A Generalized Index Theorem for Monotone Matrix-Valued Functions with Applications to Discrete Oscillation Theory

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Preprint Series: 2012 - 04

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A GENERALIZED INDEX THEOREM FOR MONOTONE MATRIX-VALUED FUNCTIONS WITH APPLICATIONS TO DISCRETE OSCILLATION THEORY

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Abstract. An index theorem is a tool for computing the change of the index (i.e., the number of negative eigenvalues) of a symmetric monotone matrix-valued function when its variable passes through a singularity. In 1995, the first author proved an index theorem in which a certain critical matrix coefficient is constant. In this paper, we generalize the above index theorem to the case when this critical matrix may be varying, but its rank, as well as the rank of some additional matrix, are constant. This includes as a special case the situation when this matrix has a constant image. We also show that the index theorem does not hold when the main assumption on constant ranks is violated. Our investigation is motivated by the oscillation theory of discrete symplectic systems with nonlinear dependence on the spectral parameter, which was recently developed by the second author and for which we obtain new oscillation theorems.

1. Motivation

In 1995, the first author proved a result called an "index theorem", which shows how to compute the change of the index of a symmetric monotone matrix-valued function when its variable passes through a singularity, see [16, Theorem 2] or [17, Theorem 3.4.1], and for comparison also [20, Proposition 2.5]. By the index of a matrix we mean the number of its negative eigenvalues. This result has been utilized in many applications, in particular in the oscillation theory of Sturm–Liouville differential equations, linear Hamiltonian systems, and discrete symplectic systems. The relevant references are [17, Sections 4.2, 5.2, 7.2] and [5, 9, 18, 20].

One of the key assumptions of the index theorem in [16, Theorem 2] is that one of the considered coefficients *is constant*, i.e., it does not depend on t. In our main result (Theorem 2.1 below) this would be the matrix $R_2 \equiv R_2(t)$. This assumption implies certain limitations in the applications. For example, in [20, Section 6.4] it was observed that the application of the index theorem in the oscillation theory of discrete symplectic systems forces one of the coefficients of the system, denoted by $\mathcal{B}_k(\lambda)$ in Section 3, to be independent of the spectral parameter λ. On the other hand, a subsequent result of the second author in [21] shows that in the scalar case, i.e., for second order Sturm–Liouville difference equations

$$\Delta(r_k(\lambda)\,\Delta x_k) + q_k(\lambda)\,x_{k+1} = 0, \quad k \in [0, N-1]_{\mathbb{Z}}, \tag{1.1}$$

the oscillation theorems in [20, Section 6.1] hold without the restriction on constancy of the coefficient $\mathcal{B}_k(\lambda) = 1/r_k(\lambda)$. The latter results in [21] were, however, derived without using the index theorem.

Those investigations in [20, 21] raised the question whether the index theorem could be derived under some more general assumptions than the constancy of that coefficient. In this paper we affirmatively answer this question. We prove a general index theorem in which the

2010 *Mathematics Subject Classification. Primary* 15B57. *Secondary* 15A42, 26A48, 39A21.

Key words and phrases. Index theorem; Rank theorem; Limit theorem; Oscillation theorem; Discrete symplectic system; Sturm–Liouville difference equation.

critical matrix coefficient is allowed to be varying with constant rank. Our index theorem is new even under the special assumption on the constant image of the critical matrix. The index theorem with the latter assumption then naturally leads to new oscillation theorems for discrete symplectic systems, whose coefficient $\mathcal{B}_k(\lambda)$ has a constant image in λ, see Section 3. This includes, in particular, second order Sturm–Liouville difference equations (1.1) considered in [21], for which we have $\mathcal{B}_k(\lambda) = 1/r_k(\lambda)$, as well as Sturm–Liouville difference equations of arbitrary even order

$$\sum_{j=0}^{n} (-1)^j \Delta^j \big(r_k^{[j]}(\lambda)\, \Delta^j y_{k+n-j}\big) = 0, \quad k \in [0, N-n]_{\mathbb{Z}}, \tag{1.2}$$

for which have

$$\mathcal{B}_k(\lambda) = (1, \ldots 1, 1)^T \cdot (0, \ldots, 0, 1) \;\cdot\; 1/r_k^{[n]}(\lambda). \tag{1.3}$$

In both cases we see that the image of $\mathcal{B}_k(\lambda)$ is constant. Therefore, the results of this paper are important not only as a new theory, but also for applications. Such applications arise e.g. in control theory, in which Hamiltonian and symplectic systems play an indispensable role.

2. Generalized index theorem

In this paper, we denote by $\operatorname{Ker} A$, $\operatorname{Im} A$, $\operatorname{rank} A$, $\operatorname{def} A$, $\operatorname{ind} A$, A^T, A^{-1}, and $A^\dagger$ the kernel, image, rank, defect (i.e., the dimension of the kernel), index, transpose, inverse, and the Moore–Penrose generalized inverse (shortly the pseudoinverse) of the matrix A, see [2]. Moreover, the notation $f(0^+)$ and $f(0^-)$ stands for the right-hand and left-hand limits of the function $f(t)$ at $t = 0$. The statement of the main result of this paper now follows.

Theorem 2.1 (Index theorem). *Let $X(t)$, $U(t)$, $R_1(t)$, $R_2(t)$ be given real $m \times m$-matrix-valued functions on $[0, \varepsilon)$ such that*

$$\left.\begin{aligned} R_1(t)\, R_2^T(t) \ \text{and}\ X^T(t)\, U(t) \ \text{are symmetric},\\ \operatorname{rank}\,(R_1(t),\ R_2(t)) = \operatorname{rank}\,(X^T(t),\ U^T(t)) = m \end{aligned}\right\} \quad \text{for } t \in [0, \varepsilon), \tag{2.1}$$

and assume that $X(t)$, $U(t)$, $R_1(t)$, $R_2(t)$ are continuous at 0, i.e.,

$$\left.\begin{aligned} &\lim_{t\to 0^+} R_1(t) = R_1 := R_1(0), \qquad \lim_{t\to 0^+} X(t) = X := X(0),\\ &\lim_{t\to 0^+} R_2(t) = R_2 := R_2(0), \qquad \lim_{t\to 0^+} U(t) = U := U(0), \end{aligned}\right\} \tag{2.2}$$

