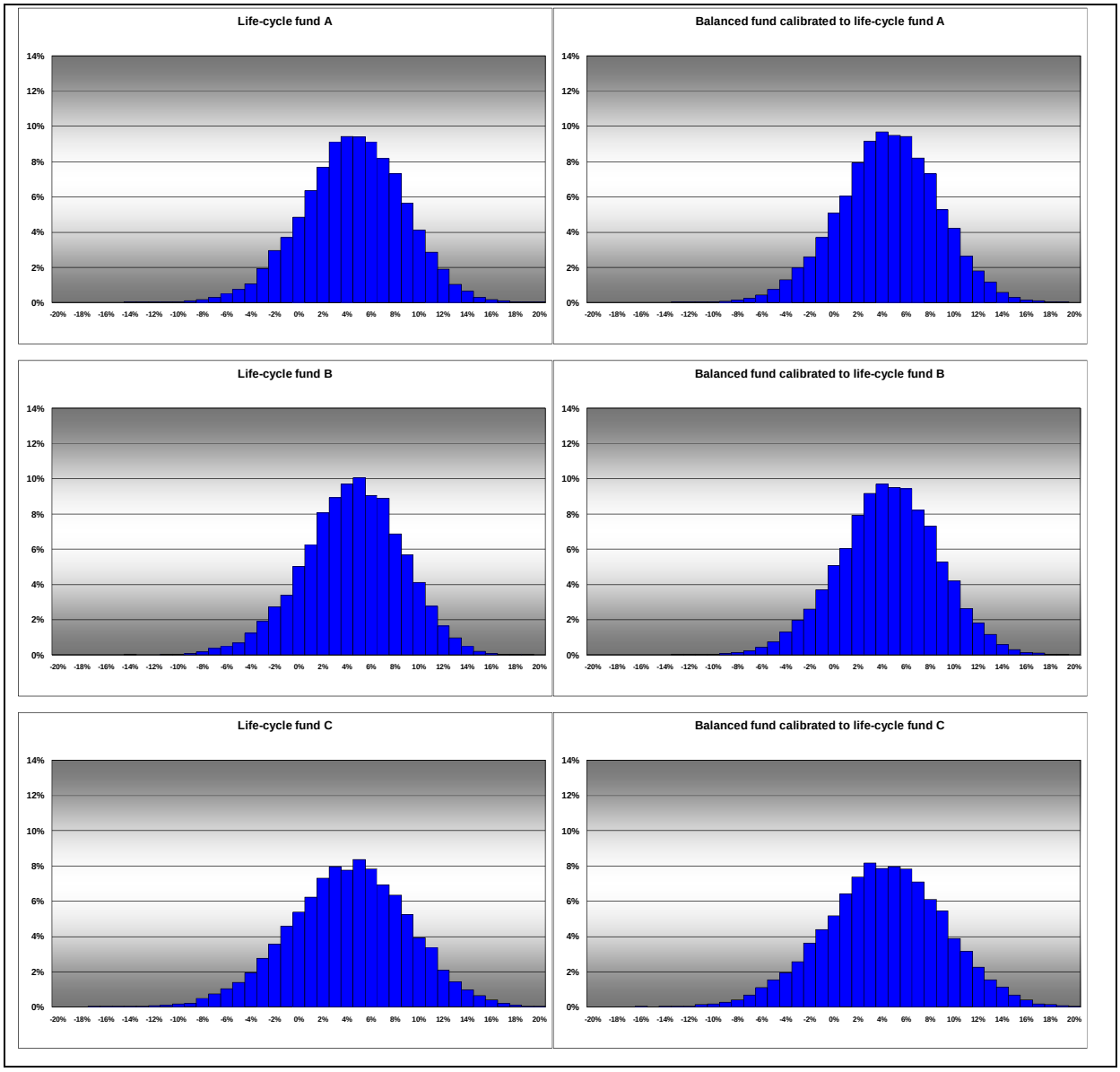


Figure 2: Empirical returns: Life-cycle v balanced fund – single contribution (CIR-SV)

Compared to Figure 1, the distribution of the considered funds are generally less 'concentrated' and a summary of some estimated percentiles in Table 6 further indicates fatter tails when compared to the (BS) model. Again, bear in mind that life-cycle fund A and life-cycle fund B are mapped to the same balanced fund (cf. Table 4) and hence the results for Balanced A and Balanced B are identical.

Strategy	5%	25%	Median	75%	95%
Balanced A	-2.99%	1.34%	4.09%	6.81%	10.64%
Life-cycle A	-2.98%	1.28%	4.09%	6.90%	10.68%
Balanced B	-2.99%	1.34%	4.09%	6.81%	10.64%
Life-cycle B	-3.02%	1.32%	4.10%	6.76%	10.42%
Balanced C	-4.66%	0.49%	3.76%	7.01%	11.63%
Life-cycle C	-4.68%	0.40%	3.75%	6.98%	11.38%

Table 6: Characteristic figures – single contribution (CIR-SV)

Comparing Table 5 and Table 6 shows that the returns in the bad (good) case[14] are generally lower (higher) than compared to the (BS) model. However, our main focus is not primarily on analysing the effects of the change of model on the observed returns but rather on investigating whether the calibrated balanced funds still provide an accurate approximation to the life-cycle funds. Figure 2 and Table 6 indicate that the distributions of the life-cycle and accordingly calibrated balanced funds are very similar. To confirm this impression, we perform statistical tests on the hypothesis of the above return distributions being drawn from the same origin distribution applying a two-sample Kolmogorov-Smirnov and a two-sample Anderson-Darling test. The statistics and p-values were calculated using the computational package R (2010) and further RExcel as developed by Baier and Neuwirth (2007). Results are summarized in Table 7.

Strategy	Kolmogorov-Smirnov		Anderson-Darling	
	Statistic	*p-Value*	*Statistic*	*p-Value*
A	0.0073	0.6609	-0.3232	0.4242
B	0.0082	0.5120	0.7061	0.1723
C	0.0066	0.7842	-0.0743	0.3521

Table 7: Test for equality of distributions – single contribution (CIR-SV)

[14] Roughly defined as less (more) than the observed median.

