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Abstract

We introduce an estimation method for multivariate Lévy processes, that can be used to model the development of different stock prices simultaneously. The method is based on historical observations that are available in discrete form. We choose a high frequency framework with a time horizon that tends to infinity. We provide estimators for the covariance matrix of the diffusion part, for the drift vector and for a specific transformation of the Lévy measure.

Keywords:
consistency, covariance matrix, covariation, critical region, drift vector, ellipsoid, high frequency, Levy measure, Merton model, random field, Skorohod space

1. Introduction

In financial mathematics stochastic processes are used to describe stock price developments. The increments of the logarithmic stock price are called log returns and are used for statistical investigations. The classical approach in the famous Black-Scholes model, introduced in Black and Scholes (1973), is to use just a Brownian motion to describe the logarithmic price development, so that the log returns are normally distributed. But various empirical studies revealed that this approach does not always match to the real data (cf. Mandelbrot, 1963; Cont, 2001). In particular the log returns of most financial data are not normally distributed, since they are skewed and have more probability mass in the tails (cf. Schoutens, 2003, Chapter 4).

One possibility to improve the modeling is using a Lévy process instead of the Brownian motion. Lévy processes are a natural generalization of the Brownian motion and form a class of stochastic processes that have stationary and independent increments and are stochastically continuous. They are a combination of a Brownian motion and a pure jump process. According to Schoutens (2003, p.43), Lévy models are much more flexible, since they can take the skewness and the excess kurtosis into account. The jump process can be used to model seldom and sudden large movements of the stock price. These asymmetric jumps can cause asymmetric and heavy tails in the distribution of the log returns. In most cases the distribution of Lévy processes is not easy to handle, because it is not given in a closed form. However, it is possible to work with the characteristic function, since it can be explicitly represented and only depends on the so called Lévy triplet. The Lévy triplet consists of the variance of the Brownian motion, a drift term and the so called Lévy measure that describes the jump behavior.

An important task is to estimate the Lévy triplet out of historical data. Usually, in Finance the data are only available at finitely many observation points. This fact is the main issue of the estimation. To recognize the jumps in the stock price development, it would be necessary to observe the process continuously. That means, we have a so called *ill-posed inverse problem*, which is a situation, where completely different input parameters lead to output data that look almost the same. Hence, it is difficult to apply standard methods like maximum likelihood procedures (cf. Honoré (1998), Cont and Tankov (2004, Chapter 7.2.1), Aït-Sahalia (2004))

In the recent years two different frameworks for the estimation of stochastic processes have been established, *low frequency* and *high frequency data*. In the context of asymptotic statistics *low frequency* means that the distance between two estimation points is asymptotically fixed. So an increasing sample size only leads to an increasing time horizon. For Lévy processes this means that a classical statistical situation with i.i.d. samples, which are independent of the sample size, is existent (because of the stationary and independent increments). E.g. in Neumann and ReiSS (2009), Gugushvili (2009) and Riesner (2006) method for univariate processes are presented using the low frequency framework. The other framework that has been established is the *high frequency* setting. In the context of asymptotical statistics *high frequency* means that the distance between two observation points asymptotically tends to zero. The advantage of this approach is that it is easier to distinguish asymptotically between diffusion and jumps. The drawback is that the framework is outside the classical statistical situation. The increments are i.i.d., however they depend on the sample size n, so a triangular array is of random variables is given.

In Gegler and Stadtmüller (2010) we have introduced an estimation method for univariate Lévy processes based on the high frequency framework. However, in Finance the development of different stock prices and their interactions are often considered simultaneously. Portfolios usually depend on different assets and a lot of options have more than one underlying. So we want to extend this method to multivariate Lévy processes. In Gegler and Stadtmüller (2010) the strategy is to use a threshold to distinguish between diffusion and jumps. In the multivariate case we use a *critical region*, instead. If an increment is located outside the critical region we consider it as a jump, if it is located within the critical region we assign it to the continuous part. Then, we estimated the diffusion and jump characteristics separately. The critical region has the form of an ellipsoid, since the level curve of density of the normal distribution forms an ellipsoid, too.

As far as we know, in literature the estimation of the complete characteristics of a general multivariate Lévy process based on high frequency data has not been considered yet. In Gobbi and Mancini (2010, 2007) related problems are considered. A method is introduced to separate the integrated covariance of the diffusion from the co-jumps. In Barndorff-Nielsen and Shephard (2004a,b) the concept of realized bipower covariation is introduced for multivariate semimartingales. In Aït-Sahalia and Jacod (2007); Barndorff-Nielsen et al. (2006); Jacod (2008); Woerner (2006); Figueroa-López (2009); Shimizu (2006); Mancini and Renò (2011); Mancini (2009, 2008, 2004, 2003, 2001) estimation methods are introduced that deal with univariate processes based on high frequency data (see also Gegler and Stadtmüller, 2010).

In Section 2 the framework is given. In Section 3 the estimation method is introduced. The strategy is to develop estimators under the assumption that the exact jumps and the continuous part are known. Then, we show that using the critical region instead of this assumption leads to estimators that behave asymptotically very similar. In Section 4 we check the finite sample properties of the estimators by simulation study.

2. The framework

We use a suitable probability space $(\Omega, \mathbb{F}, \mathbb{P})$. On this probability space we define an $\mathbb{R}^d-$valued Lévy process $\{L_t,\ t \geq 0\}$. We assume $\mathbb{E}\left[|L_1|\right] < \infty$. Then, the characteristic function is given by the following version of the Lévy Khinchin representation

$$\phi_{L_t}(u) = \exp\left(t\left(-\frac{1}{2}u^T C u + iu^T\gamma + \int_{\mathbb{R}^d}\left(\exp(iu^T x) - 1 - iu^T x\right)\nu(dx)\right)\right), \quad t \geq 0,$$

where (C, γ, ν) is the Lévy triplet. (see e.g. Sato (2007, Theorem 8.1)). We provide estimators for C, γ and, if $\mathbb{E}\left[|L_1|^{2l}\right] < \infty$, for the following finite transformation of the Lévy measure

$$\mu(z) = \mu_{A,l}(z) := \int_{\{x \leq z\}} (x^T A\, x)^l\, \nu(dx),$$

where $l \in \mathbb{N}$, $A \in \mathbb{R}^{d\times d}$, $A \neq 0$ and $z \in \overline{\mathbb{R}}^d$.

By the Lévy-Itô decomposition (e.g. see Sato (2007, Theorem 19.2)) we have

$$L_t = {}^C L_t + {}^J L_t \quad t \geq 0,$$

where $\{{}^C L_t,\ t \geq t\}$ is a nonstandard Wiener process with covariance matrix C and drift $\gamma + \int_{\mathbb{R}^d} x(1 - \mathbb{1}(\|x\| \leq 1))\nu(dx)$ ($\|\cdot\|$ denotes the supremum norm). The process $\{{}^J L_t,\ t \geq t\}$ is a pure jump Lévy process with characteristics $(0, 0, \nu)$.

The Lévy process can be used to describe a logarithmic stock price development. We do not require that the Lévy measure ν is absolute continuous in the multivariate framework. However, we require that the Lévy measures $\widetilde{\nu}_i$ of the corresponding univariate component Lévy processes $\{L_{t,i},\ t \geq 0\}$ are absolute continuous with densities f_i. In addition we require that, for all $\varepsilon > 0$, there exists $\kappa_\varepsilon > 0$ such that

$$f_i(x) < \kappa_\varepsilon, \quad x \in (-\varepsilon, \varepsilon)^c. \tag{1}$$

Keep in mind that $\widetilde{\nu}_i$ is different to the margins ν_i of the Lévy measure ν (cf. Sato (2007, Proposition 11.10)). Furthermore, to avoid technical issues, we assume that the covariance matrix C is not singular. We impose that we can observe the process at high frequent, equidistant time points

$$t_k := h\,k, \quad k = 1, \ldots, n, \qquad h := \kappa n^{-\alpha}, \tag{2}$$

where $\alpha \in (0, 1]$, $\kappa > 0$. The time horizon is given by $T = \frac{1}{\kappa} n^{1-\alpha}$. For simplification in the following we set w.l.o.g. $\kappa = 1$. In most cases we assume $\alpha \in (1/2, 1]$ to ensure that $T \to \infty$ and $h/T \to 0$. If we estimate the covariance matrix C, we will also allow $\alpha = 1$. Next we define the increments of the process at this time points.

$$X_{h,k} := L_{k\,h} - L_{(k-1)\,h} \qquad k = 1, \ldots, n.$$

3. Estimation method

Our strategy to develop estimators is as follows. First, we think about an estimator for the covariance matrix C under the unrealistic assumption that we know the increments of the diffusion part of the process which are defined as follows

$$^{C}X_{h,k} := {}^{C}L_{k\,h} - {}^{C}L_{(k-1)\,h} \qquad k = 1, \ldots, n,$$

where the $^{C}L_t$ is the nonstandard Wiener process from above that turns out by the Lévy-Itô decomposition. We call the estimator that is based under this unrealistic assumption *pre-estimator* and define it as follows

$$\widehat{C}_{pre} := \frac{1}{T} \sum_{k=1}^{n} {}^{C}X_{h,k}\, {}^{C}X_{h,k}^{T}.$$

Theorem 1(i) shows that this estimator is consistent. Analogously, we develop an *pre-estimator* for $\mu(z)$ under the unrealistic assumption that we can observe the process continuously. That means we know the exact jump times and sizes $\{\Delta L_t,\ t \in [0, T]\}$, where $\Delta L_t := L_t - \lim_{s \to t-} L_s$. This pre-estimator is defined as follows

$$\widehat{\mu}_{pre}(z) = \widehat{\mu}_{A,l,pre}(z) := \frac{1}{T} \sum_{0 \leq t \leq T} \left(\Delta L_t^T A \Delta L_t\right)^l \mathbb{1}(\Delta L_t \leq z) \quad z \in \overline{\mathbb{R}}^d.$$

Theorem 1(ii) shows the asymptotic properties of this estimator. Since $\widehat{\mu}_{pre}(z)$ depends on $z \in \mathbb{R}^d$ we interpret it as a random field in z and give a limit field.

Theorem 1. *(i) Let $1/2 < \alpha \leq 1$ in (2). Then, as $n \to \infty$,*

$$\widehat{C}_{pre} \xrightarrow{P} C.$$

(ii) Let $l \in \mathbb{N}$ and $\mathbb{E}[|L_1|^{4l}] < \infty$. Define $\vec{g} : [0, 1]^d \to \overline{\mathbb{R}}^d$, $\vec{g}(y) = \left(g\left(y^{(1)}\right), \ldots, g\left(y^{(d)}\right)\right)^T$, where $g : [0, 1] \to \overline{\mathbb{R}}$ is a strictly increasing and bijective function. Then, as $T \to \infty$,

$$\sqrt{T}\left(\widehat{\mu}_{pre} \circ \vec{g} - \mu \circ \vec{g}\right) \xrightarrow{\mathcal{D}} G,$$

where $\mathcal{D}$ denotes weak convergence in the generalized Skorohod space $D[0, 1]^d$ (cf. Bickel and Wichura, 1971). G is a Gaussian process over $[0, 1]^d$ with

$$\mathbb{E}\left[G(y)\right] = 0, \quad \operatorname{Cov}\left[G(y_1), G(y_2)\right] = \int_{\{x^{(1)} \leq g(y_1^{(1)} \wedge y_2^{(1)}), \ldots, x^{(d)} \leq g(y_1^{(d)} \wedge y_2^{(d)})\}} (x^T A x)^{2l} \nu(dx), \quad y,\, y_1,\, y_2 \in [0, 1]^d.$$

Proof. (i) Because of the normal distribution it can be easily verified that $\mathbb{E}(\widehat{C}_{pre} - C)^2 \to 0$.
(ii) For $z \in \overline{\mathbb{R}}^d$ we have

$$\widehat{\mu}_{pre}(z) = \frac{1}{T} \sum_{0<t\leq T} (\Delta L_t^T A \Delta L_t)^l \mathbb{1}(\Delta L_t \leq z) = \frac{1}{T} \sum_{k=1}^{T} \sum_{k-1<t\leq k} (\Delta L_t^T A \Delta L_t)^l \mathbb{1}(\Delta L_t \leq z).$$

Since the random variables $\sum_{k-1<t\leq k} (\Delta L_t^T A \Delta L_t)^l \mathbb{1}(\Delta L_t \leq z)$ are i.i.d. the Central Limit theorem can be applied to obtain

$$\sqrt{T}\left(\widehat{\mu}_{pre}(z_1) - \mu(z_1), \ldots, \widehat{\mu}_{pre}(z_m) - \mu(z_m)\right)^T \xrightarrow{d} Z \sim N(0, \Sigma),$$

where

$$\Sigma_{i,j} := \int_{\{x^{(1)} \leq z_i^{(1)} \wedge z_j^{(1)}, \ldots, x^{(d)} \leq z_i^{(d)} \wedge z_j^{(d)}\}} (x^T A x)^{2l} \nu(dx).$$