and that $X(t)$ is invertible for $t \in (0, \varepsilon)$. Moreover, denote

$$\left.\begin{aligned} M(t) &:= R_1(t)\, R_2^T(t) + R_2(t)\, U(t)\, X^{-1}(t)\, R_2^T(t), &&\\ \Lambda(t) &:= R_1(t)\, X(t) + R_2(t)\, U(t), && \Lambda := \Lambda(0)\\ S(t) &:= X^\dagger R_2^T(t), && S := S(0),\\ S^*(t) &:= R_2^T(t) - X S(t) = (I - XX^\dagger)\, R_2^T(t), && S^* := S^*(0), \end{aligned}\right\} \tag{2.3}$$

and suppose that the functions $U(t)\, X^{-1}(t)$ and $M(t)$ are monotone on $(0, \varepsilon)$ and that

$$\operatorname{rank} R_2(t) \equiv \operatorname{rank} R_2 \quad \text{and} \quad \operatorname{rank} S^*(t) \equiv \operatorname{rank} S^* =: m - r \tag{2.4}$$

are constant on $[0, \varepsilon)$. Finally, let $T \in \mathbb{R}^{m\times r}$ be such that

$$\operatorname{rank} T = r, \quad T^T T = I_{r\times r}, \quad \operatorname{Im} T = \operatorname{Ker} S^*, \quad \text{and} \quad Q := T^T \Lambda\, S T \in \mathbb{R}^{r\times r}. \tag{2.5}$$

Then the matrix Q is symmetric, and $\operatorname{ind} M(0^+)$, $\operatorname{ind} M(0^-)$, $\operatorname{def} \Lambda(0^+)$ *exist with*

$$\operatorname{ind} M(0^+) = \operatorname{ind} Q + m - \operatorname{rank} T + \operatorname{def} \Lambda - \operatorname{def} \Lambda(0^+) - \operatorname{def} X \tag{2.6}$$

if $M(t)$ and $U(t)\, X^{-1}(t)$ are nonincreasing on $(0, \varepsilon)$, and

$$\operatorname{ind} M(0^+) = \operatorname{ind} Q + m - \operatorname{rank} T \tag{2.7}$$

if $M(t)$ and $U(t)\, X^{-1}(t)$ are nondecreasing on $(0, \varepsilon)$.

Remark 2.2. (i) If $R_2(t) \equiv R_2$ is constant on $[0, \varepsilon)$ as in [16, Theorem 2] and [20, Proposition 2.5], then assumption (2.4) is trivially satisfied. Therefore, our Theorem 2.1 is a generalization of those results to varying $R_2(t)$.

(ii) Note that if $\operatorname{Im} R_2^T(t)$ is constant on $[0, \varepsilon)$, then (2.4) is also satisfied. Observe that this constant image assumption does not depend on X, while of course (2.4) depends in general on $R_2(t)$ and also on X. Note that by the rank formula $\operatorname{rank} AB = \operatorname{rank} B - \dim(\operatorname{Ker} A \cap \operatorname{Im} B)$, see e.g. [2, Fact 2.10.13(ii)], we have

$$\operatorname{rank} S^*(t) = \operatorname{rank} R_2(t) - \dim(\operatorname{Im} X \cap \operatorname{Im} R_2^T(t)).$$

(iii) Condition (2.4) is optimal in a sense that it cannot be further weakened to a constant rank of $R_2(t)$ alone. More precisely, we provide examples illustrating the fact that the conclusions of Theorem 2.1 do not hold when the rank of $R_2(t)$ changes. Similarly, another example shows that the conclusions of Theorem 2.1 do not hold either if the constant rank condition is satisfied by $R_2(t)$ but it is violated by $S^*(t)$, or when the rank of $R_2(t)$ is constant but the image of $R_2^T(t)$ changes.

(iv) The analysis of the proof of [16, Theorem 2] or [17, Theorem 3.4.1, pp. 102–104] shows that the main problem with varying $R_2(t)$ resides in the fact that the Moore–Penrose pseudoinverse $R_2^\dagger(t)$ is not in general continuous at 0. The key to this problem is [7, Theorem 10.5.1], which says that the Moore–Penrose pseudoinverse of a continuous matrix is continuous if and only if the matrix has constant rank. In particular, the continuity of $R_2^\dagger(t)$ now follows from assumption (2.4).

Proof of Theorem 2.1. By [7, Theorem 10.5.1] and (2.4), the continuity of $R_1(t)$ and $R_2(t)$ implies that $S^*(t) \to S^*$, $S(t) \to S$, $V(t) \to V$ as $t \to 0^+$ with

$$r = \operatorname{rank} V \equiv \operatorname{rank} V(t), \quad \operatorname{Im} V(t) = \operatorname{Ker} S^*(t) \quad \text{for } t \in [0, \varepsilon),$$

where the $m \times m$ matrices $V(t)$ and V are defined by

$$V(t) := I - [S^*(t)]^\dagger S^*(t), \quad V := V(0).$$

The monotonicity of $M(t)$ on $(0, \varepsilon)$ implies that $\operatorname{def} M(0^+)$ and $\operatorname{ind} M(0^+)$ exist and, since $M(t) = \Lambda(t)\, X^{-1}(t)\, R_2^T(t)$ on $(0, \varepsilon)$, we have by [16, formula (21)] or [17, formula (3.4.8)]

$$\operatorname{Ker} M(t) = \operatorname{Ker} \Lambda^T(t) \oplus \operatorname{Ker} R_2^T(t) \quad \text{for all } t \in (0, \varepsilon). \tag{2.8}$$