Above tests do not reject the null-hypothesis that the above empirical distributions were sampled from the same origin distributions (e.g. assuming a significance level of 5%). Obviously, this is not a proof that they actually have the same underlying probability distributions but at least a good indicator that the distributions do not differ too much. Hence, we can conclude that the balanced fund's approximations appropriately assess the life-cycle fund's risk-return profile.

The effect of additionally allowing for jumps in equity returns by applying the (CIR-SVJD) model is shown in Figure 3 and Table 8 where the empirical returns obtained by means of Monte-Carlo techniques are illustrated.

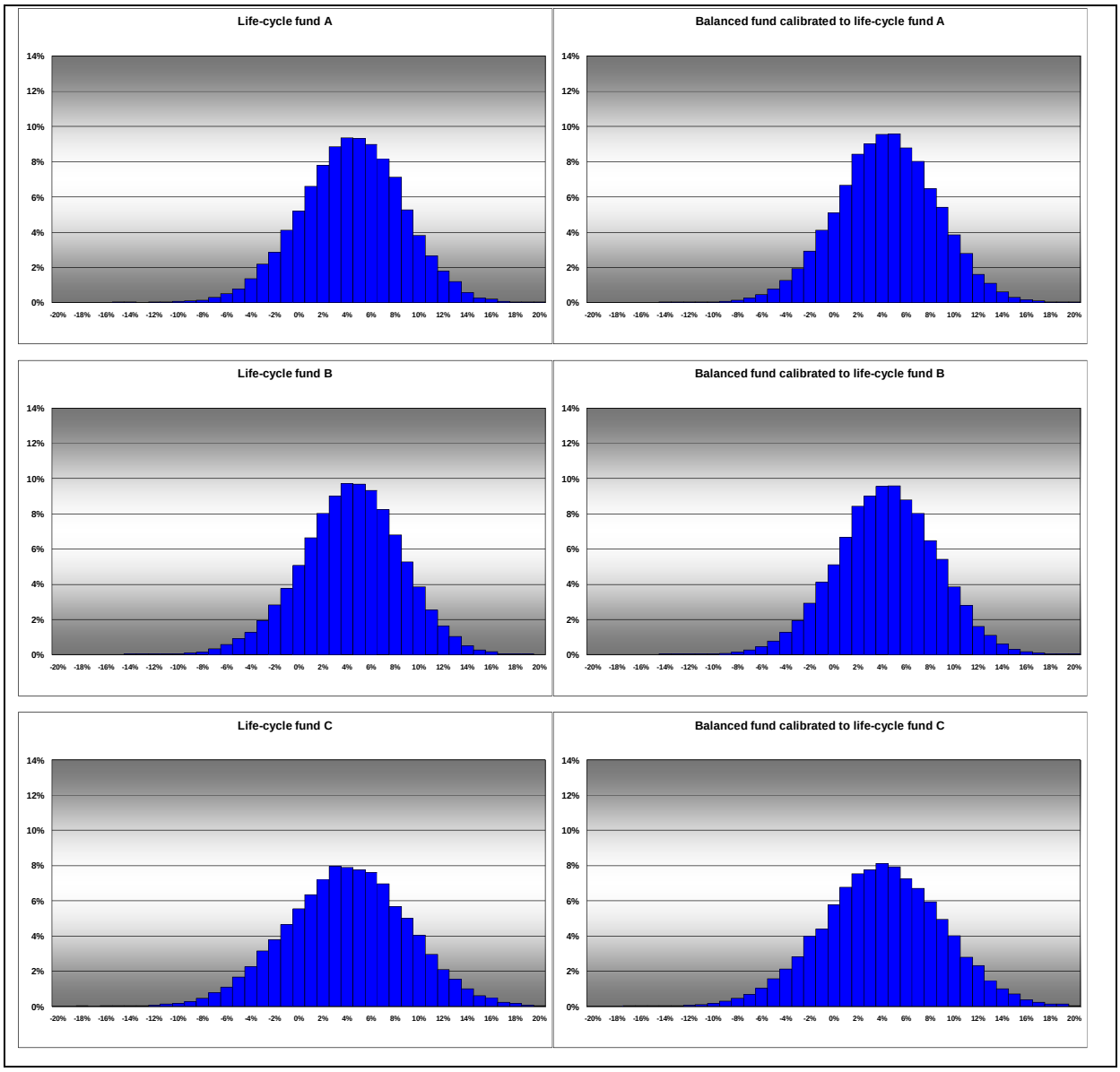


Figure 3: Empirical returns: Life-cycle v balanced fund – single contribution (CIR-SVJD)

By eye the distributions of life-cycle and balanced funds' returns again look very similar and the approximations derived in the simple (BS) model allow for a good assessment of the life-cycle funds' risk-return profiles by the balanced funds. Table 8 summarizes some estimated percentiles of the underlying distributions.

Strategy	5%	25%	Median	75%	95%
Balanced A	-3.03%	1.14%	3.89%	6.71%	10.63%
Life-cycle A	-3.17%	1.09%	3.96%	6.75%	10.67%
Balanced B	-3.03%	1.14%	3.89%	6.71%	10.63%
Life-cycle B	-3.15%	1.17%	3.94%	6.63%	10.48%
Balanced C	-4.74%	0.22%	3.52%	6.88%	11.62%
Life-cycle C	-4.88%	0.14%	3.54%	6.84%	11.57%

Table 8: Characteristic figures – single contribution (CIR-SVJD)

As expected, the additional introduction of jumps negatively affects the returns as compared to the previous (CIR-SV) model. However, the returns are not tremendously changed due to the rather long investment horizon and the relatively small (expected) jump sizes (cf. Table 3). Further, the impact of adding jumps to the equity process seems to affect balanced and life-cycle funds in a similar way and hence the distributions appear reasonably close again. The null-hypothesis that the observed returns were drawn from the same origin distribution is again not rejected (cf. Table 9) although estimated p-Values generally decrease.