The statement follows from Heinrich and Schmidt (1985, Lemma 3). □

However, in reality we do not know the increments of the diffusion part of the process and cannot observe the process continuously. To disentangle jumps from diffusion, we choose the following approach. We introduce a critical region $B_n \subset \mathbb{R}^d$. If we have $X_{k,h} \notin B_n$, we consider $X_{k,h}$ as a jump. If we have $X_{k,h} \in B_n$, we assign $X_{k,h}$ to the continuous part. For all n we define

$$\boldsymbol{B}_n(D, \beta) := \{x \in \mathbb{R}^d : x^T D_n^{-1} x \leq \beta b_n^2\},$$

where D is a symmetric $d \times d$−matrix, $\beta > 0$ and $b_n := \sqrt{2\,h\,\log n}$. We call a sequence of random sets $\{B_n,\ n \in \mathbb{N}\}$ a sequence of critical regions, if there exists $1 < \beta' < \beta'' < \infty$ such that

$$\boldsymbol{B}_n(C, \beta') \subseteq B_n[\omega] \subseteq \boldsymbol{B}_n(I, \beta''),$$

for all $n \in \mathbb{N}$ and $\omega \in \Omega$ (I is the identity matrix). An element B_n of the sequence is called critical region. For $\beta > 1$, we call $\{B_n = \boldsymbol{B}_n(C, \beta), n \in \mathbb{N}\}$ a sequence of true critical regions. In the following we assume w.l.o.g. that B_n is deterministic to avoid technical overload. Important for the proofs are the bounds $\boldsymbol{B}_n(C, \beta')$ and $\boldsymbol{B}_n(I, \beta'')$. The critical set B_n is chosen in such a way that it contains asymptotically the diffusion increments. This is shown in Lemma 1. The true critical regions form ellipsoids in $\mathbb{R}^d$. We choose that definition, since the level curve of the density of the multivariate normal distribution forms an ellipsoid, too.

The estimators for C and $\mu(z)$ that use the critical region are defined as follows

$$\widehat{C} := \frac{1}{T} \sum_{k=1}^{n} X_{h,k} X_{h,k}^T \mathbb{1}(X_{h,k} \in B_n)$$

$$\widehat{\mu}(z) = \widehat{\mu}_{A,l}(z) := \frac{1}{T} \sum_{k=1}^{n} (X_{h,k}^T A X_{h,k})^l \mathbb{1}(X_{h,k} \notin B_n,\ X_{h,k} \leq z), \quad z \in \overline{\mathbb{R}}^d.$$

The next theorems give the asymptotic properties. For $\widehat{C}$ only consistency can be shown. In general we only obtain consistency for $\widehat{\mu}(z)$, too. However, we obtain better rates, if

$$f_i(x) = O(x^{-3}),\ x \to 0, \quad \text{for all } i \in \{1, \ldots, d\}, \tag{3}$$

where f_i is the density of the of the Lévy measure $\widetilde{\nu}_i$ of the univariate component Lévy process $\{L_{t,i},\ t \geq 0\}$.

Theorem 2. *Let $1/2 < \alpha \leq 1$ in (2). Then, as $n \to \infty$,*

$$\widehat{C} - \widehat{C}_{pre} \xrightarrow{P} 0 \quad \textit{and hence} \quad \widehat{C} - C \xrightarrow{P} 0.$$

Proof. We have

$$\frac{1}{T}\sum_{k=1}^{n} X_{h,k}X_{h,k}^{T}\mathbb{1}(X_{h,k}\in B_n)-\widehat{C}_{pre}=\frac{1}{T}\sum_{k=1}^{n} {}^{C}X_{h,k}{}^{C}X_{h,k}^{T}\mathbb{1}(X_{h,k}\notin B_n) \tag{4}$$

$$+\frac{1}{T}\sum_{k=1}^{n} {}^{J}X_{h,k}{}^{J}X_{h,k}^{T}\mathbb{1}(X_{h,k}\in B_n) \tag{5}$$

$$+\frac{1}{T}\sum_{k=1}^{n} {}^{C}X_{h,k}{}^{J}X_{h,k}^{T}\mathbb{1}(X_{h,k}\in B_n) \tag{6}$$

$$+\frac{1}{T}\sum_{k=1}^{n} {}^{J}X_{h,k}{}^{C}X_{h,k}^{T}\mathbb{1}(X_{h,k}\in B_n). \tag{7}$$

We only consider the diagonal elements of the square matrices in (4) and (5). In Lemma 5 and Lemma 6 it is shown that they converges to zero in probability. For the entries of the matrix in (6) and (7), by the Cauchy Schwartz inequality we have

$$\frac{1}{T}\sum_{k=1}^{n}|{}^{C}X_{h,k,i_1}||{}^{J}X_{h,k,i_2}|\mathbb{1}(X_{h,k}\in B_n)\leq\sqrt{\frac{1}{T}\sum_{k=1}^{n}{}^{C}X_{h,k,i_1}^{2}}\sqrt{\frac{1}{T}\sum_{k=1}^{n}{}^{J}X_{h,k,i_2}^{2}\mathbb{1}(X_{h,k}\in B_n)},$$

where $i_1,\, i_2\in\{1,\dots,d\}$. By Lemma 12 the expectation of the first factor is uniform bounded and by Lemma 6 the second factor converges to zero in probability. □

Theorem 3. *(i) Let $1/2<\alpha<1$ in (2), $l\in\mathbb{N}$ and and assume $\mathbb{E}[|L_1|^{4l}]<\infty$. Then, as $n\to\infty$,*

$$\sup_{z\in\mathbb{R}^d}\left|\widehat{\mu}_{pre}(z)-\mu(z)\right|\xrightarrow{P}0\quad\text{and hence}\quad\sup_{z\in\mathbb{R}^d}\left|\widehat{\mu}(z)-\mu(z)\right|\xrightarrow{P}0\,.$$

(ii) Let $1/2<\alpha<1$ in (2). We choose

$$r\in\begin{cases}[0\vee 1/2-2\alpha/3\,,\alpha/3) & if\ \alpha\leq 3/5\\ [0\vee 1/2-2\alpha/3\,,(1-\alpha)/2) & if\ \alpha>3/5\end{cases},$$

define $l:=\lfloor 3r/(\alpha-3r)\rfloor+2$ and $p:=\lfloor l\alpha/(\alpha-3r)+1\rfloor/l$ and require $\mathbb{E}[|L_1|^{2l(2\vee p)}]$ and that (3) is satisfied. Then, as $n\to\infty$,

$$n^r\sup_{z\in\mathbb{R}^d}\left|\widehat{\mu}(z)-\mu(z)\right|\xrightarrow{P}0\quad\text{and hence}\quad n^r\sup_{z\in\mathbb{R}^d}\left|\widehat{\mu}(z)-\mu(z)\right|\xrightarrow{P}0\,.$$

(iii) Let $3/5<\alpha<1$ in (2). We define $l:=\lfloor 1-\alpha/(5\alpha/3-1)\rfloor+2$ and $p:=\lfloor 2l/(5-3/\alpha)+1\rfloor/l$ and require $\mathbb{E}\left[|L_1|^{2l(2\vee p)}\right]<\infty$ and that (3) is satisfied. Then, as $n\to\infty$

$$\sqrt{T}\sup_{z\in\mathbb{R}^d}\left|\widehat{\mu}(z)-\mu(z)\right|\xrightarrow{P}0\,.$$

Hence, for $\vec{g}:[0,1]^d\to\overline{\mathbb{R}}^d$, $\vec{g}(y)=\left(g\left(y^{(1)}\right),\dots,g\left(y^{(d)}\right)\right)^T$, where $g:[0,1]\to\overline{\mathbb{R}}$ is a strictly increasing and bijective function, we have, as $n\to\infty$

$$\sqrt{T}\,\left(\widehat{\mu}\circ\vec{g}-\mu\circ\vec{g}\right)\xrightarrow{\mathcal{D}}G,$$

where G is a Gaussian process over $[0,1]^d$ with

$$\mathbb{E}\left[G(y)\right]=0,\quad \mathrm{Cov}\left[G(y_1),G(y_2)\right]=\int_{\{x^{(1)}\leq g(y_1^{(1)}\wedge y_2^{(1)}),\dots,x^{(d)}\leq g(y_1^{(d)}\wedge y_2^{(d)})\}}(x^TAx)^{2l}\nu(dx),\quad y,\,y_1,\,y_2\in[0,1]^d.$$

Proof. We use the given r in (ii) and set $r = 0$ in (i) and $r = (1-\alpha)/2$ in (iii). We use Theorem 1(ii), so we only have to prove

$$n^r \sup_{z\in\mathbb{R}^d} \left| \frac{1}{T} \sum_{k=1}^{n} \left(X_{h,k}^T A X_{h,k} \right)^l \mathbb{1}(X_{h,k} \notin B_n,\, X_{h,k} \leq z) - \mu_{pre}(z) \right|$$

$$= n^r \sup_{z\in\mathbb{R}^d} \left| \frac{1}{T} \sum_{k=1}^{n} \left(X_{h,k}^T A X_{h,k} \right)^l \mathbb{1}(X_{h,k} \notin B_n,\, X_{h,k} \leq z) - \frac{1}{T} \sum_{0\leq t\leq T} \left(\Delta L_t^T A \Delta L_t \right)^l \mathbb{1}(\Delta L_t \leq z) \right|$$

$$n^r \sup_{z\in\mathbb{R}^d} \left| \frac{1}{T} \sum_{k=1}^{n} \left({}^J X_{h,k}^T A^J X_{h,k} + 2^J X_{h,k}^T A^C X_{h,k} + {}^C X_{h,k}^T A^C X_{h,k} \right)^l \mathbb{1}(X_{h,k} \notin B_n,\, X_{h,k} \leq z) \right.$$

$$\left. - \frac{1}{T} \sum_{0\leq t\leq T} \left(\Delta L_t^T A \Delta L_t \right)^l \mathbb{1}(\Delta L_t \leq z) \right| \xrightarrow{P} 0.$$

By Lemma 10 (iii), we obtain an upper bound that is up to a positive constant equal to

$$n^r \sup_{z\in\mathbb{R}^d} \left| \frac{1}{T} \sum_{k=1}^{n} \left({}^J X_{h,k}^T A^J X_{h,k} \right)^l \mathbb{1}(X_{h,k} \notin B_n,\, X_{h,k} \leq z) - \frac{1}{T} \sum_{0\leq t\leq T} \left(\Delta L_t^T A \Delta L_t \right)^l \mathbb{1}(\Delta L_t \leq z) \right| \tag{8}$$

$$+ n^r \frac{1}{T} \sum_{m=1}^{l} \sum_{k=1}^{n} \left({}^J X_{h,k}^T A^J X_{h,k} \right)^{l-m} \left| {}^C X_{h,k}^T A^J X_{h,k} \right|^m \mathbb{1}(X_{h,k} \notin B_n) \tag{9}$$

$$+ n^r \frac{1}{T} \sum_{m=1}^{l} \sum_{k=1}^{n} \left({}^J X_{h,k}^T A^J X_{h,k} \right)^{l-m} \left| {}^J X_{h,k}^T A^C X_{h,k} \right|^m \mathbb{1}(X_{h,k} \notin B_n) \tag{10}$$

$$+ n^r \frac{1}{T} \sum_{m=1}^{l} \sum_{k=1}^{n} \left({}^J X_{h,k}^T A^J X_{h,k} \right)^{l-m} \left({}^C X_{h,k}^T A^C X_{h,k} \right)^m \mathbb{1}(X_{h,k} \notin B_n). \tag{11}$$

The term in (8) is considered in Lemma 8(i) for (i) and 8(ii) for (ii) and (iii). There we use Proposition 2. For (9) and (10) we use the following upper bound

$$\max_{i_1,i_2\in\{1,\dots,d\}} |A_{i_1,i_2}| \cdot n^r \frac{1}{T} \sum_{m=1}^{l} \sum_{k=1}^{n} \left(\sum_{i_1,i_2=1}^{d} |{}^J X_{h,k,i_1}||{}^J X_{h,k,i_2}| \right)^{l-m} 2 \left(\sum_{i_1,i_2=1}^{d} |{}^C X_{h,k,i_1}||{}^J X_{h,k,i_2}| \right)^m \mathbb{1}(X_{h,k} \notin B_n).$$

We apply Lemma 10(ii) and 10(i) and obtain an upper bound that is up to a positive constant equal to

$$n^r \max_{i_1,i_2\in\{1,\dots,d\}} |A_{i_1,i_2}| \cdot \frac{1}{T} \sum_{m=1}^{l} \sum_{k=1}^{n} \sum_{i_1,i_2,i_3=1}^{d} |{}^J X_{h,k,i_1}|^{2l-2m} |{}^J X_{h,k,i_2}|^m |{}^C X_{h,k,i_3}|^m \mathbb{1}(X_{h,k} \notin B_n).$$

That term tends to zero by Lemma 4. For (11) we have analogously

$$2\, n^r \max_{i_1,i_2\in\{1,\dots,d\}} |A_{i,j}| \frac{1}{T} \sum_{m=1}^{l} \sum_{k=1}^{n} \sum_{i_1,i_2=1}^{d} |{}^J X_{h,k,i_1}|^{2l-2m} |{}^C X_{h,k,i_2}|^{2m} \mathbb{1}(X_{h,k} \notin B_n).$$

This term tends zero by Lemma 5 for $l = 1, m = 1$, Lemma 12(i) for $l > 1, m = l$ and Lemma 4 for all other summands. □

For the estimation of γ, the critical region is not needed. The estimator is defined as follows

$$\widehat{\gamma} := \frac{1}{T} \sum_{k=1}^{n} X_{h,k}$$

and the asymptotic properties are given in the next theorem.