It follows from (2.8) that

$$\operatorname{def} M(0^+) = \operatorname{def} \Lambda(0^+) + \operatorname{def} R_2 \tag{2.9}$$

with $\operatorname{def} R_2(t) \equiv \operatorname{def} R_2$ by (2.4). Hence, $\operatorname{def} \Lambda(0^+)$ exists as well. Moreover, $R_2^T T = (XS + S^*)\, T = XST$, since $S^* T = 0$, and therefore,

$$Q = T^T \Lambda S T = T^T (R_1 X + R_2 U)\, S T = T^T (R_1 R_2^T + S^T X^T U S)\, T$$

is symmetric. Next, by [17, Corollary 3.1.3], we have $R_1(t) = R_2(t)\, S_1(t) + S_2(t)$ for $t \in [0, \varepsilon)$, where $S_1(t)$ is symmetric, $\operatorname{rank}(R_2(t),\ S_2(t)) = m$, $\operatorname{Ker} S_2(t) = \operatorname{Im} R_2^T(t)$, i.e., $S_2(t)\, R_2^T(t) = 0$, for all $t \in [0, \varepsilon)$. In particular, we may take

$$S_1(t) := \tfrac{1}{2}\, R_1^T(t)\, [R_2^\dagger(t)]^T + \tfrac{1}{2}\, R_2^\dagger(t)\, R_1(t), \tag{2.10}$$

$$S_2(t) := \tfrac{1}{2}\, R_1(t)\, [I - R_2^\dagger(t)\, R_2(t)] + \tfrac{1}{2}\, [I - R_2(t)\, R_2^\dagger(t)]\, R_1(t), \tag{2.11}$$

as can be verified by direct calculations. Hence, by assumption (2.4), we have $S_1(t) \to S_1 := S_1(0)$ as $t \to 0^+$, and in particular $S_1(t)$ is bounded as $t \to 0^+$. It now follows from the limit theorem [15, Theorem 1] or [17, Theorem 3.3.7] by the *monotonicity* of $U(t)\, X^{-1}(t)$ that

$$\left.\begin{aligned} X^T[S_1(t) + U(t)\, X^{-1}(t)]\, X \to X^T(S_1 X + U) \quad \text{as } t \to 0^+, \\ d^T[S_1(t) + U(t)\, X^{-1}(t)]\, d \to \infty \text{ (resp. } -\infty) \quad \text{as } t \to 0^+ \text{ for all } d \notin \operatorname{Im} X. \end{aligned}\right\} \tag{2.12}$$

By $\operatorname{rank} V(t) \equiv r$, there exists $R(t) \in \mathbb{R}^{m \times r}$ such that $S^*(t)\, T(t) = 0$ and

$$T^T(t)\, T(t) = I_{r \times r} \quad \text{with} \;\; T(t) := V(t)\, R(t) \quad \text{for all } t \in (0, \varepsilon). \tag{2.13}$$

By the Bolzano–Weierstrass theorem, there exists a sequence $t_k \searrow 0$ such that

$$T := \lim_{k \to \infty} T(t_k) \quad \text{exists.} \tag{2.14}$$

Then $T \in \mathbb{R}^{m \times r}$ and $S^*\, T = 0$, $T^T T = I_{r \times r}$, $\operatorname{Im} T = \operatorname{Ker} S^*$, i.e., it satisfies the requirements in (2.5) of the theorem.

On the other hand, the matrices, R_1, R_2, X, U, S, S^*, Λ, T, Q from (2.3), (2.14), (2.5) are given as in the rank theorem [16, Theorem 1] or [17, Theorem 3.1.8], so that in particular $R_2^T d \in \operatorname{Im} X$ if and only if $d \in \operatorname{Im} T$. Since $R_2^T(t)\, T(t) = X S(t)\, T(t)$, we get

$$T^T(t_k)\, M(t_k)\, T(t_k) = T^T(t_k)\, S^T(t_k)\, X^T\, [S_1(t_k) + U(t_k)\, X^{-1}(t_k)]\, X S(t_k)\, T(t_k) \to Q$$

for $k \to \infty$, by (2.12). Moreover, if $d \notin \operatorname{Im} T$, then $x(t) := R_2^T(t)\, d \to R_2^T d \notin \operatorname{Im} X$ as $t \to 0^+$, and it follows from (2.12) and [17, Eq. (3.3.7*) of Proposition 3.3.10] that $d^T M(t) d \to \infty$ (resp. $-\infty$) as $t \to 0^+$ for all $d \notin \operatorname{Im} T$. Hence, we have shown that

$$\left.\begin{aligned} T^T(t_k)\, M(t_k)\, T(t_k) \to Q \quad \text{as } k \to \infty, \text{ and} \\ d^T M(t)\, d \to \infty \;\; \text{(resp. } -\infty) \text{ as } t \to 0^+ \text{ for all } d \notin \operatorname{Im} T. \end{aligned}\right\} \tag{2.15}$$

Let $\mu_1(t), \dots, \mu_m(t)$ and $\mu_1, \dots, \mu_r$ be the eigenvalues of $M(t)$ and Q. We will prove:

(i) $\mu_i(t) \to \mu_i$ for $i \in \{1, \dots, r\}$ and $\mu_i(t) \to \infty$ for $i \in \{r+1, \dots, m\}$ as $t \to 0^+$, when $\mu_1(t) \le \cdots \le \mu_m(t)$ and $\mu_1 \le \cdots \le \mu_r$, and if $M(t)$ and $U(t)\, X^{-1}(t)$ are nonincreasing,

(ii) $\mu_i(t) \to \mu_i$ for $i \in \{1, \dots, r\}$ and $\mu_i(t) \to -\infty$ for $i \in \{r+1, \dots, m\}$ as $t \to 0^+$, when $\mu_1(t) \ge \cdots \ge \mu_m(t)$ and $\mu_1 \ge \cdots \ge \mu_r$, and if $M(t)$ and $U(t)\, X^{-1}(t)$ are nondecreasing.

Without loss of generality we may consider only the case (i), since for the proof of the case (ii) we may take $-M(t)$ and $-U(t)\, X^{-1}(t)$. First observe that by [17, Proposition 3.2.3] the eigenvalues $\mu_1(t), \dots, \mu_m(t)$ are nonincreasing, because $M(t)$ is nonincreasing on $(0, \varepsilon)$. Moreover, $\mu_1(t), \dots, \mu_r(t)$ are bounded by (2.15), (2.13), and the minimum-maximum principle (compare with [17, Proposition 3.2.1]). Hence, the limit

$$\mu_i' := \lim_{t \to 0^+} \mu_i(t) \quad \text{exists for } i \in \{1, \dots, r\} \text{ with } \mu_1' \le \cdots \le \mu_r'. \tag{2.16}$$

We claim that $\mu_i' = \mu_i$ for all $i \in \{1, \dots, r\}$. Note that [17, Proposition 3.2.6] or [16, Proposition 3] cannot be applied, because the matrix $T(t)$ now depends on t and it is not constant in general. But we will proceed similarly as in the proof of [17, Proposition 3.2.6]. This is the crucial new aspect of this proof, because [17, Proposition 3.2.6] becomes in general false for $T = T(t)$ depending on t. On the other hand, its assertion remains true in our special situation, where we do not use the monotonicity of $M(t)$ alone, but also the monotonicity of $U(t)\,X^{-1}(t)$. We proceed as follows.