Strategy	Kolmogorov-Smirnov		Anderson-Darling	
	Statistic	*p-Value*	*Statistic*	*p-Value*
A	0.0098	0.2921	0.1033	0.3043
B	0.0080	0.5523	0.5121	0.2077
C	0.0093	0.3592	-0.0787	0.3533

Table 9: Test for equality of distributions – single contribution (CIR-SVJD)

Summarizing, the balanced funds derived by equations (1) and (2) provide accurate approximations of the considered life-cycle funds' risk-return profiles even when applying

more sophisticated financial models and hence seem an appropriate tool for assessing risk-return profiles of life-cycle funds quickly. Further, the very existence or additional value of life-cycle investment seems challenged when single contributions are considered. In the following section we therefore analyse the case where regular contributions are made to the considered life-cycle funds.

5 Regular contributions

Similar with Section 4, we start with analyzing regular contributions to the considered funds assuming a (BS) economy in Section 5.1 and then switch to more sophisticated asset models in Section 5.2.

5.1 Calibration in the (BS) model

In contrast to the single contribution case – where analytical solutions of the considered risk-return profiles exist – regular contributions do not allow for deriving closed form solutions even in the simple (BS) economy, since no tractable closed form assessment for sums of log-normal distributed random variables is available[15]. Therefore, we rely on a moment matching procedure to calibrate the balanced funds under consideration, i.e. we determine the balanced fund's equity portion and its management fee such that first and second moment of the distribution of wealth when regularly investing in the balanced fund and regularly investing in the respective life-cycle fund coincide at *the investment's maturity date*.

For ease of notation and without loss of generality we assume a yearly[16] contribution of 1 unit of currency. At time T, the result of annually contributing to the life-cycle fund is then written as $W_{LF}(T) = \sum_{i=0}^{T-1} \frac{V_{LF}(T)}{V_{LF}(i)} = \frac{V_{LF}(T)}{V_{LF}(T-1)} \cdot \left(1 + W_{LF}(T-1)\right)$ which can be calculated recursively applying $W_{LF}(1) = \frac{V_{LF}(1)}{V_{LF}(0)}$. Further, let $Z_{LF}(t) := \frac{V_{LF}(t)}{V_{LF}(t-1)}$ denote the annual returns of the life-cycle fund. Assuming a (BS) economy, we obtain $Z_{LF}(t) = \exp\left(\int_{t-1}^{t} \left(x_{S,u}(r + \lambda_S) + (1 - x_{S,u})r - c_{LF} - \frac{1}{2}(x_{S,u}\sigma_S)^2 \right) du + \int_{t-1}^{t} (x_{S,u}\sigma_S) dW_u \right)$ and hence conclude $Z_{LF}(t) \sim LN\left(\mu_{LF}(t), \sigma_{LF}^2(t)\right)$ with

[15] Cf. e.g. Dufrense (2004) for an analysis of approximations of sums of log-normal distributed random variables.

[16] The same ideas can be applied to arbitrary regular contributions such as quarterly or monthly premium contributions.

$\mu_{LF}(t) = \int_{t-1}^{t} \left(x_{S,u}(r+\lambda_S) + (1-x_{S,u})r - c_{LF} - \frac{1}{2}(x_{S,u}\sigma_S)^2 \right) du$ and $\sigma_{LF}^2(t) = \int_{t-1}^{t} (x_{S,u}\sigma_S)^2 du$. In addition, $Z_{LF}(T), Z_{LF}(T-1), \ldots, Z_{LF}(1)$ are stochastically independent random variables due to the independent increments of the underlying Brownian Motion.

The expected wealth when investing annually in the life-cycle fund at maturity T – i.e. the first moment of $W_{LF}(T)$ – is then derived as $\mathbf{E_P} W_{LF}(T) = \mathbf{E_P}[Z_{LF}(T)] \cdot (1 + \mathbf{E_P}[W_{LF}(T-1)])$ and recursively calculated using $\mathbf{E_P}[W_{LF}(1)] = \mathbf{E_P}[Z_{LF}(1)]$. Regarding the second moment of $W_{LF}(T)$ we obtain $\mathbf{E_P}[W_{LF}^2(T)] = \mathbf{E_P}[Z_{LF}^2(T)] \cdot (1 + 2 \cdot \mathbf{E_P}[W_{LF}(T-1)] + \mathbf{E_P}[W_{LF}^2(T-1)])$ which is again determined recursively using $\mathbf{E_P}[W_{LF}^2(1)] = \mathbf{E_P}[Z_{LF}^2(1)]$. Finally, the first and second moment of the distribution of wealth when investing annually in a balanced fund are derived similarly.

As shown in the Appendix, solving for the balanced fund's equity portion and its management fee in order to match the first and second moment of an investment in any given regular life-cycle fund is essentially equivalent with solving a polynomial equation set. Within the Appendix we further prove the (unique) existence of a 'matching' balanced fund, treat the solution's analytics in more detail and provide a fast and robust algorithm to actually derive the required balanced fund. Applying this calibration methodology, Table 10 displays the balanced funds calibrated to the different life-cycle strategies A, B and C as introduced in Section 2. Similar with Section 4 we set the life-cycle funds' management fee to $c_{LF} = 1.3\%$ p.a., use the (BS) model as parameterized in Section 3 and assume an investment horizon of $T = 12$ years.