Theorem 4. *Let* $0 < \alpha < 1$ *in (2). Then, as* $n \to \infty$,

$$\widehat{\gamma} \xrightarrow{a.s.} \gamma \quad \textit{and} \quad \sqrt{T}\,(\widehat{\gamma} - \gamma) \xrightarrow{d} Z \sim N\left(0, C + \int_{\mathbb{R}^d} xx^T \nu(dx)\right).$$

Proof. We assume w.l.o.g. $T \in \mathbb{N}$, then,

$$\widehat{\gamma} := \frac{1}{T}\sum_{k=1}^{n} X_{h,k} = \frac{1}{T} L_T = \frac{1}{T}\sum_{k=1}^{T} X_{1,k}.$$

The random variables $X_{1,k}$, $k = 1, \ldots, n$ are i.i.d.. So we can apply the Strong Law of Large Numbers and the Central Limit theorem. □

4. Simulation study

In this section we check the finite sample properties of the estimators by simulation study. Before we can start we have to handle a practical issue. The problem is that the definition of the critical region B_n depends on the unknown covariance C. We solve this problem by using the estimator $\widehat{C}$, instead. However, the estimator depends in turn on the critical region. This leads to an iteration method. So for $\beta > 1$ we define $\widehat{B}_{n,0}(\beta) := \mathbb{R}^d$ and, for all $i \in \mathbb{N}$, $\widehat{B}_{n,i}(\beta) := \boldsymbol{B}_n(\widehat{C}(\widehat{B}_{n,i-1}(\beta)), \beta)$ and $\widehat{B}^*_n(\beta) := \lim_{i\to\infty} \widehat{B}_{n,i}(\beta)$. It can be shown that for all $\varepsilon > 0$, $\mathbb{P}((1-\varepsilon)\boldsymbol{B}_n(C,\beta) \subseteq \widehat{B}^*_n(\beta) \subseteq (1+\varepsilon)\boldsymbol{B}_n(C,\beta)) \to 1,\ n \to \infty$. That means $\widehat{B}^*_n(\beta)$ and $\boldsymbol{B}_n(C,\beta)$ are very similar. The parameter $\beta > 1$ is chosen very close to 1.

We use the bivariate Merton model, introduced in Merton (1976), to generate bi-dimensional log returns. The setting is similar to Aït-Sahalia (2004) and Gegler and Stadtmüller (2010). We use $n = 1000$ and $h = 1/252$ (daily data) and set the parameters to

$$C := \begin{pmatrix} 0.09 & 0.03 \\ 0.03 & 0.04 \end{pmatrix}, \quad \gamma := \begin{pmatrix} 0.035 \\ 0.025 \end{pmatrix}, \quad \eta := \begin{pmatrix} 0.01 & -0.005 \\ -0.005 & 0.01 \end{pmatrix}, \quad \beta := \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad \lambda = 10.$$

where η and β control the normally distributed jumps and λ is the jump intensity. We give estimators for C, and the term

$$qcv := \int_{\mathbb{R}^2} x_1\, x_2\, \nu(dx) = \mu_{1,A}, \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This quantity is equal to the expectation of the quadratic covariation of the jump part of the Lévy process in a time interval $[0, 1]$. The true value in this scenario is 0.05. We perform 2000 runs and give the result in Table 1. We realize that the the bias and the variance of all estimators are very small.

$n = 1000$	$\widehat{C}_{1,1}$	$\widehat{C}_{2,2}$	$\widehat{C}_{1,2}$	$\widehat{qcv}$
True	0.0900	0.0400	0.0300	-0.0500
Mean	0.0884	0.0394	0.0294	-0.0491
Stdev	0.0042	0.0019	0.0022	0.0203
MSE	0.0000	0.0000	0.0000	0.0004
Min	0.0757	0.0336	0.0227	-0.1376
0.25-q.	0.0854	0.0380	0.0279	-0.0617
Median	0.0883	0.0394	0.0293	-0.0477
0.75-q.	0.0912	0.0407	0.0309	-0.0344
Max	0.1045	0.0463	0.0386	0.0057

Table 1: Simulation result using the bivariate Merton model

Appendix A. Technical details

Lemma 1. *Let $\{{}^{C}X_{h,k},\ k=1,\dots,n\}$ be the increments of a Wiener process with covariance matrix C and drift γ and let $\alpha \in (1/2,1]$ in (2). Define*

$$M_1^{(n)} := \{\exists k \in \{1,\dots,n\},\ s.t.\ {}^{C}X_{h,k} \notin B_n\},$$

then, as $n \to \infty$,

$$\mathbb{P}(M_1^{(n)}) \longrightarrow 0.$$

Proof. We use the increments of the centered Wiener process ${}^{C'}L_t = {}^{C}L_t - t(\gamma + \int_{\mathbb{R}^d} x(1-\mathbb{1}(\|x\|\le 1))\nu(dx))$ and have

$$\begin{aligned}
&\mathbb{P}({}^{C}X_{h,1} \notin B_n) = \mathbb{P}({}^{C'}X_{h,1} + h(\gamma + \int_{\mathbb{R}^d} x(1-\mathbb{1}(\|x\|\le 1))\nu(dx)) \notin B_n)\\
&\le \mathbb{P}({}^{C'}X_{h,1} + h(\gamma + \int_{\mathbb{R}^d} x(1-\mathbb{1}(\|x\|\le 1))\nu(dx)) \notin \boldsymbol{B}_n(C,\beta'))\\
&= \mathbb{P}({}^{C'}X_{h,1} + h(\gamma + \int_{\mathbb{R}^d} x(1-\mathbb{1}(\|x\|\le 1))\nu(dx)) \notin \{x:\ x^T C^{-1} x \le \beta'\, b_n^2\})\\
&= \mathbb{P}({}^{C'}X_{h,1} \notin \{x:\ (x - h(\gamma + \int_{\mathbb{R}^d} x(1-\mathbb{1}(\|x\|\le 1))\nu(dx)))^T C^{-1}(x - h(\gamma + \int_{\mathbb{R}^d} x(1-\mathbb{1}(\|x\|\le 1))\nu(dx))) \le \beta'\, b_n^2\}).
\end{aligned}$$

Because $b_n^2 = 2h\log n$, there exists a $\widetilde{\beta'} \in (1,\beta')$ such that an upper bound is given by

$$\mathbb{P}({}^{C'}X_{h,1} \notin \{x:\ x^T\, C^{-1}\, x \le \widetilde{\beta'}\, b_n^2\}) = \mathbb{P}({}^{C'}X_{h,1}^T\, C^{-1}\, {}^{C'}X_{h,1} > \widetilde{\beta'}\, b_n^2) = \mathbb{P}({}^{C'}X_{1,1}^T\, C^{-1}\, {}^{C'}X_{1,1} > \widetilde{\beta'}\, 2\log n),$$

where we have use the selfsimilarity property of the Wiener process (see e.g. Sato, 2007, Theorem. 5.4). The term ${}^{C'}X_{1,1}^T\, C^{-1}\, {}^{C'}X_{1,1}$ is χ_d^2–distributed. Thus, the probability form above is equal to

$$\int_{\widetilde{\beta'} 2\log n}^{\infty} \frac{t^{d/2-1} e^{-t/2}}{2^{d/2}(\gamma + \int_{\mathbb{R}^d} x(1-\mathbb{1}(\|x\|\le 1))\nu(dx))(d/2)} dt$$

and there exists a $\xi \in (1/\widetilde{\beta}, 1)$ such that an upper bound is given by

$$\int_{\widetilde{\beta'}\, 2\log n}^{\infty} \exp\left(\frac{-\xi t}{2}\right) dt = \frac{2}{\xi}\exp\left(-\xi\widetilde{\beta'}\log n\right) = \frac{2}{\xi} n^{-\xi\widetilde{\beta'}}.$$

By the Binomial distribution and Bernoulli's inequality we have

$$\mathbb{P}(M_1^{(n)}) = 1 - \left(1 - \mathbb{P}({}^{C}X_{h,1} \notin B_n)\right)^n \le n\,\mathbb{P}({}^{C}X_{h,1} \notin B_n) \le \frac{2}{\xi} n^{1-\xi\widetilde{\beta'}} \to 0.$$

□

Definition 1. (i) Let $0 < t_1 \le t_2 < \infty$, $B \in \mathbb{B}(\mathbb{R}^d)$ and $z_1,\ z_2 \in \overline{\mathbb{R}}^d$. We define

$$\begin{aligned}
N_{(t_1,t_2]}(B) &:= \sum_{t_1<t\le t_2} \mathbb{1}(\Delta L_t \in B,\ \Delta L_t \ne 0)\\
N_{(t_1,t_2]}(z_1,z_2) &:= N_{(t_1,t_2]}(\{x \in \mathbb{R}^d :\ z_1 \le x \le z_2\})\\
N_{(t_1,t_2]}(z_1) &:= N_{(t_1,t_2]}(-\infty, z_1),\\
N_{t_1}(\cdot) &:= N_{(-\infty,t_1]}(\cdot).
\end{aligned}$$

(ii) Let $0 < t_1 \le t_2 < \infty$, $i \in \{1,\dots,d\}$ and $B \in \mathbb{B}(\mathbb{R})$. We denote

$$N_{(t_1,t_2],i}(B) := \sum_{t_1<t\le t_2} \mathbb{1}(\Delta L_{t,i} \in B,\ \Delta L_{t,i} \ne 0).$$

For $z_1,\ z_2 \in \overline{\mathbb{R}}$, the quantities $N_{(t_1,t_2],i}(z_1,z_2)$, $N_{(t_1,t_2],i}(z_1)$ and $N_{t_1,i}(\cdot)$ are defined analogously to (i).

Lemma 2. *Let $\alpha \in (1/2, 1]$ in (2). Define with some $u_n > 0$*

$$M_2^{(n)}(u_n) := \left\{\exists k \in \{1, ..., n\}\ s.t.\ N_{(k-1)h,kh}(\{x \in \mathbb{R}^d : \|x\| \geq u_n\}) > 1\right\}.$$

(i) Assume that $n^{1-2\alpha}\nu^2(\{x \in \mathbb{R}^d : \|x\| \geq \xi_n\}) \to 0$ $(n \to \infty)$. Then, as $n \to \infty$,

$$\mathbb{P}\left(M_2^{(n)}(\xi_n)\right) \longrightarrow 0.$$

(ii) Assume that $n^{1-2\alpha}/\xi_n^4 \to 0$, $n \to \infty$ and that (3) is satisfied. Then, as $n \to \infty$,

$$\mathbb{P}\left(M_2^{(n)}(\xi_n)\right) \longrightarrow 0.$$

(iii) Let $\xi > 0$ be a constant, then, as $n \to \infty$,

$$\mathbb{P}\left(M_2^{(n)}(\xi)\right) \longrightarrow 0.$$

Proof. Ad (i), we have

$$\begin{aligned}\mathbb{P}\left(M_2^{(n)}\right) &\leq n(1 - \exp(-\nu(\{x \in \mathbb{R}^d : \|x\| \geq \xi_n\})h)(1 + \nu(\{x \in \mathbb{R}^d : \|x\| \geq \xi_n\})h)) \\ &\leq n(1 - (1 - \nu(\{x \in \mathbb{R}^d : \|x\| \geq \xi_n\}))h)(1 + \nu(\{x \in \mathbb{R}^d : \|x\| \geq \xi_n\})h) = n\,h^2\nu^2(\{x \in \mathbb{R}^d : \|x\| \geq \xi_n\}) \longrightarrow 0.\end{aligned}$$