First, let $d_1,\dots,d_r\in\mathbb{R}^r$ be orthonormal eigenvectors of Q, i.e., $Qd_i=\mu_i d_i$ for $i\in\{1,\dots,r\}$. For these indices and for $t\in(0,\varepsilon)$ we define the vectors $c_i(t):=T(t)\,d_i$. Then $c_1(t),\dots,c_r(t)\in\mathbb{R}^m$ are also orthonormal, since $T^T(t)\,T(t)=I_{r\times r}$ for $t\in(0,\varepsilon)$. From the extremal properties of the eigenvalues by the minimum-maximum principle [17, Proposition 3.2.1(ii)], from $\lim_{k\to\infty}T^T(t_k)\,M(t_k)\,T(t_k)=Q$ by (2.15), and from $\mu_i'=\lim_{k\to\infty}\mu_i(t_k)$ by (2.16) it follows that for the given $\varepsilon>0$ there exists an index $N\in\mathbb{N}$ such that

$$\begin{aligned}
\mu_i &= \max\left\{\frac{d^TQd}{\|d\|^2},\ \ d\in\mathbb{R}^r,\ d\perp d_{i+1},\dots,d_r\right\}\\
&\geq \max\left\{\frac{d^T\,T^T(t_k)\,M(t_k)\,T(t_k)\,d}{\|d\|^2}-\varepsilon,\ \ d\in\mathbb{R}^r,\ d\perp d_{i+1},\dots,d_r\right\}\\
&= \max\left\{\frac{c^TM(t_k)\,c}{\|c\|^2}-\varepsilon,\ \ c\in\mathbb{R}^m,\ c\perp c_{i+1}(t_k),\dots,c_r(t_k),\ c\in\operatorname{Im}T(t_k)\right\}\\
&\geq \mu_i(t_k)-\varepsilon
\end{aligned}$$

for all $k\geq N$ and $i\in\{1,\dots,r\}$. Hence, it follows that

$$\mu_i\geq\mu_i'\qquad\text{for every } i\in\{1,\dots,r\}. \tag{2.17}$$

The opposite inequality is the crucial part. Therefore, let $d_1(t),\dots,d_r(t)\in\mathbb{R}^m$ be orthonormal eigenvectors of $M(t)$ corresponding to the eigenvalues $\mu_1(t),\dots,\mu_r(t)$, i.e., $M(t)\,d_i(t)=\mu_i(t)\,d_i(t)$ for $i\in\{1,\dots,r\}$ and $t\in(0,\varepsilon)$. Set $D(t):=(d_1(t),\dots,d_r(t))\in\mathbb{R}^{m\times r}$. Then, by the Bolzano–Weierstrass theorem, $D(\tau_j)\to D\in\mathbb{R}^{m\times r}$ as $j\to\infty$ for a subsequence $\{\tau_j\}$ of $\{t_k\}$ with $\tau_j\searrow 0$, and with $D^TD=I_{r\times r}$, since $D^T(t)\,D(t)\equiv I_{r\times r}$ on $(0,\varepsilon)$. Moreover, by the definition, $D^T(t)\,M(t)\,D(t)=\operatorname{diag}\{\mu_1(t),\dots,\mu_r(t)\}$, so that

$$\lim_{t\to0^+}D^T(t)\,M(t)\,D(t)=Q':=\operatorname{diag}\{\mu_1',\dots,\mu_r'\}. \tag{2.18}$$

Since $D^T(t)\,M(t)\,D(t)$ is bounded as $t\to0^+$, we have by [17, Eq. (3.3.7*)] that

$$\lim_{j\to\infty}R_2^T(\tau_j)\,D(\tau_j)=R_2^TD\quad\text{ with }\quad\operatorname{Im}R_2^TD\subseteq\operatorname{Im}X,$$

and therefore $\operatorname{Im}D\subseteq\operatorname{Im}T$. Since $D^TD=T^TT=I_{r\times r}$, we obtain that

$$D=TP\quad\text{for some orthogonal matrix } P\in\mathbb{R}^{r\times r}. \tag{2.19}$$

Since $U(t)\,X^{-1}(t)$ is *nonincreasing* and (2.18) holds, we get for fixed $k\in\mathbb{N}$ (observe that $\tau_j\searrow0$ for $j\to\infty$, i.e., we get eventually $j\geq k$)

$$\begin{aligned}
Q'=\operatorname{diag}\{\mu_1',\dots,\mu_r'\}&=\lim_{j\to\infty}D^T(\tau_j)\,R_2(\tau_j)\,[S_1(\tau_j)+U(\tau_j)\,X^{-1}(\tau_j)]\,R_2^T(\tau_j)\,D(\tau_j)\\
&\geq D^TR_2\,[S_1+U(\tau_k)\,X^{-1}(\tau_k)]\,R_2^TD\\
&\overset{(2.19)}{=} P^TT^T(XS+S^*)^T\,[S_1+U(\tau_k)\,X^{-1}(\tau_k)]\,(XS+S^*)\,TP\\
&= P^TT^TS^TX^T\,[S_1+U(\tau_k)\,X^{-1}(\tau_k)]\,XSTP\\
&\overset{(2.12)}{\to} P^TT^TS^T\,(X^TS_1X+X^TU)\,STP=P^TQP\qquad\text{for } k\to\infty.
\end{aligned}$$

