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The Impact of Inflation Risk on Financial Planning and Risk-Return Profiles

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Abstract

The importance of funded private or occupational old age provision is expected to increase due to the demographic changes and the resulting problems for government-run pay-as-you-go systems. Clients and advisors therefore need reliable methodologies to match offered products with clients' needs and risk appetite. In Graf et al. (2012) the authors have introduced a methodology based on stochastic modeling to properly assess the risk-return profiles – i.e. the probability distribution of future benefits – of various old age provision products. In this paper, additionally to the methodology proposed so far, we consider the impact of inflation on the risk-return profile of old age provision products. In a model with stochastic interest rates, stochastic inflation and equity returns including stochastic equity volatility, we derive risk-return-profiles for various types of existing unit-linked products with and without embedded guarantees and especially focus on the difference between nominal and real returns. We find that typical “rule of thumb” approximations for considering inflation risk are inappropriate and further show that products that are considered particularly safe by practitioners because of nominal guarantees may bear significant inflation risk. Finally, we propose product designs suitable to reduce inflation risk and investigate their risk-return profile in real terms.

Keywords: Stochastic Modeling, Financial Planning, Inflation, Product Design.

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1 Introduction

The demographic transition constitutes a severe challenge for government-run pay-as-you-go pension systems in many countries. Therefore, the importance of funded private and/or occupational old age provision has been increasing and will very likely continue to increase. Competing for clients' money, product providers (e.g. life insurers and asset managers) have come up with a variety of "packaged products" for old age provision that often consist of equity investments combined with certain nominal maturity value and/or minimum death benefit death guarantees. Graf et al (2012) provide an overview over the most important types of such products and also show that information that is typically provided by product providers is not sufficient to assess and to compare the downside risk and the return potential (the "risk-return profile") of different products. They also argue that results provided in the literature on financial planning (cf. the literature overview given in their paper) can often not be applied in practice. Therefore, they introduce a methodology suitable to be implemented in practice to assess the risk-return profile of such products. In a model with stochastic interest rates, equity returns and stochastic equity volatility, they derive risk-return-profiles for various products and investigate the impact of premium payment mode and investment horizon.

Graf et al. (2012) are solely concerned with analyzing nominal returns. However, the relevant criterion should not be the number of units of some currency that is provided as a benefit but rather the purchasing power of the benefit. Hence, an appropriate assessment of inflation risk should be included in their analyses.

This is currently of particular interest since recent quantitative easings issued by several governments in order to stimulate capital markets to overcome the recent subprime and current Euro debt crisis have revived the issue of a potential inflation and its impact on financial planning. For assessing the protection against inflation risk of different asset classes, the empirical analyses of Amenc et al. (2009) provide a good starting point. Further, recent academic literature deals with portfolio optimization problems explicitly taking inflation risk into account: Brière and Signore (2012) solve a portfolio allocation problem in a risk-reward setting focusing on real (i.e. inflation adjusted) investment return whereas Weiyyin (2012) solves the asset allocation problem applying the expected utility approach and taking inflation into account.

However, these important theoretical results are often too complex for a practical use by clients and/or advisors and highly depend on the particular choice of a utility function. Furthermore, the results are not applicable in practice because assumptions about available

products are often oversimplifying. For example, charges are usually not considered at all and premium payments are typically reduced to single premiums. Additionally, a practical implementation of such approaches would often require a continuous management of clients' accounts which is often complex, not feasible for rather small contract volumes or might in some countries result in tax disadvantages upon each transaction. Therefore, in practice often "packaged products" where certain strategies are implemented that do not require any action on the client's side during the lifetime of the product, are offered and many financial advisors try to find the most suitable product out of a variety of such products for each client.

In the present paper, we include inflation risk in the framework suitable for practical implementation that has been proposed by Graf et al. (2012) and particularly analyze the difference between nominal and real returns observed. We especially focus on common old age provision products that are considered particularly safe by practitioners because of certain investment guarantee on a nominal basis. Finally, we propose adjustments to the existing products and investigate their impact.

In the academic literature, three approaches for a stochastic modeling of inflation are common. The probably most common approach usually applied in pricing is given by the "Jarrow-Yildirim" approach. Among others², Jarrow and Yildirim (2003) propose a model based on the idea of linking nominal and real units of currency with a foreign exchange approach. They derive the dynamics of some consumer price index (CPI) along with the instantaneous nominal and real interest rates in a Heath-Jarrow-Morton framework. The CPI is then interpreted as the exchange rate between the nominal (i.e. domestic) and real (i.e. foreign) "currencies". Additionally, e.g. Belgrade et al. (2004) and Mercurio (2005) introduce alternative approaches, proposing the use of market models based on traded inflation derivatives similar to the popular market models that are available for interest rate modeling. Note that both, the Jarrow-Yildirim and the market model approach are primarily designed for pricing inflation-linked derivatives (under some risk-neutral probability measure) and may therefore not directly be applied to a (real-world) analysis under an objective probability measure. Finally, e.g. Ahlgrim et al. (2005) develop an economic scenario generator capable of simulating a variety of economic variables including inflation rates by using a one-factor diffusion model designed for analyses in the actuarial sector. Since we are also interested in real-world analyses, we adopt the approach introduced by Ahlgrim et al. (2005) in the following.