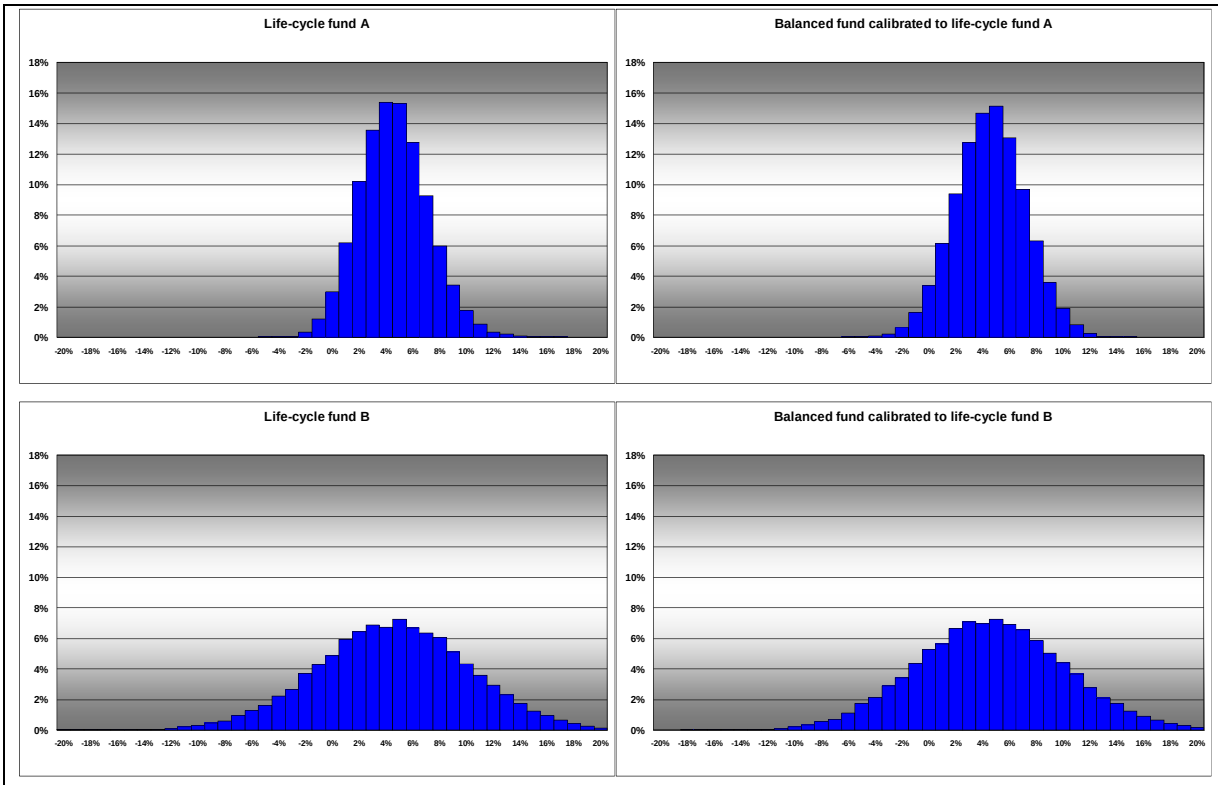
Strategy	**Equity portion** x_S	**Management fee** c_{BF}
A	35.88%	1.32% p.a.
B	76.08%	1.63% p.a.
C	67.25%	1.91% p.a.

Table 10: Balanced funds' calibration – regular contributions

Comparing Table 10 and the calibration results in the single contribution case (cf. Table 4) shows the massive impact of different premium payment modes on the calibrated balanced funds and hence on the risk-return profiles of considered life-cycle funds. In contrast to the

single contribution setting where life-cycle strategy A and B resulted in exactly the same risk-return profile and hence were matched with the same balanced fund, the classical life-cycle strategy A is now associated with a far more conservative balanced fund than strategy B. This results from the amount of capital that is exposed to the risky asset: Strategy A (B) decreases (increases) its equity exposure and hence the more capital is invested throughout the fund's term the less (more) risky it is allocated. Strategy B is now even more aggressive than strategy C which was identified as the 'riskiest' life-cycle strategy in the single contribution case. Further it is worthwhile noting that – in order to match moments – similar with the results in Section 4, all balanced funds require higher management fees than the considered life-cycle funds.

Figure 4 and Table 11 summarize the empirical distribution of returns of annually investing in the considered funds obtained by applying 20,000 Monte-Carlo trajectories.



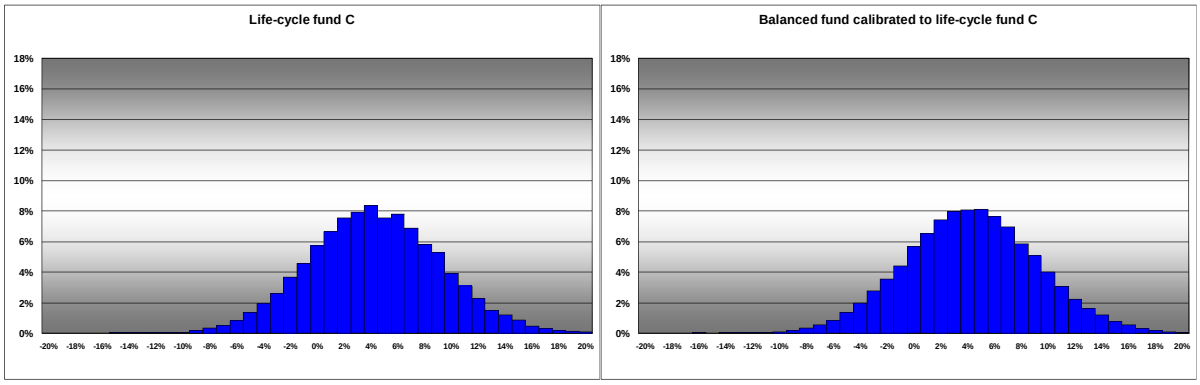


Figure 4: Empirical returns: Life-cycle v balanced fund – regular contributions (BS)

Table 10 already indicated the massive impact of regularly investing on the risk-return profiles especially compared to their single contribution counterparts. This indication is confirmed by Figure 4 which uncovers the tremendous differences of the considered life-cycle funds. By eye the calibrated balanced fund seem to deliver appropriate and accurate approximations for the life-cycle funds' risk-return profiles. However, Table 11 shows that e.g. the balanced funds' 5th percentile under (over-) estimates the life-cycle funds' 5th percentile regarding strategy A and B respectively.

Strategy	5%	25%	Median	75%	95%
Balanced A	-0.26%	2.31%	4.06%	5.80%	8.42%
Life-cycle A	0.08%	2.33%	4.00%	5.75%	8.40%
Balanced B	-5.04%	0.34%	4.05%	7.76%	13.37%
Life-cycle B	-5.38%	0.24%	4.06%	7.85%	13.39%
Balanced C	-4.29%	0.46%	3.74%	7.01%	11.95%
Life-cycle C	-4.15%	0.48%	3.71%	7.05%	11.96%

Table 11: Characteristic figures – regular contributions (BS)

Hence, although the balanced funds' (first two) moments match the life-cycle funds' (first two) moments, the probability distribution of returns is not entirely described by its (first two) moments and therefore the balanced funds fail to completely match the life-cycle funds' risk-return profiles. Table 12 confirms this indication above by rejecting the null hypothesis that the above samples were drawn from the same origin distribution at least for strategy A (e.g. assuming a confidence level of 5%).