Hence, $Q'\geq P^TQP$. Since P is orthogonal, the matrices Q and P^TQP have the same eigenvalues. Hence, we conclude that

$$\mu_i\leq\mu_i'\qquad\text{for every } i\in\{1,\dots,r\}. \tag{2.20}$$

Therefore, by (2.16), (2.17), and (2.20) we have that the statement of part (i) holds. Now, our index theorem follows exactly in the same way as in [16, Theorem 2] or [17, Theorem 3.4.1] by using the monotonicity of $M(t)$. In particular, if $M(t)$ is nonincreasing, then the number of negative eigenvalues of $M(t)$ for positive t close to 0 is equal to the number of its eigenvalues which are negative at 0 plus the number of its eigenvalues which are zero at 0 but which are negative in the right neighborhood of 0, i.e.,

$$\operatorname{ind} M(0^+) = \operatorname{ind} Q + \operatorname{def} Q - \operatorname{def} M(0^+). \tag{2.21}$$

On the other hand, if $M(t)$ is nondecreasing, then the number of negative eigenvalues of $M(t)$ for positive t close to 0 is equal to the number of its eigenvalues which are negative at 0 plus the number of its eigenvalues which tend to $-\infty$ as $t \to 0^+$, i.e.,

$$\operatorname{ind} M(0^+) = \operatorname{ind} Q + m - r. \tag{2.22}$$

Since by the rank theorem [16, Theorem 1] or [17, Theorem 3.1.8] we have

$$\operatorname{rank} Q = 2 \operatorname{rank} T + \operatorname{rank} \Lambda + \operatorname{rank} R_2 - \operatorname{rank} X - 2m, \tag{2.23}$$

and since $\operatorname{def} Q = r - \operatorname{rank} Q$, $\operatorname{rank} T = r$, $\operatorname{rank} \Lambda = m - \operatorname{def} \Lambda$, $\operatorname{rank} X = m - \operatorname{def} X$, and $\operatorname{rank} R_2 = m - \operatorname{def} R_2$, it follows from (2.23) that

$$\operatorname{def} Q = \operatorname{def} \Lambda - \operatorname{rank} T + \operatorname{def} R_2 - \operatorname{def} X + m. \tag{2.24}$$

Hence, combining formulas (2.21), (2.24), and (2.9), we obtain for nonincreasing $M(t)$

$$\begin{aligned} \operatorname{ind} M(0^+) &= \operatorname{ind} Q + [\operatorname{def} \Lambda - \operatorname{rank} T + \operatorname{def} R_2 - \operatorname{def} X + m] - [\operatorname{def} \Lambda(0^+) + \operatorname{def} R_2] \\ &= \operatorname{ind} Q + m - \operatorname{rank} T + \operatorname{def} \Lambda - \operatorname{def} \Lambda(0^+) - \operatorname{def} X, \end{aligned}$$

while from (2.22) and $\operatorname{rank} T = r$ we get for nondecreasing $M(t)$

$$\operatorname{ind} M(0^+) = \operatorname{ind} Q + m - \operatorname{rank} T.$$

This shows that formulas (2.6) and (2.7) hold and the proof is complete. ∎

The most significant disadvantage of condition (2.4) is that it depends also on the matrix $X = X(0)$, which makes it “not really” suitable in practical applications. The following special case of Theorem 2.1 removes this disadvantage, since the crucial assumption is formulated only in terms of $R_2(t)$. Note that the result below is still more general than [16, Theorem 2] or [20, Proposition 2.5], since it allows $R_2(t)$ to be varying.

Corollary 2.3 (Index theorem)**.** *With the notation* (2.1)–(2.3) *and assumptions of Theorem 2.1, suppose that*

$$\operatorname{Im} R_2^T(t) \equiv \operatorname{Im} R_2^T \quad \textit{is constant on } [0, \varepsilon) \tag{2.25}$$

instead of (2.4)*. Then formulas* (2.6) *and* (2.7) *hold.*

Proof. The result follows from Remark 2.2(ii). ∎

The gap between the rank conditions (2.4) and the image condition (2.25) will be discussed in Remark 2.8. Next we present a two-sided index theorem, which follows from Theorem 2.1 by a simple reflection argument $t \mapsto -t$.

Corollary 2.4 (Index theorem)**.** *Assume that the relevant quantities $X(t)$, $U(t)$, $R_1(t)$, $R_2(t)$, $M(t)$, $\Lambda(t)$, $S(t)$, $S^*(t)$ and X, Λ, T, Q in Theorem 2.1 are defined on the interval $(-\varepsilon, \varepsilon)$, the right-hand limits in* (2.2) *are replaced by the corresponding limits, $X(t)$ is invertible on*

$(-\varepsilon,\varepsilon)\setminus\{0\}$, *and* (2.4) *holds on* $(-\varepsilon,\varepsilon)$. *If* $U(t)\,X^{-1}(t)$ *and* $M(t)$ *are nonincreasing on* $(-\varepsilon,0)$ *and on* $(0,\varepsilon)$, *then the equations*

$$\operatorname{ind} M(0^+) = \operatorname{ind} Q + m - \operatorname{rank} T + \operatorname{def} \Lambda - \operatorname{def} \Lambda(0^+) - \operatorname{def} X, \tag{2.26}$$

$$\operatorname{ind} M(0^-) = \operatorname{ind} Q + m - \operatorname{rank} T \tag{2.27}$$

hold, and moreover

$$\operatorname{ind} M(0^+) - \operatorname{ind} M(0^-) = \operatorname{def} \Lambda - \operatorname{def} \Lambda(0^+) - \operatorname{def} X. \tag{2.28}$$

The above result will be used in the following examples which illustrate the applicability or nonapplicability of the new index theorem.

Remark 2.5. Analogously to [16, Theorem 2] and [17, Theorem 3.4.1], the new index theorem in Corollary 2.3 can be formulated under a slightly different assumption than the monotonicity of $M(t)$ and $U(t)\,X^{-1}(t)$, namely under the condition that the matrix

$$S_1(t) + U(t)\,X^{-1}(t) \ \text{ is monotone on } \ (0,\varepsilon), \tag{2.29}$$

where $S_1(t)$ is given in (2.10). This follows by the observation that, under the assumption (2.25), condition (2.29) implies that a matrix congruent (i.e., with the same inertia) to $M(t)$ is monotone on $(0,\varepsilon)$, and then the proof of [16, Theorem 2] can be modified so that it leads to the same conclusion as in Corollary 2.3. Observe that our proof of Theorem 2.1 with the weaker assumption (2.4) instead of (2.25) uses the monotonicity of $U(t)\,X^{-1}(t)$ and also the same monotonicity of $M(t)$.