Ad (ii), because of (3) there exists a $\kappa > 0$ such that

$$\begin{aligned}n\,h^2\nu^2(\{x \in \mathbb{R}^d : \|x\| \geq \xi_n\}) &\leq n\,h^2\left(\sum_{i=1}^{d}\widetilde{\nu}_i((-\xi_n, \xi_n)^c)\right)^2 \\ \leq n\,h^2\left(\sum_{i=1}^{d}\int_{(-\xi_n,\xi_n)^c} f_i(x)dx\right)^2 &\leq \kappa\,n\,h^2\left(\int_{(-\xi_n,\xi_n)^c}\frac{1}{x^3}dx\right)^2 = \kappa\,n\,h^2(1/\xi_n^2)^2 \longrightarrow 0.\end{aligned}$$

Ad (iii), analogously to (i) and (ii). □

Definition 2. Let $\{L_t,\ t \geq 0\}$ be a Lévy process with triplet (C, γ, ν) and J the corresponding Poisson measure (see e.g. Sato (2007, Theorem 19.2)).
(i) The large jump part is defined by

$${}^{CP}L_t(\omega)[\xi_1, \xi_2] := \int_0^t \int_{\xi_1 < \|x\| \leq \xi_2} x\,J((ds, dx), \omega)$$

and contains all jumps with size in $(\xi_1, \xi_2]$, where $\xi_1, \xi_2 \in [0, \infty]$ in case of $\int_{\|x\|\leq 1}\|x\|\nu(dx) < \infty$ and $\xi_1, \xi_2 \in (0, \infty]$ in case $\int_{\|x\|\leq 1}\|x\|\nu(dx) = \infty$. Denote ${}^{CP}L_t[\xi] := {}^{CP}L_t[\xi, \infty]$ and in case $\int_{\|x\|\leq 1}\|x\|\nu(dx) < \infty$, ${}^{CP}L_t := {}^{CP}L_t[0]$.
(ii) The small jump part is defined by

$${}^{sJ}L_t(\omega)[\xi_1, \xi_2] := \int_0^t \int_{\xi_1 \leq \|x\| \leq \xi_2} (x\,J((ds, dx), \omega) - x\,ds\,\nu(dx))$$

and contains all jumps with size in $(\xi_1, \xi_2]$ and the corresponding compensating drift term, where $\xi_1, \xi_2 \in [0, \infty]$ in case of $\int_{\|x\|>1}\|x\|\nu(dx) < \infty$ and $\xi_1, \xi_2 \in [0, \infty)$ in case of $\int_{\|x\|>1}\|x\|\nu(dx) = \infty$. Define ${}^{sJ}L_t[\xi] := {}^{sJ}L_t[0, \xi]$ and ${}^{sJ}L_t := {}^{sJ}L_t[1]$.
(iii) The compensating drift term is defined by

$$\gamma_t^c[\xi_1, \xi_2] := t\int_{\xi_1 \leq \|x\| \leq \xi_2} x\,\nu(dx)$$

and compensates the jumps with size in $(\xi_1, \xi_2]$, where $\xi_1, \xi_2 \in [0, \infty]$ in case of $\int_{\|x\|\leq 1}\|x\|\nu(dx) < \infty$, $\int_{\|x\|>1}\|x\|\nu(dx) < \infty$ and $\xi_1, \xi_2 \in [0, \infty)$ in case of $\int_{\|x\|\leq 1}\|x\|\nu(dx) < \infty$, $\int_{\|x\|>1}\|x\|\nu(dx) = \infty$ and $\xi_1, \xi_2 \in (0, \infty]$ in case of $\int_{\|x\|\leq 1}\|x\|\nu(dx) =$

∞, $\int_{\|x\|>1}\|x\|\nu(dx)<\infty$ and $\xi_1,\xi_2\in(0,\infty)$ in case of $\int_{\|x\|\leq 1}\|x\|\nu(dx)=\infty$, $\int_{\|x\|>1}\|x\|\nu(dx)=\infty$.
(iv) The large jump parts of the component processes are defined by

$$^{CP_i}L_t(\omega)[\xi_1,\xi_2]:=\int_0^t\int_{\xi_1<|x_i|\leq\xi_2}x\,J((ds,dx),\omega),\quad i\in\{1,\cdots,d\}.$$

Please note the difference to the component of the large jump part that is denoted by $^{CP}L_{t,i}[\xi_1,\xi_2]$. The quantities $^{sJ_i}L_t(\omega)[\xi_1,\xi_2]$ and $\gamma_t^{c_i}[\xi_1,\xi_2]$ are defined analogously.
(v) The increments of the corresponding processes are denoted by

$$\{^{CP}X_{h,k}[\xi_1,\xi_2],\ k=\{1,\cdots,d\}\},\qquad\{^{sJ}X_{h,k}[\xi_1,\xi_2],\ k=\{1,\cdots,d\}\},$$
$$\{^{CP_i}X_{h,k}[\xi_1,\xi_2],\ k=\{1,\cdots,d\}\},\qquad\{^{sJ_i}X_{h,k}[\xi_1,\xi_2],\ k=\{1,\cdots,d\}\}.$$

Lemma 3. *Let $\alpha\in(1/2,1]$ in (2) and κ_1, $\kappa_2>0$. We define*

$$M_2^{(n)}(\xi_n,\xi_n',\chi,i):=\{\exists z\in\mathbb{R}\backslash(-\xi_n,\xi_n):N_{T,i}(z-\kappa_1\xi_n',z+\kappa_1\xi_n')\geq\kappa_2 n^{\chi}\}.$$

(i) Let $0<\rho<\alpha/2-1/4$ and $\xi_n':=n^{-\rho}$. Then, there exists a sequence $\xi_n\to 0$ such that, as $n\to\infty$,

$$\frac{\xi_n'}{\xi_n}\to 0,\quad\frac{\sqrt{n}h\nu\left(\{x\in\mathbb{R}^d:\|x\|\geq\xi_n\}\right)}{{\xi_n'}^2}\to 0,\quad\mathbb{P}\left(M_2^{(n)}(\xi_n,\xi_n',1-\alpha,i)\right)\to 0,\quad\forall i\in\{1,...,d\}.$$

(ii) Let (3) be satisfied, choose

$$r\in\begin{cases}[0\vee 1/2-2\alpha/3,\alpha/3) & if\ \alpha\leq 3/5\\ [0\vee 1/2-2\alpha/3,(1-\alpha)/2] & if\ \alpha>3/5\end{cases}$$

and define $\xi_n:=n^{(3r-\alpha)/6}log(n)$ and $\xi_n':=n^{(r-\alpha)/2}log(n)$. Let $\theta\in[r,\alpha/3)$, then, as $n\to\infty$,

$$\mathbb{P}\left(M_2^{(n)}(\xi_n,\xi_n',1-\alpha-\theta,i)\right)\longrightarrow 0,\quad\forall i\in\{1,...,d\}.$$

Proof. We set w.l.o.g. $\kappa_1=\kappa_2=1$ and assume $n^{\chi}\in\mathbb{N}$.
Ad (i), for all positive zero sequences ξ_n and ξ_n' satisfying $\xi_n'/\xi_n\to 0$ we define

$$A_m^{(n)}:=\{N_{T,i}\left((2(m-1)\,\xi_n',2(m+1)\,\xi_n']\setminus(-\xi_n/2,\xi_n/2)^c\right)\geq n^{\chi}\}\quad\text{and obtain}\quad M_2^{(n)}\subset\bigcup_{m\in\mathbb{Z}}A_m^{(n)}.$$

We have

$$\begin{aligned}\mathbb{P}\left(A_m^{(n)}\right)&=\sum_{k=n^{\chi}}^{\infty}\exp\left(-\widetilde{\nu}_i\left((2(m-1)\,\xi_n',2(m+1)\,\xi_n']\setminus(-\xi_n/2,\xi_n/2)^c\right)n^{1-\alpha}\right)\cdot\\&\quad\cdot\frac{\left(\widetilde{\nu}_i\left((2(m-1)\,\xi_n',2(m+1)\,\xi_n']\setminus(-\xi_n/2,\xi_n/2)^c\right)n^{1-\alpha}\right)^k}{k!}\\&\leq\frac{\left(\widetilde{\nu}_i\left((2(m-1)\,\xi_n',2(m+1)\,\xi_n']\setminus(-\xi_n/2,\xi_n/2)^c\right)n^{1-\alpha}\right)^{n^{\chi}}}{n^{\chi}!}.\end{aligned}$$

Since (1), there exits a $\kappa_{\xi_n}>0$ such that

$$\begin{aligned}&\widetilde{\nu}_i\left((2(m-1)\,\xi_n',2(m+1)\,\xi_n']\setminus(-\xi_n/2,\xi_n/2)^c\right)n^{1-\alpha}=n^{1-\alpha}\int_{x\in(2(m-1)\,\xi_n',2(m+1)\,\xi_n']\setminus(-\xi_n/2,\xi_n/2)^c}f_i(x)\,dx\\&\leq 4\,n^{1-\alpha}\max_{(2(m-1)\,\xi_n',2(m+1)\,\xi_n']\setminus(-\xi_n/2,\xi_n/2)^c}f_i(x)\cdot\xi_n'=:\kappa_{m,\xi_n}\,n^{1-\alpha-\rho}\leq\kappa_{\xi_n}\,n^{1-\alpha-\rho}.\end{aligned}$$

Then,

$$\mathbb{P}\left(M_2^{(n)}\right) \leq \sum_{m\in\mathbb{Z}} \mathbb{P}(A_m^{(n)}) \leq \sum_{m\in\mathbb{Z}} \frac{\left(\widetilde{\nu}_i\left(\,(2(m-1)\,\xi_n', 2(m+1)\,\xi_n'] \setminus (-\xi_n/2, \xi_n/2)^c\,\right) n^{1-\alpha}\right)^{n^\chi}}{n^\chi!}$$

$$= \sum_{m\in\mathbb{Z}} \frac{\left(\widetilde{\nu}_i\left(\,(2(m-1)\,\xi_n', 2(m+1)\,\xi_n'] \setminus (-\xi_n/2, \xi_n/2)^c\,\right) n^{1-\alpha}\right)^{n^\chi-1}}{n^\chi!} \widetilde{\nu}_i\left(\,(2(m-1)\,\xi_n', 2(m+1)\,\xi_n'] \setminus (-\xi_n/2, \xi_n/2)^c\,\right) n^{1-\alpha}$$

$$\leq \sum_{m\in\mathbb{Z}} \frac{\left(\kappa_{\xi_n}\, n^{1-\alpha-\rho}\right)^{n^\chi-1}}{n^\chi!} \widetilde{\nu}_i\left(\,(2(m-1)\,\xi_n', 2(m+1)\,\xi_n'] \setminus (-\xi_n/2, \xi_n/2)^c\,\right) n^{1-\alpha} \leq \frac{\left(\kappa_{\xi_n}\, n^{1-\alpha-\rho}\right)^{n^\chi-1}}{n^\chi!}\, n^{1-\alpha}\, 2\,\widetilde{\nu}_i((-\xi_n/2, \xi_n/2)^c).$$

For each $i \in \{1, \cdots, d\}$ we can find a sequence ξ_n such that the last term tends to zero by Stirling's formula. Obviously we can also find a sequence ξ_n such that $\sqrt{n}h\nu\left(\{x : \|x\| \geq \xi_n\}\right)/\xi_n'^2 \to 0$. Then we choose the slowest of all sequences.
Ad (ii), since (3), we have analogously to (i)

$$\widetilde{\nu}_i\left(\,(2(m-1)\,\xi_n', 2(m+1)\,\xi_n'] \setminus (-\xi_n/2, \xi_n/2)^c\,\right) n^{1-\alpha} = n^{1-\alpha} \int_{x\in(2(m-1)\,\xi_n', 2(m+1)\,\xi_n'] \setminus (-\xi_n/2, \xi_n/2)^c} f_i(x)\,dx$$

$$\leq n^{1-\alpha} \int_{x\in(2(m-1)\,\xi_n', 2(m+1)\,\xi_n'] \setminus (-\xi_n/2, \xi_n/2)^c} 1/(\xi_n/2)^3\,dx \leq n^{1-\alpha}\, 4\,\xi_n'/(\xi_n/2)^3.$$

Then, analogously to (i),

$$\mathbb{P}\left(M_2^{(n)}\right) \leq \frac{\left(n^{1-\alpha}\, 4\,\xi_n'/(\xi_n/2)^3\right)^{n^{1-\alpha-r}}}{(n^{1-\alpha-r})!}\, n^{1-\alpha}\, 2\,\widetilde{\nu}_i((-\xi_n/2, \xi_n/2)^c)$$

tends to zero by Stirling's formula. □

Lemma 4. *(i) Let* $1/2 < \alpha \leq 1$ *in (2),* $i_1, i_2 \in \{1, ..., d\}$ *and* $\mathbb{E}[|L_1|^2] < \infty$*. Then, as* $n \to \infty$

$$\frac{1}{T}\sum_{k=1}^{n} |{}^J X_{h,k,i_1}||{}^C X_{h,k,i_2}|\mathbb{1}(X_{h,k} \notin B_n) \xrightarrow{P} 0.$$