Strategy	Kolmogorov-Smirnov		Anderson-Darling	
	Statistic	*p-Value*	*Statistic*	*p-Value*
A	0.015	0.018	7.883	0.000
B	0.009	0.359	1.215	0.104
C	0.006	0.890	-0.503	0.478

Table 12: Test for equality of distributions – regular contributions (BS)

The null hypothesis is mainly rejected due to the difference of the probability distribution in the lower tail (i.e. returns below the 5th percentile). Roughly speaking, the balanced fund under- (over-) estimates the lower tail of the life-cycle fund's distribution if the life-cycle fund's glide path decreases (increases) its equity exposure over time. Since strategy C applies a permanent reallocation of risky and riskless asset, both effects 'cancel out' partly and hence the estimate of the balanced fund's lower tail is more close to the life-cycle fund's lower tail.

Bearing these limitations in mind, the approach derived may still help financial advisors and clients to easily assess the risk and upside potential of investing in life-cycle funds by just considering their balanced fund counterparts. Similar to the single contribution case, the additional value of actually investing in a life-cycle strategy seems challenged, in particular when taking into account that above balanced funds closely match the life-cycle funds' risk-return profile and in addition come at the cost of higher management fees (cf. Table 10). Matching the first two moments of the balanced funds requires higher management fees as the life-cycle funds apply and hence balanced funds that were instead equipped with the life-cycle fund's management fee (1.3% p.a. in our example) resulted in a higher expected return as the life-cycle funds deliver. Further, it should be possible to manage balanced funds at the same cost of life-cycle funds and hence charge the same management fee. Hence, even in this setting balanced funds may dominate the life-cycle fund investment.

Section 4 already revealed the shortcomings of the rule of thumb just using the average equity portion of the life-cycle funds' glide path over time as approximation for the respective balanced funds. Following the results of Figure 4 and Table 10, the rule of thumb's misspecification should even be more pronounced when regular contributions are considered instead. Therefore, similar with Figure 1 and Table 5 we now analyse a balanced fund with equity portion of 50% and management fee of 1.3% p.a. and derive its empirical returns using Monte-Carlo techniques. Table 13 gives some estimated percentiles in comparison to the life-cycle funds' estimates (cf. Table 11).

Strategy	5%	25%	Median	75%	95%
Life-cycle A	0.08%	2.33%	4.00%	5.75%	8.40%
Life-cycle B	-5.38%	0.24%	4.06%	7.85%	13.39%
Life-cycle C	-4.15%	0.48%	3.71%	7.05%	11.96%
Rule Of Thumb	-1.74%	1.83%	4.28%	6.71%	10.38%

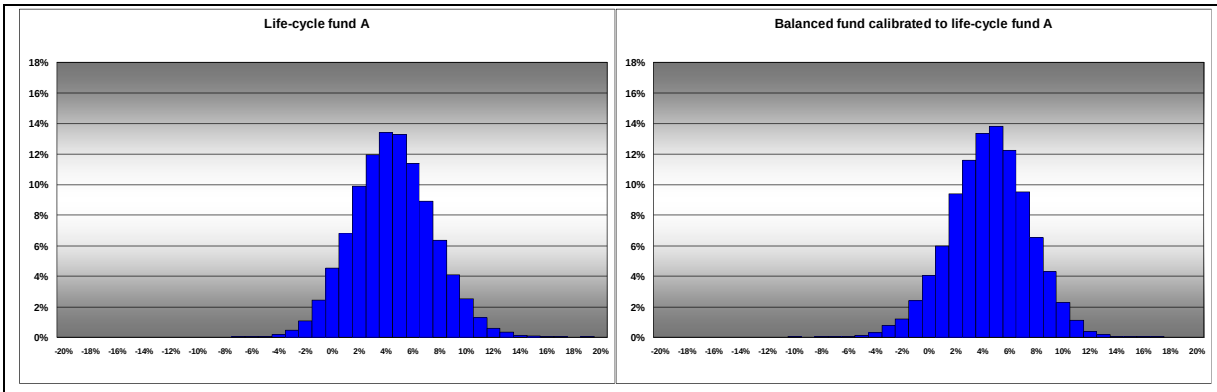
Table 13: Characteristic figures using the "rule of thumb" – regular contributions (BS)

In summary, the rule of thumb approximation completely fails to describe the considered life-cycle funds' return distribution and is hence of only very limited explanatory value to financial advisors and their clients. In contrast, our introduced calibration methodology allows for a more robust assessment of life-cycle fund at least in the (BS) economy.

5.2 Model risk

In line with Section 4, we now analyse above methodologies' accurateness when more sophisticated asset models are considered. In what follows, we only display the results from the (CIR-SVJD) model since for the (CIR-SV) model similar results (in terms of accurateness, of the approximation not in terms of final wealth distributions) are obtained.

Figure 5 shows the empirical returns of the considered life-cycle and their calibrated balanced funds (cf. Table 10) assuming the (CIR-SVJD) model as introduced in Section 3 after applying 20,000 Monte-Carlo trajectories.