Next we provide several examples which illustrate the "optimality" of our new index theorem. First we show that Theorem 2.1 does not hold when $\operatorname{rank} R_2(t)$ is not constant.

Example 2.6. Let $m=1$ and fix $a>0$. For $t\in[0,\varepsilon)$, where $\varepsilon := 1/(2a)$, we consider the functions $R_1(t)\equiv R_1=-1$, $R_2(t)=t$, $U(t)\equiv U=1$, and $X(t)\equiv X=1/a$. Then $U(t)\,X^{-1}(t)\equiv a$ is constant (hence nonincreasing), $M(t)=at^2-t$ is decreasing on $[0,\varepsilon)$, and the rank of $R_2(t)$ is not constant on $[0,\varepsilon)$, since $R_2=0$. Moreover, $\Lambda(t)=t-1/a$ is nonzero on $[0,\varepsilon)$, $S(t)=at$ and $S^*(t)\equiv S^*=0$ on $[0,\varepsilon)$, and $Q=0$, $\Lambda=-1/a$, $S=0$. Since $\operatorname{rank} S^*=0$, we have $r=1$ and $T=1$. It follows that $\operatorname{ind} M(0^+)=1$ because $M(t)<0$ on $(0,\varepsilon)$, while $\operatorname{ind} Q=0$, $\operatorname{rank} T=1$, $\operatorname{def}\Lambda=0$, $\operatorname{def}\Lambda(0^+)=0$, and $\operatorname{def} X=0$. This shows that formula (2.6) of Theorem 2.1 does not hold ($1\neq 0$). Note that in this case the function $S_1(t)$ from (2.10) in the proof of Theorem 2.1 has the form $S_1(t)=-1/t$ for $t\in(0,\varepsilon)$ and $S_1(0)=0$, i.e., it is not continuous at 0 as required in the proof.

In the following example we show that the index theorem, i.e., Theorem 2.1 or Corollary 2.4, does not hold when $R_2(t)$ does have constant rank, but the rank of $S^*(t)$ changes. Note that in this example the image of $R_2^T(t)$ is not constant either.

Example 2.7. Let $m=2$ and $\varepsilon>0$ be given. For $\alpha\in\{-2,2\}$ and $t\in(-\varepsilon,\varepsilon)$ we define the 2×2 matrices

$$R_1(t) = \begin{pmatrix} 0 & 0 \\ \sqrt{|t|} & 1 \end{pmatrix}, \quad R_2(t) = \begin{pmatrix} 1 & -\sqrt{|t|} \\ 0 & 0 \end{pmatrix}, \quad X(t) = \operatorname{diag}\{\alpha, t\},$$

$$U(t)\equiv U = I_{2\times 2}, \quad R_1=\operatorname{diag}\{0,1\}, \quad R_2=\operatorname{diag}\{1,0\}, \quad X=\operatorname{diag}\{\alpha,0\}.$$

Then these matrices satisfy the assumptions of Corollary 2.4 with $\operatorname{rank} R_2(t) \equiv 1$ constant on $(-\varepsilon, \varepsilon)$, but with $\operatorname{rank} S^*(t)$ changing at 0. Indeed, from (2.3) we have

$$M(t) = \operatorname{diag}\{m(t), 0\} \qquad \text{with } \ m(t) := 1/\alpha + \operatorname{sgn} t,$$

$$\Lambda(t) = \begin{pmatrix} 1 & -\sqrt{|t|} \\ \alpha\sqrt{|t|} & t \end{pmatrix}, \quad S^*(t) = \begin{pmatrix} 0 & 0 \\ -\sqrt{|t|} & 0 \end{pmatrix}, \quad S(t) \equiv S = \operatorname{diag}\{1/\alpha, 0\},$$

$$\Lambda = \operatorname{diag}\{1, 0\}, \quad S^* = 0, \quad T = I_{2\times 2}, \quad Q = \operatorname{diag}\{1/\alpha, 0\}.$$

It follows that the functions $U(t)\, X^{-1}(t) = \operatorname{diag}\{1/\alpha, 1/t\}$ and $M(t)$ are nonincreasing on $(-\varepsilon, 0)$ and $(0, \varepsilon)$. Moreover, $\operatorname{ind} M(0^+) = 0$, $\operatorname{ind} M(0^-) = 1$, $r = \operatorname{rank} T = 2$, $\operatorname{def} \Lambda = 1$, $\operatorname{def} \Lambda(0^+) = 0$, and $\operatorname{def} X = 1$. Now, if $\alpha = 2$, then $\operatorname{def} Q = 0$ and equation (2.26) is satisfied $(0 = 0)$, while equation (2.27) does not hold $(1 \neq 0)$. But if $\alpha = -2$, then $\operatorname{def} Q = 1$ and equation (2.27) holds $(1 = 1)$, while (2.26) does not hold $(0 \neq 1)$. In both cases equation (2.28) is false $(-1 \neq 0)$.

Remark 2.8. In the last part of this section we provide a comparison of the rank condition (2.4) and the image condition (2.25) used in Theorem 2.1 and Corollary 2.3. In particular, we discuss the gap between (2.4) and (2.25). Of course, condition (2.25) implies (2.4), as already noted in Remark 2.2(ii) above. Next, it can be easily seen that the constancy of the rank of $R_2(t)$ in (2.4) is equivalent with (2.25) if and only if $R_2(t) \equiv 0$ or $R_2(t)$ is invertible. Moreover, the second rank condition in (2.4) is always satisfied or follows from the first one if X is invertible or if $X = 0$. Hence, the "smallest" general situation may occur when $m = 2$ and $r = \operatorname{rank} R_2(t) = \operatorname{rank} X = 1$. Below we give examples with these dimensions. One may construct, of course, similar examples in all higher dimensions.