(ii) Let $1/2 < \alpha \leq 1$ *in (2),* $i_1, i_2, i_3 \in \{1, ..., d\}$*,* $r \in [0, \alpha/2)$ *and* $j_1,\ j_2,\ j_3 \in \mathbb{N}$ *satisfying* $j_1 + j_2 = 1,\ j_3 \geq 2$ *or* $j_1 + j_2 \geq 2,\ j_3 \geq 1$*. If* $\mathbb{E}[|L_1|^{2j_1}] < \infty$ *and* $\mathbb{E}[|L_1|^{2j_2}] < \infty$*, then, as* $n \to \infty$*,*

$$n^r \frac{1}{T}\sum_{k=1}^{n} |{}^J X_{h,k,i_1}|^{j_1}|{}^J X_{h,k,i_2}|^{j_2}|{}^C X_{h,k,i_3}|^{j_3} \xrightarrow{P} 0.$$

Proof. Ad (i), by Cauchy Schwartz inequality we obtain an upper bound

$$\sqrt{\frac{1}{T}\sum_{k=1}^{n} {}^C X_{h,k,i_2}^2 \mathbb{1}(X_{h,k} \notin B_n)}\ \sqrt{\frac{1}{T}\sum_{k=1}^{n} {}^J X_{h,k,i_1}^2}.$$

The first factor tends to zero in probability by Lemma 5. The expectation of the second term is given by Lemma 12 and is finite and independent of n. Thus, the second term is stochastically bounded.
Ad (ii), we consider the expectation and apply Lemma 12. The cases $j_1 + j_2 = 1,\ j_3 \geq 2,\ j_1 \geq 2,\ j_2 = 0,\ j_3 \geq 1,\ j_1 = 0,\ j_2 \geq 2,\ j_3 \geq 1$ follow directly. For $j_1 \geq 1,\ j_2 \geq 1\ j_3 \geq 1$ we can use Hölder's inequality and obtain

$$\mathbb{E}\left[|{}^J X_{h,1,i_1}^{j_1}||{}^J X_{h,1,i_2}^{j_2}|\right] \leq \sqrt{\mathbb{E}\left[{}^J X_{h,1,i_1}^{2j_1}\right]\mathbb{E}\left[{}^J X_{h,1,i_2}^{2j_2}\right]} = O(h).$$

□

Lemma 5. *Let $1/2 < \alpha \le 1$ in (2) and $i \in \{1, \cdots, d\}$, then, as $n \to \infty$*

$$\frac{1}{T}\sum_{k=1}^{n} {}^{C}X_{h,k,i}^{2}\mathbb{1}(X_{h,k} \notin B_n) \ \xrightarrow{P}\ 0.$$

Proof. We have $\boldsymbol{B}(C,\beta') \subseteq B_n$. We choose $\widetilde{\beta'} \in (1,\beta')$, then

$$\begin{aligned}&\frac{1}{T}\sum_{k=1}^{n} {}^{C}X_{h,k,i}^{2}\,\mathbb{1}(X_{h,k} \notin B_n)\\&= \frac{1}{T}\sum_{k=1}^{n} {}^{C}X_{h,k,i}^{2}\,\mathbb{1}\left(X_{h,k} \notin B_n, {}^{C}X_{h,k} \in \boldsymbol{B}_n(C,\widetilde{\beta'})\right) + \frac{1}{T}\sum_{k=1}^{n} {}^{C}X_{h,k,i}^{2}\,\mathbb{1}\left(X_{h,k} \notin B_n, {}^{C}X_{h,k} \notin \boldsymbol{B}_n(C,\widetilde{\beta'})\right)\end{aligned}$$

The probability that the latter term is different from zero tends to zero by Proposition 1. For the first term there exists $\delta > 0$ such that an upper bound is given by

$$\frac{1}{T}\sum_{k=1}^{n} {}^{C}X_{h,k,i}^{2}\,\mathbb{1}\left(\|{}^{J}X_{h,k}\| > \delta b_n\right).$$

We consider the expectation and obtain

$$\begin{aligned}&n^{1-(1-\alpha)}\,\mathbb{E}\left[{}^{C}X_{h,k,i}^{2}\right]\mathbb{P}\left(\|{}^{J}X_{h,k}\| > \delta b_n\right) \le n^{1-(1-\alpha)}\,\mathbb{E}\left[{}^{C}X_{h,k,i}^{2}\right]\left(\mathbb{P}\left(\|{}^{sJ}X_{h,k}\| > \delta b_n\right) + \mathbb{P}\left(\|{}^{CP}X_{h,k}[1]\| > 0\right)\right)\\&\le n^{1-(1-\alpha)}\,\mathbb{E}\left[{}^{C}X_{h,k,i}^{2}\right]\left(\left(\sum_{i=1}^{d}\frac{\mathrm{Var}\left({}^{sJ}X_{h,k,i}\right)}{\delta^2 b_n^2}\right) + (1-\exp(-\nu(\{x : \|x\| > 1\})h))\right).\end{aligned}$$

The term converges to zero by Lemma 12 and by $1 - \exp(-\nu(\{x : \|x\| > 1\})h) = O(h)$. □

Lemma 6. *Let $1/2 < \alpha \le 1$ in (2) and $i \in \{1, \cdots, d\}$, then, as $n \to \infty$*

$$\frac{1}{T}\sum_{k=1}^{n} {}^{J}X_{h,k,i}^{2}\mathbb{1}(X_{h,k} \in B_n) \ \xrightarrow{P}\ 0.$$

Proof. We have

$$\frac{1}{T}\sum_{k=1}^{n} {}^{J}X_{h,k,i}^{2}\mathbb{1}(X_{h,k} \in B_n) \le \frac{1}{T}\sum_{k=1}^{n} {}^{J}X_{h,k,i}^{2}\mathbb{1}(X_{h,k} \in B_n, {}^{C}X_{h,k} \notin B_n) + \frac{1}{T}\sum_{k=1}^{n} {}^{J}X_{h,k,i}^{2}\mathbb{1}(X_{h,k} \in B_n, {}^{C}X_{h,k} \in B_n)$$

The probability that the first term is different from zero converge to zero by Proposition 1. An upper bound for the second term is given by

$$\frac{1}{T}\sum_{k=1}^{n} {}^{J}X_{h,k,i}^{2}\mathbb{1}({}^{J}X_{h,k} \in 2B_n) \le \frac{1}{T}\sum_{k=1}^{n} {}^{sJ}X_{h,k,i}^{2}\mathbb{1}({}^{sJ}X_{h,k} \in 2B_n) + \frac{1}{T}\sum_{k=1}^{n} 4(\beta'' b_n)^2\mathbb{1}(\|{}^{CP}X_{h,k}[1]\| > 0).$$

The first term is considered in Lemma 7 and the expectation of the second term is equal to

$$4\frac{n}{T}(\beta'' b_n)^2\left(1 - \exp\left(-\frac{\nu(\{x : \|x\| > 1\})}{n^{\alpha}}\right)\right) \to 0.$$

□

Lemma 7. *Let $1/2 < \alpha \le 1$ in (2), $i \in \{1, ..., d\}$ and let δ_n be a positive zero sequence. Then, as $n \to \infty$,*

$$\frac{1}{T}\sum_{k=1}^{n} {}^{sJ}X_{h,k,i}^{2}\mathbb{1}(\|{}^{sJ}X_{h,k}\| < \delta_n) \xrightarrow{L_1} 0.$$

Proof. We can find a sequence $\delta'_n > 0$ satisfying $\delta'_n \to 0$, $\delta_n^2/\delta'_n \to 0$ and $h\left(\int_{\sqrt{\delta'_n}\le\|x\|\le 1}\|x\|\nu(dx)\right)^2 \to 0$. Then the left hand side of (i) is equal to

$$\frac{1}{T}\sum_{k=1}^{n} {}^{sJ}X_{h,k,i}^2 \mathbb{1}\Big(\|{}^{sJ}X_{h,k}\| < \delta_n, \sum_{(k-1)h<t\le kh} \|\Delta^{sJ}L_t\|^2 \le \delta'_n\Big) \tag{A.1}$$

$$+\frac{1}{T}\sum_{k=1}^{n} {}^{sJ}X_{h,k,i}^2 \mathbb{1}\Big(\|{}^{sJ}X_{h,k}\| < \delta_n, \sum_{(k-1)h<t\le kh} \|\Delta^{sJ}L_t\|^2 > \delta'_n\Big). \tag{A.2}$$

For the expectation of (A.1) an upper bound is given by

$$\frac{n}{T}\mathbb{E}\left[{}^{sJ}X_{h,1,i}^2 \mathbb{1}(\|\Delta^{sJ}L_t\| \le \sqrt{\delta'_n}\,,\ \forall t\ \in ((k-1)h, kh])\right]$$
$$\le \frac{2n}{T}\mathbb{E}\left[{}^{sJ}X_{h,1,i}^2[0,\ \sqrt{\delta'_n}]\right] + \frac{2n}{T}\mathbb{E}\left[{}^{sJ}X_{h,1,i}^2[\ \sqrt{\delta'_n}, 1]\ \mathbb{1}(\|\Delta^{sJ}L_t\| \le \sqrt{\delta'_n}\,,\ \forall t\ \in ((k-1)h, kh])\right].$$

By Lemma 12, the first term tends to zero, since

$$\frac{n}{T}\mathbb{E}\left[{}^{sJ}X_{h,1,i}^2[0,\ \sqrt{\delta'_n}]\right] = \frac{n}{T}h\mathbb{E}\left[{}^{sJ}X_{1,1,i}^2[0,\ \sqrt{\delta'_n}]\right] = \int_{\|x\|\le\sqrt{\delta'_n}} x_i^2\nu(dx) \longrightarrow 0.$$

The second term consists only of the compensating drift that is equal to

$$\frac{n}{T}h^2\left(\int_{\sqrt{\delta'_n}\le\|x\|\le 1} x_i\nu(dx)\right)^2 \le \frac{n}{T}h^2\left(\int_{\sqrt{\delta'_n}\le\|x\|\le 1} \|x\|\nu(dx)\right)^2 \longrightarrow 0.$$

For the expectation of (A.2) an upper bound is given by

$$\mathbb{E}\left[\frac{1}{T}\sum_{k=1}^{n}\delta_n^2\mathbb{1}\left(\sum_{(k-1)h<t\le kh}\|\Delta^{sJ}L_t\|^2 > \delta'_n\right)\right]$$
$$= \frac{n}{T}\,\delta_n^2\,\mathbb{P}\left(\sum_{0<t\le h}\|\Delta^{sJ}L_t\|^2 > \delta'_n\right) \le \frac{n}{T}\delta_n^2\frac{\mathbb{E}\left[\sum_{0<t\le h}\|\Delta^{sJ}L_t\|^2\right]}{\delta'_n} = \frac{\delta_n^2 n}{\delta'_n T}O(h),$$

where we have used Markov's Inequality and thus, the convergence follows from the assumption on δ'_n. □

Lemma 8. *(i) Let* $1/2 < \alpha \le 1$ *in (2),* $l \in \mathbb{N}$ *and* $\mathbb{E}\left[|L_1|^{4l}\right] < \infty$. *Then, as* $n \to \infty$

$$\sup_{z\in\mathbb{R}^d}\left|\frac{1}{T}\sum_{k=1}^{n}\left({}^{J}X_{h,k}^T A\,{}^{J}X_{h,k}\right)^l \mathbb{1}(X_{h,k}\notin B_n,\ X_{h,k}\le z) - \frac{1}{T}\sum_{0\le t\le T}\left(\Delta L_t^T A\Delta L_t\right)^l \mathbb{1}(\Delta L_t \le z)\right| \xrightarrow{P} 0.$$

(ii) Let $1/2 < \alpha \le 1$ *in (2), assume (3), choose*

$$r \in \begin{cases} [0\vee 1/2-2\alpha/3, \alpha/3) & if\ \alpha \le 3/5 \\ [0\vee 1/2-2\alpha/3, (1-\alpha)/2] & if\ \alpha > 3/5 \end{cases},$$

define $l := \lfloor 3r/(\alpha-3r)\rfloor + 2$ *and* $p := \lfloor l\alpha/(\alpha-3r)+1\rfloor/l$ *and require* $\mathbb{E}[|L_1|^{2l(2\vee p)}] < \infty$. *Then, as* $n \to \infty$,

$$n^r\sup_{z\in\mathbb{R}^d}\left|\frac{1}{T}\sum_{k=1}^{n}\left({}^{J}X_{h,k}^T A\,{}^{J}X_{h,k}\right)^l \mathbb{1}(X_{h,k}\notin B_n,\ X_{h,k}\le z) - \frac{1}{T}\sum_{0\le t\le T}\left(\Delta L_t^T A\Delta L_t\right)^l \mathbb{1}(\Delta L_t \le z)\right| \xrightarrow{P} 0.$$