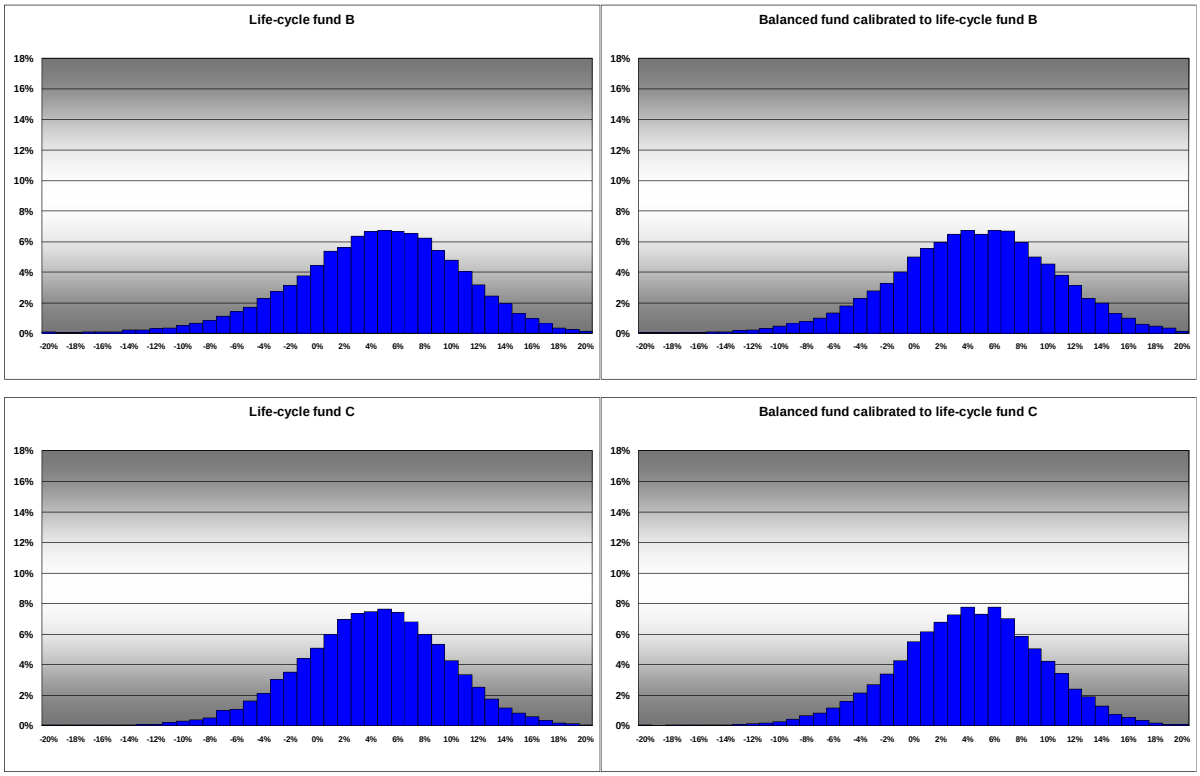


Figure 5: Empirical returns: Life-cycle v balanced fund – regular contributions (CIR-SVJD)

As in Figure 4, the empirical distributions look very similar and the balanced funds seem to appropriately approximate the life-cycle funds' risk-return profile. However, in line with the results in the (BS) model, the distributions do not coincide entirely which is indicated by comparing some empirical percentiles in Table 14 and partly confirmed by the statistics in Table 15.

Strategy	5%	25%	Median	75%	95%
Balanced A	-1.00%	2.05%	4.04%	5.95%	8.75%
Life-cycle A	-0.81%	1.95%	3.92%	5.94%	9.02%
Balanced B	-6.19%	0.11%	4.12%	7.99%	13.55%
Life-cycle B	-6.74%	0.13%	4.26%	8.11%	13.36%
Balanced C	-5.29%	0.24%	3.79%	7.22%	12.14%
Life-cycle C	-5.28%	0.22%	3.81%	7.25%	12.08%

Table 14: Characteristic figures – regular contributions (CIR-SVJD)

Changing models from the (BS) economy to the (CIR-SVJD) model has a similar effect on the returns as in the case of single contributions (cf. Table 5 and Table 8). The probability of lower and higher returns as compared to the (BS) model increases in the (CIR-SVJD) model due to the additional sources of risk.

Comparing the life-cycle funds with their corresponding balanced funds yields to similar results as within the (BS) model concerning the discrepancies in the lower percentiles. Further, the upper percentiles (e.g. the 95th-percentile) now also show some differences. In line with the (BS) model, the 5th- and 95th-percentile are roughly (under-) overestimated when a decreasing (increasing) equity share in the life-cycle funds' glide path is applied. Table 15 confirms the empirical return distributions not being sampled from the same original distribution at least for strategies A and B by rejecting the null hypothesis.

	Kolmogorov-Smirnov		**Anderson-Darling**	
Strategy	*Statistic*	*p-Value*	*Statistic*	*p-Value*
A	0.022	0.000	8.184	0.000
B	0.011	0.149	3.302	0.015
C	0.006	0.830	-0.950	0.613

Table 15: Test for equality of distributions – regular contributions (CIR-SVJD)

In summary, the calibration algorithm as introduced relying on a moment matching procedure is able to capture a major part (e.g. 5th– 95th-percentile) of the considered life-cycle funds' return distributions, however fails in exactly reproducing the tails of the underlying life-cycle investments. Therefore, the approximation's appropriateness from a statistical point of view is rejected (cf. Table 15). These differences may however be insignificant from a practical point of view and hence still a reasonably well characterization of life-cycle funds – superior to the rule of thumb – is provided.

6 Conclusion and Outlook

In this paper we have analyzed the risk-return profiles – i.e. the probability distribution of returns – of investing in life-cycle and balanced funds. In contrast to previous research, we have not focussed on comparing the return distribution of artificial life-cycle and balanced funds, but provided methodologies to construct balanced funds approximating the risk-return profile of any given life-cycle fund.

Starting with a single contribution and a simple Black-Scholes economy, we have provided closed form solutions for balanced funds matching the risk-return profile of any given life-cycle fund. Further, using Monte-Carlo techniques we have shown that, even when more sophisticated asset models are considered, the derived approximations show appropriate results and no difference of the underlying risk-return profiles is detected from a statistical point of view. Regarding regular contributions we have provided a fast and robust algorithm to derive balanced funds matching the first two moments of the wealth distribution of an investment in any given life-cycle fund assuming a Black-Scholes economy. These approximations were able to explain a major part of the life-cycle fund's risk-return profile even under more complex asset models. However, due to some differences in the lower and upper tail of the distributions, the applied statistical tests rejected the 'equality' of life-cycle and appropriately calibrated balanced fund. Nevertheless, the approximations still seem useful from a practical point of view, since the difference was of minor magnitude especially compared to rule of thumb approximations used by practitioners so far.

Therefore, our results on the one side facilitate financial planning by means of easily assessing the risk-return profile of 'complex' life-cycle funds by their balanced fund counterparts and on the other side challenge the very existence of (the considered types of) life-cycle funds since balanced funds delivering a very similar risk-return profile are available. Further, balanced funds stochastically dominating life-cycle funds were constructed in the single contribution case and seem available for regular contributions as well.