Example 2.9. Let $m = 2$ and put $X := \operatorname{diag}\{1, 0\}$ and

$$R_2(t) := \begin{pmatrix} 1 & -t \\ 0 & 0 \end{pmatrix}, \quad R_1(t) := \begin{pmatrix} 0 & 0 \\ t & 1 \end{pmatrix}, \quad S^*(t) = (I - XX^\dagger)\, R_2^T(t) = \begin{pmatrix} 0 & 0 \\ -t & 0 \end{pmatrix}. \tag{2.30}$$

Then $R_1(t)\, R_2^T(t) \equiv 0$ is symmetric, $\operatorname{rank}(R_1(t),\ R_2(t)) \equiv 2$, and $\operatorname{Im} R_2^T(t)$ depends on t, i.e., condition (2.25) does not hold. Moreover, the first condition in (2.4) is satisfied, since $\operatorname{rank} R_2(t) \equiv 1$ is constant, but the second condition in (2.4) is violated as $\operatorname{rank} S^*(t) \equiv 1$ for $t \neq 0$ and $\operatorname{rank} S^*(0) = 0$. On the other hand, if we put

$$X := \operatorname{diag}\{0, 1\}, \quad S^*(t) = (I - XX^\dagger)\, R_2^T(t) = \operatorname{diag}\{1, 0\}, \tag{2.31}$$

then $\operatorname{rank} S^*(t) \equiv 1$ is constant. This shows that with $R_2(t)$ and X as in (2.30) and (2.31) condition (2.4) is satisfied, but condition (2.25) is not. Therefore, in this case Theorem 2.1 can be applied, while Corollary 2.3 cannot be applied.

Finally, we show that the rank conditions (2.4) and the image condition (2.25) are actually equivalent when the matrix X is taken to be arbitrary. This result describes the most exact relationship between the two conditions. Note that we do not even assume the continuity of the function $R_2(t)$ at 0.

Proposition 2.10. *Let be given an $m \times m$ matrix-valued function $R_2(t)$ for $t \in [0, \varepsilon)$ with some $\varepsilon > 0$ and put $R_2 := R_2(0)$. Then the following two conditions are equivalent:*

(i) $\operatorname{rank}(I - XX^\dagger)\, R_2^T(0^+) = \operatorname{rank}(I - XX^\dagger)\, R_2^T$ *for every $X \in \mathbb{R}^{m\times m}$, i.e., for every X there exists $\delta \in (0, \varepsilon)$ such that* $\operatorname{rank}(I - XX^\dagger)\, R_2^T(t) \equiv \operatorname{rank}(I - XX^\dagger)\, R_2^T$ *is constant on $[0, \delta)$,*

(ii) $\operatorname{Im} R_2^T(0^+) = \operatorname{Im} R_2^T$, *i.e., there exists* $\delta \in (0,\varepsilon)$ *such that* $\operatorname{Im} R_2^T(t) \equiv \operatorname{Im} R_2^T$ *is constant on* $[0,\delta)$.

Proof. We know that (ii) implies (i) by Remark 2.2(ii). Now we assume (i). Then for $X = 0$ there exists $\delta \in (0,\varepsilon)$ such that

$$\operatorname{rank} R_2(t) \equiv \operatorname{rank} R_2 \quad \text{is constant on } [0,\delta). \tag{2.32}$$

Suppose that the condition (ii) does not hold, i.e., there exists a sequence $\{t_k\}_{k=1}^{\infty} \subseteq (0,\delta)$, $t_k \searrow 0$ for $k \to \infty$, such that $\operatorname{Im} R_2^T(t_k) \neq \operatorname{Im} R_2^T$ for all $k \in \mathbb{N}$. Hence, since $[\operatorname{Im} R_2^T(t)]^{\perp} = \operatorname{Ker} R_2(t)$, we get $\operatorname{Ker} R_2(t_k) \neq \operatorname{Ker} R_2$ for all $k \in \mathbb{N}$. More precisely, $\operatorname{Ker} R_2 \not\subseteq \operatorname{Ker} R_2(t_k)$, since by (2.32) we have $\operatorname{def} R_2(t_k) = \operatorname{def} R_2$ for $k \in \mathbb{N}$. Now we put

$$X := R_2^T = R_2^T(0).$$

Then $(I - XX^{\dagger})\, R_2^T = 0$ and $\operatorname{Im}(I - XX^{\dagger}) = \operatorname{Ker} R_2$ with $\operatorname{rank}(I - XX^{\dagger}) = \operatorname{def} R_2(t_k)$ for all $k \in \mathbb{N}$ by (2.32). But since $\operatorname{Im}(I - XX^{\dagger}) = \operatorname{Ker} R_2 \not\subseteq \operatorname{Ker} R_2(t_k)$, we obtain that $R_2(t_k)\,(I - XX^{\dagger}) \neq 0$ or, by transposing, $(I - XX^{\dagger})\, R_2^T(t_k) \neq 0$ for all $k \in \mathbb{N}$. Hence,

$$0 = \operatorname{rank}(I - XX^{\dagger})\, R_2^T < 1 \leq \operatorname{rank}(I - XX^{\dagger})\, R_2^T(t_k) \quad \text{for all } k \in \mathbb{N}.$$

Upon taking $k \to \infty$ we get a contradiction with the assumption (i). Hence, condition (ii) holds, and the proof is complete. ∎

3. Applications in discrete oscillation theory

In this section we consider an important application of the index theorem in discrete oscillation theory. In particular, as in [20] we consider the discrete symplectic system

$$x_{k+1} = \mathcal{A}_k(\lambda)\, x_k + \mathcal{B}_k(\lambda)\, u_k, \quad u_{k+1} = \mathcal{C}_k(\lambda)\, x_k + \mathcal{D}_k(\lambda)\, u_k, \quad k \in [0,N]_{\mathbb{Z}}, \tag{S$_\lambda$}$$

where $\lambda \in \mathbb{R}$ is the spectral parameter, and for $k \in [0,N]_{\mathbb{Z}}$ the $n \times n$ matrix-valued functions $\mathcal{A}_k(\lambda)$, $\mathcal{B}_k(\lambda)$, $\mathcal{C}_k(\lambda)$, $\mathcal{D}_k(\lambda)$ are differentiable (hence continuous) in the variable λ. The name "symplectic system" refers to the fact that the coefficient matrix $\mathcal{S}_k(\lambda)$ of system (S$_\lambda$) is assumed to be symplectic, i.e., for all $k \in [0,N]_{\mathbb{Z}}$ and $\lambda \in \mathbb{R}$

$$\mathcal{S}_k^T(\lambda)\, \mathcal{J} \mathcal{S}_k(\lambda) = \mathcal{J}, \quad \mathcal{S}_k(\lambda) := \begin{pmatrix} \mathcal{A}_k(\lambda) & \mathcal{B}_k(\lambda) \\ \mathcal{C}_k(\lambda) & \mathcal{D}_k(\lambda) \end{pmatrix}, \quad \mathcal{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