Proof. We set $r = 0$ in (i), in (ii) r is defined. We use the notation

$$u_k := \left({}^J X_{h,k}^T A\, {}^J X_{h,k}\right)^l, \quad k = 1, \cdots, n, \qquad v := \sum_{0\le t\le T} \left(\Delta L_t^T A \Delta L_t\right)^l \mathbb{1}\left(\Delta L_t \le z\right).$$

We decompose the term in the indicator function and keep in mind ${}^J X_{h,k} = {}^{CP}X_{h,k}[\xi_n] + \gamma^c_{h,k}[\xi, 1] + {}^{sJ}X_{h,k}[\xi_n]$ and

$$B_n \subseteq \boldsymbol{B}_n(I, \beta'') = \{x \in \mathbb{R}^d : x^T x \le \beta'' b_n^2\}, \quad \text{where} \quad b_n = \sqrt{2\,h\,\log n}.$$

Furthermore we define $\vec{b}_n := (1, \ldots, 1)^T b_n$. In (i) and (ii), respectively, we use the sequences $\{\xi_n, n \in \mathbb{N}\}$ and $\{\xi'_n, n \in \mathbb{N}\}$ given in Lemma 3(i) and (ii), respectively. It can easily verified that $\{\xi_n, n \in \mathbb{N}\}$ also satisfies the condition of Lemma 2(i) and (ii), respectively. We have

$$\begin{aligned}
&\left|\sum_{k=1}^n u_k \mathbb{1}\left(X_{h,k} \notin B_n,\, X_{h,k} \le z\right) - v\right| \le \left|\sum_{k=1}^n u_k \mathbb{1}\left({}^J X_{h,k} \le z\right) - v\right| + \sum_{k=1}^n |u_k| \mathbb{1}\left({}^J X_{h,k} \in (z - \sqrt{\beta''}\vec{b}_n, z + \sqrt{\beta''}\vec{b}_n)\right) \\
&+ \sum_{k=1}^n |u_k| \mathbb{1}\left({}^J X_{h,k} \in (-2\sqrt{\beta''}\vec{b}_n,\, \sqrt{\beta''}\vec{b}_n)\right) + 2\sum_{k=1}^n |u_k| \mathbb{1}\left({}^C X_{h,k} \notin B_n\right) \\
&\le \left|\sum_{k=1}^n u_k \mathbb{1}\left({}^J X_{h,k} \le z,\, \|{}^{sJ}X_{h,k}[\xi_n]\| < \xi'_n/2,\, \left\|\gamma^c_{h,k}[\xi_n, 1]\right\| < \xi'_n/2\right) - v\right| \\
&+ \sum_{k=1}^n |u_k| \mathbb{1}\left({}^J X_{h,k} \in (z - \sqrt{\beta''}\vec{b}_n, z + \sqrt{\beta''}\vec{b}_n),\, \|{}^{sJ}X_{h,k}[\xi_n]\| < \xi'_n/2, \left\|\gamma^c_{h,k}[\xi_n, 1]\right\| < \xi'_n/2\right) \\
&+ \sum_{k=1}^n |u_k| \mathbb{1}\left({}^J X_{h,k} \in (-2\sqrt{\beta''}\vec{b}_n,\, \sqrt{\beta''}\vec{b}_n),\, \|{}^{sJ}X_{h,k}[\xi_n]\| < \xi'_n/2, \left\|\gamma^c_{h,k}[\xi_n, 1]\right\| < \xi'_n/2\right) \\
&+ \sum_{k=1}^n |u_k| \mathbb{1}\left(\left\|\gamma^c_{h,k}[\xi_n, 1]\right\| \ge \xi'_n/2\right) + \mathbb{1}\left(\|{}^{sJ}X_{h,k}[\xi_n]\| \ge \xi'_n/2\right) + 2\sum_{k=1}^n |u_k|\, \mathbb{1}\left({}^C X_{h,k} \notin B_n\right) \\
&\le \left|\sum_{k=1}^n u_k \mathbb{1}\left({}^{CP}X_{h,k}[\xi_n] \le z,\, {}^J X_{h,k} \le z,\, \|{}^{sJ}X_{h,k}[\xi_n]\| < \xi'_n/2\right) \mathbb{1}\left(\left\|\gamma^c_{h,k}[\xi_n, 1]\right\| < \xi'/2\right) - v\right| \\
&+ \sum_{k=1}^n |u_k| \mathbb{1}\left({}^{CP}X_{h,k}[\xi_n] > z,\, {}^J X_{h,k} \le z,\, \|{}^{sJ}X_{h,k}[\xi_n]\| < \xi'_n/2, \left\|\gamma^c_{h,k}[\xi_n, 1]\right\| < \xi'_n/2\right) \\
&+ \sum_{k=1}^n |u_k| \mathbb{1}\left({}^{CP}X_{h,k}[\xi_n] \in (z - 2\xi'_n, z + 2\xi'_n)\right) + \sum_{k=1}^n |u_k| \mathbb{1}\left(\left\|{}^{CP}X_{h,k}[\xi_n]\right\| \le 2\xi'_n\right) \\
&+ \sum_{k=1}^n |u_k| \mathbb{1}\left(\left\|\gamma^c_{h,k}[\xi_n, 1]\right\| \ge \xi'_n/2\right) + \sum_{k=1}^n |u_k| \mathbb{1}\left(\|{}^{sJ}X_{h,k}[\xi_n]\| \ge \xi'_n/2\right) + 2\sum_{k=1}^n |u_k| \mathbb{1}\left({}^C X_{h,k} \notin B_n\right) \\
&\le \left|\sum_{k=1}^n u_k \mathbb{1}\left({}^{CP}X_{h,k}[\xi_n] \le z,\, \|{}^{sJ}X_{h,k}[\xi_n]\| < \xi'_n/2, \left\|\gamma^c_{h,k}[\xi_n, 1]\right\| < \xi'_n/2\right) - v\right| \\
&+ \sum_{k=1}^n |u_k| \mathbb{1}\left({}^{CP}X_{h,k}[\xi_n] \le z,\, {}^J X_{h,k} > z,\, \|{}^{sJ}X_{h,k}[\xi_n]\| < \xi'_n/2, \left\|\gamma^c_{h,k}[\xi_n, 1]\right\| < \xi'_n/2\right) \\
&+ \sum_{k=1}^n |u_k| \mathbb{1}\left({}^{CP}X_{h,k}[\xi_n] \in (z, z + \xi'_n)\right) + \sum_{k=1}^n |u_k| \mathbb{1}\left({}^{CP}X_{h,k}[\xi_n] \in (z - 2\xi'_n, z + 2\xi'_n)\right) + \sum_{k=1}^n |u_k| \mathbb{1}\left(\left\|{}^{CP}X_{h,k}[\xi_n]\right\| \le 2\xi'_n\right) \\
&+ \sum_{k=1}^n |u_k| \mathbb{1}\left(\left\|\gamma^c_{h,k}[\xi_n, 1]\right\| \ge \xi'_n/2\right) + \sum_{k=1}^n |u_k| \mathbb{1}\left(\|{}^{sJ}X_{h,k}[\xi_n]\| \ge \xi'_n/2\right) + 2\sum_{k=1}^n |u_k| \mathbb{1}\left({}^C X_{h,k} \notin B_n\right) \\
&\le \left|\sum_{k=1}^n u_k \mathbb{1}\left({}^{CP}X_{h,k}[\xi_n] \le z\right) - v\right| + 2\sum_{k=1}^n |u_k| \mathbb{1}\left({}^{CP}X_{h,k}[\xi_n] \in (z - 2\xi'_n, z + 2\xi'_n)\right)
\end{aligned}$$

$$+\sum_{k=1}^{n}|u_k|\mathbb{1}\left(\left\|{}^{CP}X_{h,k}[\xi_n]\right\|\leq 2\xi_n'\right)+2\sum_{k=1}^{n}|u_k|\mathbb{1}\left(\left\|\gamma_{h,k}^{c}[\xi_n,1]\right\|\geq \xi_n'/2\right)$$
$$+2\sum_{k=1}^{n}|u_k|\mathbb{1}\left(\|{}^{sJ}X_{h,k}[\xi_n]\|\geq \xi_n'/2\right)+2\sum_{k=1}^{n}|u_k|\mathbb{1}\left({}^{C}X_{h,k}\notin B_n\right).$$

By inserting this decomposition in our statement it remains to show

$$1.)\ n^r\sup_{z\in\mathbb{R}^d}\left|\frac{1}{T}\sum_{k=1}^{n}\left({}^{J}X_{h,k}^{T}A\,{}^{J}X_{h,k}\right)^l\mathbb{1}({}^{CP}X_{h,k}[\xi_n]\leq z)-\frac{1}{T}\sum_{0\leq t\leq T}\left(\Delta L_t^T A\Delta L_t\right)^l\mathbb{1}(\Delta L_t\leq z)\right|\xrightarrow{p}0,$$

$$2.)\ n^r\sup_{z\in\mathbb{R}^d}\left|\frac{1}{T}\sum_{k=1}^{n}\left|{}^{J}X_{h,k}^{T}A\,{}^{J}X_{h,k}\right|^l\left(2\mathbb{1}({}^{CP}X_{h,k}[\xi_n]\in(z-2\xi_n',z+2\xi_n'))+\mathbb{1}(\|{}^{CP}X_{h,k}[\xi_n]\|\leq 2\xi_n')\right)\right|\xrightarrow{p}0,$$

$$3.)\ n^r\frac{1}{T}\sum_{k=1}^{n}\left|{}^{J}X_{h,k}^{T}A\,{}^{J}X_{h,k}\right|^l\mathbb{1}\left(\|\gamma^c[\xi_n,1]\|\geq\xi_n'/2\right)\xrightarrow{p}0,$$

$$4.)\ n^r\frac{1}{T}\sum_{k=1}^{n}\left|{}^{J}X_{h,k}^{T}A\,{}^{J}X_{h,k}\right|^l\mathbb{1}(\|{}^{sJ}X_{h,k}[\xi_n]\|\geq\xi_n'/2)\xrightarrow{p}0,$$

$$5.)\ n^r\frac{1}{T}\sum_{k=1}^{n}\left|{}^{J}X_{h,k}^{T}A\,{}^{J}X_{h,k}\right|^l\mathbb{1}({}^{C}X_{h,k}\notin B_n)\xrightarrow{p}0.$$

Ad 1.), we have

$$({}^{J}X_{h,k}^{T}\,A\,{}^{J}X_{h,k})^l=\Big({}^{CP}X_{h,k}^{T}[\xi_n]\,A\,{}^{CP}X_{h,k}[\xi_n]+{}^{sJ}X_{h,k}^{T}[\xi_n]\,A\,{}^{sJ}X_{h,k}[\xi_n]$$
$$+\gamma_h^c[\xi_n,1]^T\,A\ \gamma_h^c[\xi_n,1]+{}^{CP}X_{h,k}^{T}[\xi_n]\,A\,{}^{sJ}X_{h,k}[\xi_n]+{}^{sJ}X_{h,k}^{T}[\xi_n]\,A\,{}^{CP}X_{h,k}[\xi_n]$$
$$+{}^{CP}X_{h,k}^{T}[\xi_n]\,A\ \gamma_h^c[\xi_n,1]+\gamma_h^c[\xi_n,1]^T\,A\,{}^{CP}X_{h,k}[\xi_n]+{}^{sJ}X_{h,k}^{T}[\xi_n]\,A\ \gamma_h^c[\xi_n,1]+\gamma_h^c[\xi_n,1]^T\,A\,{}^{sJ}X_{h,k}[\xi_n]\Big)^l.$$

We use the decomposition in 1.) and use Lemma 10(iii) and Lemma 11. Then there exists a $\kappa>0$ such that the following upper bound holds true.