Of course our research allows for refinement and amendments in the future. The approximations may be derived applying a Black-Scholes model with time-dependant drift in order to match a given term structure of interest rates at outset. Further, challenging the approximations' appropriateness using some historical data e.g. within a Bootstrap approach seems worthwhile studying. Finally, our findings were based on the investment vehicles' returns at maturity neglecting what happens during the investment phase. In the case of regular contributions, there might be reasons for favouring life-cycle funds over balanced funds from a client's point of view when taking into account the volatility of the client's account over the whole investment phase.

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Appendix

Within this Section we investigate the calibration of first and second moment of the wealth distribution when regularly investing in a balanced fund to some life-cycle fund's first and second moment, respectively. We consider a Black-Scholes model as defined in Section 3 and further assume annual contributions of 1 unit of currency to some life-cycle fund in the following.

Similar with the notation in Section 5 let $Z_{BF}(t) := \frac{V_{BF}(t)}{V_{BF}(t-1)} \overset{d}{=} : Z_{BF}$ denote the annual return of an arbitrary balanced fund with equity portion x_S and management fee c_{BF}. Then $Z_{BF} \sim LN\left(\mu_{BF}, \sigma_{BF}^2\right)$ holds with $\mu_{BF} = x_S \lambda_S + r - c_{BF} - \frac{1}{2}\left(x_S \sigma_S\right)^2$ and $\sigma_{BF}^2 = \left(x_S \sigma_S\right)^2$.

Our analysis is now split in three parts: First, we state the required calibration procedure as a problem of solving a polynomial equation set. Second, we derive sufficient conditions to obtain a unique and valid[17] balanced fund to solve these equations and then close the analysis with some treatment of the sufficient conditions itself.

Solving a polynomial equation set to calibrate first and second moment

In the following, let $W_{BF}(T) = \sum_{i=0}^{T-1} \frac{V_{BF}(T)}{V_{BF}(i)} = \frac{V_{BF}(T)}{V_{BF}(T-1)} \cdot \left(1 + W_{BF}(T-1)\right)$ denote the outcome of an annual investment of 1 in the underlying balanced fund and further set $x := \mathbf{E_P}\left[Z_{BF}\right] = \exp\left\{\mu_{BF} + \frac{1}{2}\sigma_{BF}^2\right\}$ and $y := \mathbf{E_P}\left[Z_{BF}^2\right] = \exp\left\{2\mu_{BF} + 2\sigma_{BF}^2\right\}$. It is easily shown (e.g. by induction) that $\mathbf{E_P}\left[W_{BF}(T)\right] = \sum_{i=1}^{T} x^i$ holds. Further, the second moment of $W_{BF}(T)$ is characterized as follows:

[17] i.e. solutions with 'admissible' equity portion (i.e. not less than 0% and not greater than 100%) and admissible management fee (i.e. not less than 0% p.a.).

Using the notations defined above, the second moment of $W_{BF}(T)$ is a polynomial of *y* with coefficients depending on *x*. In particular, we obtain

$$\mathbf{E_P}\left[W_{BF}^2(T)\right] = y^T + \sum_{i=1}^{T-1} \underbrace{\left(2\sum_{j=1}^{T-i} x^j + 1\right)}_{:=a_i(x)} y^i \ .$$

Proof by induction:

T=1

$\mathbf{E_P}\left[W_{BF}^2(1)\right] = \mathbf{E_P}\left[Z_{BF}^2\right] = y$ holds by definition.

T-1 → T

Similar calculus as performed in Section 5 gives (applying the induction hypothesis)

$$\begin{aligned}
\mathbf{E_P}\left[W_{BF}^2(T)\right] &= y\left(1 + 2\mathbf{E_P}\left[W_{BF}(T-1)\right] + \mathbf{E_P}\left[W_{BF}^2(T-1)\right]\right) \\
&= \left(2\sum_{i=1}^{T-1} x^i + 1\right) y + y \cdot \left(y^{T-1} + \sum_{i=1}^{T-2}\left(2\sum_{j=1}^{T-1-i} x^j + 1\right) y^i\right) \\
&= \left(2\sum_{i=1}^{T-1} x^i + 1\right) y + y^T + \sum_{i=1}^{T-2}\left(2\sum_{j=1}^{T-1-i} x^j + 1\right) y^{i+1} \\
&= \left(2\sum_{i=1}^{T-1} x^i + 1\right) y + y^T + \sum_{i=2}^{T-1}\left(2\sum_{j=1}^{T-1-(i-1)} x^j + 1\right) y^i \\
&= y^T + \sum_{i=1}^{T-1}\left(2\sum_{j=1}^{T-i} x^j + 1\right) y^i
\end{aligned}$$

which then completes the proof.

Therefore, the first and second moment of $W_{BF}(T)$ match the first and second moment of $W_{LF}(T)$ if (real) roots of the following polynomials can be computed.

$$f_T(x) := \sum_{i=1}^{T} x^i - \mathbf{E_P}\left[W_{LF}(T)\right]$$

$$g_{x,T}(y) := y^T + \sum_{i=1}^{T-1}\left(2\sum_{j=1}^{T-i} x^j + 1\right) y^i - \mathbf{E_P}\left[W_{LF}^2(T)\right]$$

If roots (e.g. x_0 and y_0) are found, the balanced fund's equity portion x_S and its management fee c_{BF} is easily derived from x_0 and y_0. However, it is not yet guaranteed if x_S and c_{BF} are then indeed 'valid', that is $0\% \le x_S \le 100\%$ and $c_{BF} \ge 0\%$ p.a.

The following section derives sufficient conditions for the (unique) existence of above roots and the validity of the obtained balanced fund.