The same property is then satisfied by the fundamental matrix of the system (S$_\lambda$). The main monotonicity assumption for the whole theory in [20] is

$$\Psi_k(\lambda) := \mathcal{J} \dot{\mathcal{S}}_k(\lambda)\, \mathcal{J} \mathcal{S}_k^T(\lambda)\, \mathcal{J} \geq 0 \quad \text{for all } k \in [0,N]_{\mathbb{Z}} \text{ and } \lambda \in \mathbb{R},$$

where the dot stands for the derivative with respect to λ. It is well known that discrete symplectic systems constitute a natural analogue of continuous time (differential) linear Hamiltonian systems, see [6, 8, 17, 19].

According to [18, Definition 1], the number of focal points of a conjoined basis $(X(\lambda), U(\lambda))$ of (S$_\lambda$) in the interval $(k, k+1]$ is given by the number

$$m_k(\lambda) := \operatorname{rank} M_k(\lambda) + \operatorname{ind} P_k(\lambda),$$

where the $n \times n$ matrices $M_k(\lambda)$ and $P_k(\lambda)$ are defined through the functions $X_k(\lambda)$, $X_{k+1}(\lambda)$, and $\mathcal{B}_k(\lambda)$. In particular, the matrix $P_k(\lambda)$ is symmetric. Roughly speaking, the number $\operatorname{rank} M_k(\lambda)$ counts the multiplicity of the focal point at $k+1$, while the number $\operatorname{ind} P_k(\lambda)$ counts the multiplicity of the focal point in the interval $(k, k+1)$. One can see from this definition that when λ varies, the change of the number of focal points in $(k, k+1]$ contains as one part the change of the index of the symmetric matrix $P_k(\lambda)$. This provides the connection of discrete oscillation theory with the index theorem. In [20], the main results are oscillation theorems which relate the number of focal points of conjoined bases of (S_λ) in $(0, N+1]$ with the number of eigenvalues which lie in the interval $(-\infty, \lambda]$. Such a result is very valuable in Sturmian theory, since it gives the number of generalized zeros of corresponding eigenfunctions.

Our generalized index theorem yields new results for associated symplectic eigenvalue problems with the Dirichlet, separated, and jointly varying endpoints, including the periodic and antiperiodic boundary conditions. This follows from the proofs of the corresponding results in [20, Sections 6–7] and [10, 22] upon replacing the condition on constant $\mathcal{B}_k(\lambda) \equiv \mathcal{B}_k$ by the assumption

$$\operatorname{Im} \mathcal{B}_k(\lambda) \ \text{ is constant in } \lambda \text{ on } \mathbb{R} \tag{3.1}$$

and applying Theorem 2.1 instead of [16, Theorem 2] or [20, Proposition 2.5]. The most interesting examples of special symplectic systems, for which we obtain new oscillation results, are those corresponding to Sturm–Liouville difference equations and linear Hamiltonian difference systems.

Example 3.1. Consider the Sturm–Liouville difference equation (1.1) with the coefficients

$$r_k(\lambda) \neq 0, \quad \dot{r}_k(\lambda) \leq 0, \quad \dot{q}_k(\lambda) \geq 0,$$

see [20, Example 7.6]. Then the corresponding oscillation theorem in [20, Theorem 6.3] with $\mathcal{B}_k(\lambda) = 1/r_k(\lambda)$ reduces to [21, Theorem 2.8]. However, the proof of the latter result did not use the index theorem. Note that the function $\mathcal{B}_k(\lambda)$ has constant image as required in (3.1).

Example 3.2. For the higher order Sturm–Liouville difference equations (1.2) with

$$r_k^{[n]}(\lambda) \neq 0, \quad \dot{r}_k^{[n]}(\lambda) \geq 0, \quad \dot{r}_k^{[i]}(\lambda) \leq 0 \quad \text{for all } i \in \{0, \dots, n-1\},$$

the corresponding oscillation theorem in [20, Theorem 6.3] with $\mathcal{B}_k(\lambda)$ given by (1.3) is new. Note that also in this case the matrix $\mathcal{B}_k(\lambda)$ has constant image as required in (3.1).

Example 3.3. The results in this section apply also to linear Hamiltonian systems

$$\Delta x_k = A_k(\lambda)\, x_{k+1} + B_k(\lambda)\, u_k, \quad \Delta u_k = C_k(\lambda)\, x_{k+1} - A_k^T(\lambda)\, u_k \quad k \in [0, N]_{\mathbb{Z}}, \tag{3.2}$$

where the $n \times n$ matrix-valued functions $A_k(\lambda)$, $B_k(\lambda)$, $C_k(\lambda)$ are differentiable and such that $B_k(\lambda)$ and $C_k(\lambda)$ are symmetric, $I - A_k(\lambda)$ is invertible with $\tilde{A}_k(\lambda) := [I - A_k(\lambda)]^{-1}$,

$$\dot{H}_k(\lambda) \geq 0, \quad H_k(\lambda) := \begin{pmatrix} -C_k(\lambda) & A_k^T(\lambda) \\ A_k(\lambda) & B_k(\lambda) \end{pmatrix}, \quad k \in [0, N]_{\mathbb{Z}}, \ \lambda \in \mathbb{R},$$

see [20, Example 7.9] and [3]. In this case, we have $\mathcal{B}_k(\lambda) = \tilde{A}_k(\lambda)\, B_k(\lambda)$ and the corresponding oscillation theorem in [20, Theorem 6.3] holds under the assumption that the matrix $\tilde{A}_k(\lambda)\, B_k(\lambda)$ has constant image in λ.

Further applications of the above new oscillation theorems for systems (S_λ), (3.2) and equations (1.1), (1.2) can be derived in connection with associated discrete quadratic functionals and Riccati equations and inequalities. We refer to [1, 4, 11–14] for an inspiration.

Acknowledgements

The first author is grateful to the Masaryk University for the hospitality provided when conducting a part of this project. This research was supported by the Czech Science Foundation under grant P201/10/1032.

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