$$n^r\sup_{z\in\mathbb{R}^d}\left|\frac{1}{T}\sum_{k=1}^{n}\left({}^{CP}X_{h,k}^{T}[\xi_n]A\,{}^{CP}X_{h,k}[\xi_n]\right)^l\mathbb{1}({}^{CP}X_{h,k}[\xi_n]\leq z)-\frac{1}{T}\sum_{0\leq t\leq T}\left(\Delta L_t^T A\Delta L_t\right)^l\mathbb{1}(\Delta L_t\leq z)\right|\tag{A.3}$$

$$+\kappa\,n^r\frac{1}{T}\sum_{m=1}^{l}\sum_{k=1}^{n}|{}^{CP}X_{h,k}[\xi_n]|^{2l-2m}(|{}^{sJ}X_{h,k}[\xi_n]||{}^{sJ}X_{h,k}[\xi_n]|)^m\tag{A.4}$$

$$+\kappa\,n^r\frac{1}{T}\sum_{m=1}^{l}\sum_{k=1}^{n}|{}^{CP}X_{h,k}[\xi_n]|^{2l-2m}(|{}^{CP}X_{h,k}[\xi_n]||{}^{sJ}X_{h,k}[\xi_n]|)^m\tag{A.5}$$

$$+\kappa\,n^r\frac{1}{T}\sum_{m=1}^{l}\sum_{k=1}^{n}|{}^{CP}X_{h,k}[\xi_n]|^{2l-2m}(|\gamma_{h,k}^c[\xi_n,1]||\gamma_{h,k}^c[\xi_n,1]|)^m\tag{A.6}$$

$$+\kappa\,n^r\frac{1}{T}\sum_{m=1}^{l}\sum_{k=1}^{n}|{}^{CP}X_{h,k}[\xi_n]|^{2l-2m}(|{}^{CP}X_{h,k}^{T}[\xi_n]||\gamma_{h,k}^c[\xi_n,1]|)^m\tag{A.7}$$

$$+\kappa\,n^r\frac{1}{T}\sum_{m=1}^{l}\sum_{k=1}^{n}|{}^{CP}X_{h,k}[\xi_n]|^{2l-2m}(|{}^{sJ}X_{h,k}^{T}[\xi_n]||\gamma_{h,k}^c[\xi_n,1]|)^m.\tag{A.8}$$

Ad (A.3), by Lemma 11 there exists a $\kappa > 0$ such that

$$n^r \mathbb{E}\left[\frac{1}{T}\sum_{0\le t\le T}(\Delta L_t^T A \Delta L_t)^l \mathbb{1}(\|\Delta L_t\| \le \xi_n)\right]$$
$$\le \kappa\, n^r \mathbb{E}\left[\frac{1}{T}\sum_{0\le t\le T}|\Delta L_t|^{2l}\mathbb{1}(\|\Delta L_t\| \le \xi_n)\right] = \kappa\, n^r \int_{\|x\|\le\xi_n}|x|^{2l}\nu(dx) \le \kappa\, n^r \sum_{i=1}^d \int_{|x_i|\le\xi_n}|x_i|^{2l}\nu_i(dx_i).$$

Thus, in (i) this term tends to zero, because $r = 0$ and in (ii) we use (3) and obtain an upper bound

$$n^r \xi_n^2 \int_{-\xi_n}^{\xi_n} x^2 \nu(dx) \to 0.$$

Finally, by Lemma 2 we obtain, for all $\varepsilon > 0$,

$$\mathbb{P}\left(n^r \sup_{z\in\mathbb{R}^d}\left|\frac{1}{T}\sum_{k=1}^n \left({}^{CP}X_{h,k}^T[\xi_n]A^{CP}X_{h,k}^T[\xi_n]\right)^l \mathbb{1}({}^{CP}X_{h,k}[\xi_n] \le z) - \frac{1}{T}\sum_{0\le t\le T}\left(\Delta L_t^T A \Delta L_t\right)^l \mathbb{1}(\Delta L_t \le z,\, \|\Delta L_t\| > \xi_n)\right| > \varepsilon\right)$$
$$\le \mathbb{P}(M_2^{(n)}(\xi_n)) \to 0.$$

The terms (A.4), (A.5), (A.6) and (A.7) are considered in Lemma 9. The term in (A.8) is bounded by (A.4) and (A.6). Ad 2.), obviously the following upper bound is given by

$$2\, n^r \sup_{z\in\mathbb{R}^d}\left|\frac{1}{T}\sum_{k=1}^n |{}^J X_{h,k}^T A^J X_{h,k}|^l \mathbb{1}({}^{CP}X_{h,k}[\xi_n] \in (z - 2\xi_n', z + 2\xi_n'), {}^{CP}X_{h,k}[\xi_n] \neq 0)\right| + 2n^r \frac{1}{T}\sum_{k=1}^n |{}^{sJ}X_{h,k}^T A^{sJ}X_{h,k}|^l.$$

By Lemma 11, for the second term, an upper bound is given by

$$2n^r \frac{1}{T}\sum_{k=1}^n |{}^{sJ}X_{h,k}|^{2l}.$$

This term is considered in Lemma 9. For the first term we use Hölder's inequality; in (i) we set $p = 2$ and in (ii) we use the given p. Then we obtain

$$\left(\frac{1}{T}\sum_{k=1}^n |{}^J X_{h,k}^T A^J X_{h,k}|^{lp}\right)^{1/p} \sup_{z\in\mathbb{R}^d}\left(n^{r/(1-1/p)}\frac{1}{T}\sum_{k=1}^n \mathbb{1}({}^{CP}X_{h,k}[\xi_n] \in (z - 2\xi_n', z + 2\xi_n'), {}^{CP}X_{h,k}[\xi_n] \neq 0)\right)^{1-1/p}.$$

The first term is stochastically bounded, because we can use Lemma 11, Lemma 12 and the given moment condition to realize that the absolute moment is uniform bounded in n. For the second term we define $\theta := 0$ in (i) and $\theta := \alpha/3 - \delta'$ in (ii), where $\delta := \lfloor l\alpha/(\alpha - 3r) + 1\rfloor / l - \alpha/(\alpha - 3r)$ and $\delta' := \delta(\alpha^2 - 6\alpha\, r + 9\, r^2)/(3\delta\,\alpha - 9\,\delta\, r + 9\, r) > 0$. Thus, $\theta \in (r, \alpha/3)$. Then, for all $\varepsilon > 0$,

$$\mathbb{P}\left(n^r \sup_{z\in\mathbb{R}^d}\left|\frac{1}{T}\sum_{k=1}^n \mathbb{1}({}^{CP}X_{h,k}[\xi_n] \in (z - 2\xi_n', z + 2\xi_n'),\ {}^{CP}X_{h,k}[\xi_n] \neq 0)\right| > \varepsilon\right)$$
$$\le \mathbb{P}\left(n^r \sup_{z\in\mathbb{R}^d}\left|\frac{1}{T}\sum_{k=1}^n \mathbb{1}\left(\bigcap_{i=1}^d \left\{{}^{CP}X_{h,k,i}[\xi_n] \in (z^{(i)} - 2\xi_n', z^{(i)} + 2\xi_n'), {}^{CP}X_{h,k}[\xi_n] \neq 0\right\}\right)\right| > \varepsilon\right)$$
$$\le \mathbb{P}\left(n^r \sup_{z\in\mathbb{R}^d}\left|\frac{1}{T}\sum_{k=1}^n \mathbb{1}\left(\exists i \in \{1, ..., d\} :\ {}^{CP}X_{h,k,i}[\xi_n] \in (z^{(i)} - 2\xi_n', z^{(i)} + 2\xi_n'), {}^{CP_i}X_{h,k}[\xi_n] \neq 0\right)\right| > \varepsilon\right)$$
$$\le \mathbb{P}\left(n^r \sup_{z\in\mathbb{R}^d}\left|\frac{1}{T}\sum_{k=1}^n \mathbb{1}\left(\bigcup_{i=1}^d \left\{{}^{CP}X_{h,k,i}[\xi_n] \in (z^{(i)} - 2\xi_n', z^{(i)} + 2\xi_n'), {}^{CP_i}X_{h,k}[\xi_n] \neq 0\right\}\right)\right| > \varepsilon\right)$$

$$\le \mathbb{P}\Big(\sum_{i=1}^{d} n^r \sup_{z^{(i)}\in\mathbb{R}} \Big|\frac{1}{T}\sum_{k=1}^{n} \mathbb{1}\Big({}^{CP}X_{h,k,i}[\xi_n] \in (z^{(i)} - 2\xi_n', z^{(i)} + 2\xi_n'), {}^{CP_i}X_{h,k}[\xi_n] \neq 0\Big)\Big| > \varepsilon\Big)$$

$$\le \sum_{i=1}^{d} \mathbb{P}\Big(n^r \sup_{z^{(i)}\in\mathbb{R}} \Big|\frac{1}{T}\sum_{k=1}^{n} \mathbb{1}\Big({}^{CP}X_{h,k,i}[\xi_n] \in (z^{(i)} - 2\xi_n', z^{(i)} + 2\xi_n'), {}^{CP_i}X_{h,k}[\xi_n] \neq 0\Big)\Big| > \varepsilon/d\Big)$$

$$\le \mathbb{P}(M_3^{(n)}(\xi_n)) + \sum_{i=1}^{d} \mathbb{P}(M_2^{(n)}(\xi_n, \xi_n', 1-\alpha-\theta, i)) \longrightarrow 0.$$

The convergence follows from Lemma 2 and 3.
Ad 3.), in (i) the term is equal to zero for large n because we have

$$\frac{h\int_{\xi_n \le \|x\| \le 1} \|x\| \nu(dx)}{\xi_n'/4} \le \frac{h\nu((-\xi_n,\xi_n)^c)}{\xi_n'/4} \le \frac{\sqrt{n} h\nu((-\xi_n,\xi_n)^c)}{\xi_n'^2} \to 0.$$

In (ii), by (3) there exists a $\kappa > 0$ such that

$$\frac{h\int_{\xi_n \le \|x\| \le 1} \|x\| \nu(dx)}{\xi_n'/4} \le \sum_{i=1}^{d} \frac{h\int_{(-1,-\xi_n]\cup[\xi_n,1]} |x_i|\widetilde{\nu}_i(dx_i)}{\xi_n'/4} \le \sum_{i=1}^{d} \frac{\kappa\, h \int_{(-1,-\xi_n]\cup[\xi_n,1]} 1/x_i^2 dx_i}{\xi_n'} \le \frac{\kappa\, h}{\xi_n\, \xi_n'} \to 0.$$

Ad 4.), we use the decomposition of 1.) and only have to show

$$n^r \frac{1}{T}\sum_{k=1}^{n} \left|{}^{CP}X_{h,k}^T[\xi_n] A\, {}^{CP}X_{h,k}[\xi_n]\right|^l \mathbb{1}(\|{}^{sJ}X_{h,k,i}[\xi_n]\| \ge \xi_n'/2)$$

$$\le n^r \frac{1}{T}\sum_{k=1}^{n} \left|{}^{CP}X_{h,k}^T[\xi_n] A\, {}^{CP}X_{h,k}[\xi_n]\right|^l \left(\sum_{i=1}^{d} \mathbb{1}(|{}^{sJ_i}X_{h,k,i}[\xi_n]| \ge \xi_n'/2)\right).$$

We use Lemma 11 and Lemma 12 and have for the expectation

$$n^r O(1)\cdot 4 \sum_{i=1}^{d} \mathrm{Var}({}^{cJ_i}X_{h,k}[\xi_n])/\xi_n'^2 = n^r O(1)\cdot 4 \sum_{i=1}^{d} \int_{-\xi_n}^{\xi_n} x_i^2 \widetilde{\nu}(dx_i)/\xi_n'^2 = n^r h \cdot O(1)/\xi_n'^2 \to 0.$$

Ad 5.), the probability that this term is different from zero is considered in Lemma 1. □

Lemma 9. *We use the setting given in Lemma 8 (i) and (ii) and consider $j_1, j_2 \in \{0,\ldots,2l\}$ satisfying either $j_1 > 0$, $j_2 \in \{1,...,2l\}$ or $j_1 = 0$, $j_2 = 2l$. Then, as $n \to \infty$,*

$$1.)\ n^r \frac{1}{T}\sum_{k=1}^{n} \left|{}^{CP}X_{h,k}[\xi_n]\right|^{j_1} \left|{}^{sJ}X_{h,k}[\xi_n]\right|^{j_2} \xrightarrow{P} 0,$$

$$2.)\ n^r \frac{1}{T}\sum_{k=1}^{n} \left|{}^{CP}X_{h,k}[\xi_n]\right|^{j_1} \left|\gamma_{h,k}^c[\xi_n, 1]\right|^{j_2} \xrightarrow{P} 0.$$