Sufficient conditions for obtaining unique and valid balanced funds

We first prove the (unique) existence of x_0 and y_0 such that $f_T(x_0) = 0$ and $g_{x_0,T}(y_0) = 0$ holds. It is easily shown that $\mathbf{E_P}[W_{LF}(T)] > 0$ and $\mathbf{E_P}[W_{LF}^2(T)] > 0$ holds. We further obtain $f_T(0) = 0$ and $\lim_{x \mapsto \infty} f_T(x) = \infty$ and therefore (at least) one real solution $x_0 > 0$ with $f_T(x_0) = \mathbf{E_P}[W_{LF}(T)]$ exists applying the intermediate value theorem. Similar arguments yield to (at least one) real solution $y_0 > 0$ with $g_{x_0,T}(y_0) = \mathbf{E_P}[W_{LF}^2(T)]$. Further, Descartes' rule of signs[18] gives at most one positive real solution for the polynomials considered. Therefore, the solutions x_0 and y_0 are unique. In addition, fast and robust algorithms exist to solve for roots of polynomials and hence quick solutions can be obtained.

Although x_0 and y_0 guarantee to match the first moments of the life-cycle fund, it is not yet ensured if x_0 and y_0 are actually moments of a valid balanced fund or even generated by a valid lognormal distributed random variable. Note, for defining a valid log-normal distributed random variable $\sigma_{BF}^2 > 0$ is essentially required in this setting. Solving x_0 and y_0 for μ_{BF} and σ_{BF}^2 gives $\mu_{BF} = \log x_0 - \frac{1}{2}\sigma_{BF}^2$ and $\sigma_{BF}^2 = \log y_0 - 2\log x_0$. Hence, if $y_0 > x_0^2$ holds, we obtain $\sigma_{BF}^2 > 0$ and thus a valid lognormal distribution. Further, the lognormal distribution under consideration is then generated by a valid balanced fund if additionally $y_0 \le x_0^2 \exp\{\sigma_S^2\}$ and $x_0 \le \exp\left\{r + \lambda_S \sqrt{\sigma_{BF}^2 / \sigma_S^2}\right\}$ hold.

[18] Published by René Descartes in his work "La Géometrie" in 1637 and often revisited nowadays e.g. in Anderson et al. (1998).

These conditions are easily checked and the required balanced fund's equity portion and its management fee are then consequently set as $x_S = \sqrt{\sigma_{BF}^2 / \sigma_S^2}$ and $c_{BF} = r + \lambda_S x_S - \log x_0$.

Some thoughts on the sufficient conditions

We start with investigating the condition $y_0 > x_0^2$. First note that $g'_{x,T}(y) > 0 \ \forall y > 0$ for any fixed $x > 0$, i.e. $g_{x,T}(y)$ is monotonically increasing for positive *x* and *y.* Hence for ensuring $y_0 > x_0^2$ it is sufficient to show $g_{x_0,T}(y_0) > g_{x_0,T}(x_0^2)$. First, we show $g_{x,T}(x^2) = f_T^2(x)$ again using induction.

T=1

$$f_1^2(x) = x^2 = g_{x,1}(x^2)$$

T-1 → T

$g_{x,T}(x^2)$ is written as

$$\begin{aligned} g_{x,T}(x^2) &= x^{2T} + \sum_{i=1}^{T-1}\left(2\sum_{j=1}^{T-i} x^j + 1\right)x^{2i} \\ &= x^{2T} + \sum_{i=1}^{T-1} x^{2i} + 2\sum_{i=1}^{T-1} x^{2i}\sum_{j=1}^{T-i} x^j \end{aligned}$$

Using the induction hypothesis then gives

$$\begin{aligned} f_T^2(x) &= \left(\sum_{i=1}^{T} x^i\right)^2 = x^{2T} + 2x^T\sum_{i=1}^{T-1} x^i + \left(\sum_{i=1}^{T-1} x^i\right)^2 \\ &= x^{2T} + 2x^T\sum_{i=1}^{T-1} x^i + x^{2(T-1)} + \sum_{i=1}^{T-2}\left(2\sum_{j=1}^{T-1-i} x^j + 1\right)x^{2i} \\ &= x^{2T} + 2x^T\sum_{i=1}^{T-1} x^i + x^{2(T-1)} + 2\sum_{i=1}^{T-2}\sum_{j=1}^{T-1-i} x^{j+2i} + \sum_{i=1}^{T-2} x^{2i} \\ &= x^{2T} + \sum_{i=1}^{T-1} x^{2i} + 2x^T\sum_{i=1}^{T-1} x^i + 2\sum_{i=1}^{T-2} x^{2i}\sum_{j=1}^{T-1-i} x^j \end{aligned}$$

Fairly simple algebra (e.g. using Geometric row expansion) then yields $\sum_{i=1}^{T-1} x^{2i} \sum_{j=1}^{T-i} x^j = x^T \sum_{i=1}^{T-1} x^i + \sum_{i=1}^{T-2} x^{2i} \sum_{j=1}^{T-1-i} x^j$ which completes the proof.

If the life-cycle fund's wealth distribution $W_{LF}(T)$ further allows for a positive variance[19], we finally obtain $g_{x_0,T}(y_0) = \mathbf{E_P}\left[W_{LF}^2(T)\right] > \left(\mathbf{E_P}\left[W_{LF}(T)\right]\right)^2 = f_T^2(x_0) = g_{x_0,T}(x_0^2)$ and hence $y_0 > x_0^2$.

Regarding the condition $y_0 \le x_0^2 \exp\{\sigma_S^2\}$, we argue from an economic point of view:

The condition is equivalent with the balanced fund's equity portion being at most 100%, i.e. no leveraging of the risky asset is required. In the considered (BS) model the 'risky part' or 'variability' of the considered life-cycle and balanced fund only stems from the equity portion used. In contrast, the management fee just deterministically reduces the performance and therefore has no direct influence on the fund investment's variability. Further, a pure equity fund – i.e. a balanced fund with equity portion of 100% – admits at least the same variability as the considered life-cycle funds due to $\left(x_{S,t}\right)_t \in [0,1]$ (cf. Section 2) which indicates that at most 100% equity portion is necessary to calibrate the balanced fund's variance in accordance with the life-cycle fund's variance. Our numerical experiments beyond this study further indicate, the condition $x_0 \le \exp\left\{r + \lambda_S \sqrt{\sigma_{BF}^2 / \sigma_S^2}\right\} = \exp\{r + \lambda_S x_S\}$ being fulfilled 'in general'.

[19] which in the (BS) model is fulfilled when the underlying asset strategy does not consist of a pure investment in the risk-free asset only.