Proof. Ad 1.), in the case $j_1 > 0$, $j_2 \in \{1,...,2l\}$ we have

$$\mathbb{P}\Big(n^r \frac{1}{T}\sum_{k=1}^{n} \left|{}^{CP}X_{h,k}[\xi_n]\right|^{j_1} \left|{}^{sJ}X_{h,k}[\xi_n]\right|^{j_2} > \varepsilon\Big)$$

$$\le \mathbb{P}\Big(n^r \frac{1}{T}\sum_{k=1}^{n} \left|{}^{CP}X_{h,k}[1]\right|^{2\lfloor (j_1+1)/2 \rfloor} + N_{((k-1)h,hk]}((-\xi_n,\xi_n)^c) \left|{}^{sJ}X_{h,k}[\xi_n]\right|^{j_2} > \varepsilon\Big) \tag{A.9}$$

$$+ \mathbb{P}\left(M_2^{(n)}(\xi_n)\right), \tag{A.10}$$

where Lemma 2 defines the set $M_2^{(n)}(\xi_n)$ and proves that (A.10) converges to zero. In (A.9) we have used that on $\Omega\backslash M_2^{(n)}(\xi_n)^c$ each ${}^{CP}X_{h,k}[\xi_n]$ contains at most one jump, that is either larger or smaller than one. We consider the expectation of the term in (A.9)

$$\begin{aligned}
&n^r \frac{n}{T}\mathbb{E}\left[\left(\left|{}^{CP}X_{h,k}[1]\right|^{2\lfloor (j_1+1)/2\rfloor} + N_{0,h}((-\xi_n,\xi_n)^c)\right)\right]\mathbb{E}\left[\left|{}^{sJ}X_{h,1}[\xi_n]\right|^{j_2}\right]\\
&\le n^r \frac{n}{T}\left(O(h) + h\,\nu\left(\{x : \|x\| \ge \xi_n\}\right)\right)\sqrt{\mathbb{E}\left[{}^{sJ}X_{h,1}^{2j_2}[\xi_n]\right]}\\
&\le O(1)\,n^{r-\alpha/2}\,\nu\left(\{x : \|x\| \ge \xi_n\}\right)\sqrt{\sum_{i=1}^d\left(\int_{\|x\|\le\xi_n} x_i^{2j_2}\nu(dx) + O(h)\right)},
\end{aligned}$$

where we have used Lemma 12. In the setting (i) of Lemma 8 this term tends to zero by the conditions of Lemma 3(i), for setting (ii) we can find an upper bound by using (3)

$$\begin{aligned}
&O(1)\,n^{r-\alpha/2}\sum_{i=1}^d \widetilde{\nu}_i((-\xi_n,\xi_n)^c)\sqrt{\sum_{i=1}^d\int_{\|x\|\le\xi_n} x_i^{2j_2}\nu(dx)} \le O(1)\,n^{r-\alpha/2}\sum_{i=1}^d\int_{|x_i|\ge\xi_n} f_i(x_i)dx_i\sqrt{\sum_{i=1}^d\int_{\|x\|\le\xi_n} x_i^{2j_2}\nu(dx)}\\
&\le O(1)\,n^{r-\alpha/2}\,\xi_n^{-2}\sqrt{\sum_{i=1}^d\int_{\|x\|\le\xi_n} x_i^{2j_2}\nu(dx)} \longrightarrow 0.
\end{aligned}$$

In the case $j_1 = 0,\ j_2 = 2l$ there exists by Lemma 10(ii) and Lemma 12(ii) a $\kappa > 0$ such that,

$$\begin{aligned}
&n^r\mathbb{E}\left[\frac{1}{T}\sum_{k=1}^n\left|{}^{sJ}X_{h,k}[\xi_n]\right|^{2l}\right] = n^r\mathbb{E}\left[\frac{1}{T}\sum_{k=1}^n\left(\sum_{i=1}^d {}^{sJ}X_{h,k,i}^2[\xi_n]\right)^l\right]\\
&\le \kappa\, n^r\frac{n}{T}\sum_{i=1}^d\mathbb{E}\left[{}^{sJ}X_{h,1,i}^{2l}[\xi_n]\right] \le n^r\frac{n\,h}{T}\sum_{i=1}^d\left(\int_{-\xi_n}^{\xi_n} x_i^{2l}\widetilde{\nu}_i(dx_i) + O(h)\right).
\end{aligned}$$

This term tends to zero in setting (i), since

$$n^0\int_{-\xi_n}^{\xi_n} x_i^2\widetilde{\nu}_i(dx_i) \to 0$$

and in setting (ii), because

$$n^r\int_{-\xi_n}^{\xi_n} x_i^{2l}\widetilde{\nu}_i(dx_i) \le n^r\xi_n^{2l-2}\int_{-\xi_n}^{\xi_n} x_i^2\widetilde{\nu}_i(dx_i) \to 0.$$

Ad 2.), in setting (i) we have

$$\left|\gamma_h^c[\xi_n,1]\right| \le h\int_{\xi_n\le\|x\|\le 1}\|x\|\nu(dx) \le h\nu\left(\{x : \|x\| \ge \xi_n\}\right)$$

and in setting (ii) we have

$$\left|\gamma_h^c[\xi_n,1]\right| \le h\int_{\xi_n\le\|x\|\le 1}\|x\|\nu(dx) \le \sum_{i=1}^d\int_{\xi_n\le|x_i|\le 1}|x_i|\widetilde{\nu}_i(dx_i) \le h/\xi_n.$$

So the term in 2.) converges analogously to 1.) to zero by using the conditions on ξ_n given in Lemma 3. □

Lemma 10. *Let $r, s \in \mathbb{N}$, $n \in \mathbb{N}$, $k \in \{1, \cdots, n\}$ and $i \in \{1, \cdots, n\}$. Then, there exists $\kappa(r, s) > 0$ such that, for all $a_{k,i,n} \in \mathbb{R}^+$, $b_{k,i,n} \in \mathbb{R}^+$, $c_{k,n} \in \mathbb{R}^+$, $d_n \in \mathbb{R}^+$ the following inequalities hold true.*

$$(i) \quad \sum_{k=1}^{n}\left(c_{k,n}\left(\sum_{i_1,i_2=1}^{r} a_{k,i_1,n}b_{k,i_2,n}\right)^s\right) \leq \kappa(r,s) \sum_{i_1,i_2=1}^{r}\sum_{k=1}^{n} c_{k,n}a_{k,i,n}^s b_{k,i,n}^s,$$

$$(ii) \quad \sum_{k=1}^{n}\left(c_{k,n}\left(\sum_{i_1,i_2=1}^{r} a_{k,i_1,n}a_{k,i_2,n}\right)^s\right) \leq \kappa(r,s) \sum_{i=1}^{r}\sum_{k=1}^{n} c_{k,n}a_{k,i,n}^{2s},$$

$$(iii) \quad \left|\sum_{k=1}^{n} c_{k,n}\left(\sum_{i=1}^{r} a_{k,i,n}\right)^s - d_n\right| \leq \left|\sum_{k=1}^{n} c_{k,n}a_{k,1,n}^s - d_n\right| + \kappa(r,s) \sum_{i=2}^{r}\sum_{m=1}^{s}\sum_{k=1}^{n} c_{k,n}a_{k,1,n}^{s-m}a_{k,i,n}^m.$$

Proof. Ad (i), because $x \mapsto x^s/r^{2s}$ is a convex function on $\mathbb{R}^+$, we have

$$\left(\frac{\sum_{i_1,i_2=1}^{r} a_{k,i_1,n}b_{k,i_2,n}}{r^2}\right)^s \leq \frac{\sum_{i_1,i_2=1}^{r} a_{k,i_1,n}^s b_{k,i_2,n}^s}{r^{2s}}.$$

Hence,

$$\sum_{k=1}^{n}\left(c_{k,n}\left(\sum_{i_1,i_2=1}^{r} a_{k,i_1,n}b_{k,i_2,n}\right)^s\right) \leq r^{2s-2} \sum_{k=1}^{n} c_{k,n} \sum_{i_1,i_2=1}^{r} a_{k,i_1,n}^s b_{k,i_2,n}^s.$$

Ad (ii), we have

$$\sum_{k=1}^{n}\left(c_{k,n}\left(\sum_{i_1,i_2=1}^{r} a_{k,i_1,n}a_{k,i_2,n}\right)^s\right) \leq \sum_{k=1}^{n}\left(c_{k,n}r^s \max_{i=1,\ldots,r}\{a_{k,i,n}^{2s}\}\right) \leq r^s \sum_{k=1}^{n} c_{k,n}\sum_{i=1}^{r} a_{k,i_1,n}^{2s} = \kappa(r,s) \sum_{i=1}^{r}\sum_{k=1}^{n} c_{k,n}a_{k,i_1,n}^{2s}.$$

Ad (iii), we have

$$\begin{aligned}
&\left|\sum_{k=1}^{n} c_{k,n}\left(\sum_{i=1}^{r} a_{k,i,n}\right)^s - d_n\right| = \left|\sum_{k=1}^{n} c_{k,n}\sum_{m=0}^{s}\binom{s}{m}a_{k,1,n}^{s-m}\left(\sum_{i=2}^{r} a_{k,i,n}\right)^m - d_n\right| \\
&\leq \left|\sum_{k=1}^{n} c_{k,n}a_{k,1,n}^s - d_n\right| + \sum_{k=1}^{n} c_{k,n}\sum_{m=1}^{s}\binom{s}{m}a_{k,1,n}^{s-m}\left(\sum_{i=2}^{r} a_{k,i,n}\right)^m \\
&\leq \left|\sum_{k=1}^{n} c_{k,n}a_{k,1,n}^s - d_n\right| + \kappa(r,s) \sum_{k=1}^{n} c_{k,n}\sum_{m=1}^{s} a_{k,1,n}^{s-m}(r-1)^m \max_{i=2,\ldots,r}\{a_{k,i,n}^m\} \\
&\leq \left|\sum_{k=1}^{n} c_{k,n}a_{k,1,n}^s - d_n\right| + \kappa(r,s) \sum_{k=1}^{n} c_{k,n}\sum_{m=1}^{s} a_{k,1,n}^{s-m}\sum_{i=2}^{r} a_{k,i,n}^m = \left|\sum_{k=1}^{n} c_{k,n}a_{k,1,n}^s - d_n\right| + \kappa(r,s) \sum_{i=2}^{r}\sum_{m=1}^{s}\sum_{k=1}^{n} c_{k,n}a_{k,1,n}^{s-m}a_{k,i,n}^m.
\end{aligned}$$

□

Lemma 11. *Let $A \in \mathbb{R}^{d\times d}$, $A = (a_{i,j})_{i,j=1}^d$ and $x, y \in \mathbb{R}^d$. Then,*

$$x^T Ay \leq d! \max_{i,j=1,\cdots,d} |a_{i,j}|\,|x|\,|y|.$$

Proof. We denote by Π the set of all permutations of $\{1, \cdots, d\}$. Define $\pi_0 \in \Pi$ such that the following inequality is satisfied $\sum_{i=1}^d |x_i y_{\pi(i)}| \leq \sum_{i=1}^d |x_i y_{\pi_0(i)}|$ for all $\pi \in \Pi$. Then we use Cauchy-Schwarz inequality and obtain

$$x^T Ay = \sum_{i,j=1}^{d} x_i\, a_{i,j}\, y_j \leq \max_{i,j=1,\cdots,d}|a_{i,j}| \sum_{i,j=1}^{d} |x_i y_j| = \max_{i,j=1,\cdots,d}|a_{i,j}| \sum_{i\in\{1,\ldots,d\},\pi\in\Pi} |x_i y_{\pi(i)}| \leq d! \max_{i,j=1,\cdots,d}|a_{i,j}| \sum_{i=1}^{d} |x_i y_{\pi_0(i)}| \leq \kappa|x||y|.$$

□

Lemma 12. *Let $\{L_t,\ t \geq 0\}$ be an $\mathbb{R}-$valued Lévy process and $\{{}^C L'_t,\ t \geq 0\}$ an $\mathbb{R}-$valued nonstandard Wiener process. Then, as $h \to 0$,*

$$(i)\ \mathbb{E}\left[|L_h|\right] = O(\sqrt{h}),$$
$$(ii)\ \mathbb{E}\left[L_h{}^j\right] = h\, c_n(1) + O(h^2) = O(h), \quad j \in \mathbb{N},\ j\ even,$$
$$(iii)\ \mathbb{E}\left[|L_h|^j\right] = O(h), \quad j \in \mathbb{N},\ j \geq 3,\ j\ odd,$$
$$(iv)\ \mathbb{E}\left[{}^C L_h^j\right] = O(h^{j/2}), \quad j \in \mathbb{N},\ j\ even,$$

where we have assumed that $\mathbb{E}\left[L_1{}^{2\lfloor (j+1)/2 \rfloor}\right] < \infty$ and have denoted by $c_j(1)$ the j-th cumulant of L_1.

Proof. The connection between cumulants and moments is given by (cf. Lukacs, 1970, p.27)

$$\mathbb{E}\left[L'_h{}^j\right] = c_j(h) + \sum_{k=1}^{n-1} \binom{n-1}{k-1} c_k(h) \mathbb{E}\left[L_h^{j-k}\right].$$

So (ii) follows directly. For (i) the Lyapunov inequality can be applied and for (iii) we have

$$\mathbb{E}\left[|L_h|^j\right] \leq \mathbb{E}\left[L_h{}^{j-1}\right] + \mathbb{E}\left[L_h{}^{j+1}\right].$$

The cumulants $c_j(h)$ of a Wiener process are equal to zero for $j \geq 3$, so (iv) follows directly. □

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