² See e.g. Beletski and Korn (2006)

The remainder of this paper is organized as follows. Section 2 introduces the products that are analyzed in a first step. We particularly investigate their nominal and real returns assuming the financial model as introduced in Section 3. Section 4 then provides a quantitative assessment of the resulting nominal and in particular real risk-return profiles. Since we find that the products introduced in Section 2 (including products with nominal investment guarantee) bear significant exposure to inflation risk, we propose product modifications that might be suitable to reduce inflation risk in Section 5. Section 6 carries out some sensitivity analyses to the obtained results and Section 7 finally concludes.

2 Standard Products

In the first part of our analyses, we consider several product types that are (sometimes in different variants) common in retirement planning and offered by various financial institutions such as banks, insurers or asset managers in many countries. The base contracts described in this section are similar to those analyzed in Graf et al. (2012).³ We distinguish products with and without embedded nominal investment guarantees and consider products with single premium P and with regular monthly premium payment P and a term of T years.⁴ The following fee structure is applied to all products:

- Premium proportional charges $\beta \cdot P$ reduce the amount invested to $(1 - \beta) \cdot P$.
- Account proportional charges γ – quoted as an annual fee – are deducted on a monthly basis from the client's account.
- Additionally, fund management charges c (also quoted as an annual charge but deducted daily) are applied within mutual funds if such funds are used in the packaged product. Additional guarantee charges may apply for the products with embedded investment guarantee (see below).

With A_t denoting the client's account value at time t and $Perf_{t, t+\frac{1}{12}}$ denoting the performance of the considered products from t to $t+\frac{1}{12}$, we obtain $A_0 = (1 - \beta) \cdot P$ and define the client's account value (immediately before the beginning of the next month) as

$$A_{(t+\frac{1}{12})-} = A_t \cdot Perf_{t, t+\frac{1}{12}} \cdot (1 - \gamma)^{\frac{1}{12}}.$$

³ The following description of the products closely follows Graf et al. (2012) which contains more details.

⁴ Although we refrain from presenting detail results for regular contributions in Sections 4 and 5, we still state the corresponding formulae.

Further, the account value at the beginning of the month is then given as $A_{(t+\frac{1}{12})} = A_{(t+\frac{1}{12})-} + (1 - \beta) \cdot P$ in case of regular contributions and $A_{(t+\frac{1}{12})} = A_{(t+\frac{1}{12})-}$ in case of a single contribution.

2.1 Mutual fund and fixed income investment

In these simple products, the client's account value A_t is either completely invested in an equity fund or in a fixed income instrument modeled as a zero-coupon bond with maturity T where $p(t, T)$ denotes the zero-bond's price at time t .

2.2 Products with nominal investment guarantee

The following products equipped with a "money back guarantee" provide the guarantee that at least the client's contributions are paid back at maturity.⁵ However, the way of generating this guarantee varies throughout the considered products: We let G_t denote the so-called guarantee basis at time t which is however only valid at the contract's maturity T . We let $G_t = P \forall t$ for the single and $G_t = \begin{cases} ([t \cdot 12] + 1) \cdot P, & t < T \\ T \cdot 12 \cdot P, & t = T \end{cases}$ for the regular contribution case.

2.2.1 Static option based product

For this product, the premium is invested into an underlying fund (in our numerical analyses the introduced equity fund). Additionally, a guarantee is provided by purchasing a suitable option that covers for losses of the fund value at maturity. As in practice, e.g. within variable annuity contracts, to finance this option, an account proportional guarantee fee⁶ g – quoted as an annual charge – is deducted from the client's account (additionally to the fees introduced above). This results in

$$A_{(t+\frac{1}{12})-} = A_t \cdot Perf_{t, t+\frac{1}{12}} \cdot (1 - \gamma)^{\frac{1}{12}} \cdot (1 - g)^{\frac{1}{12}}.$$

2.2.2 Zero plus underlying

This rather trivial but in many markets highly relevant product consists of a riskless asset (in our approach a zero-bond⁷) and a risky financial instrument (in our approach the above

⁵ Note, it is very common in the old age provision market to offer products with 100% guarantee of the contributions made. Hence, we do not consider products with different guarantee levels.

⁶ g is fixed throughout the contract's term at outset and not adjusted later on. The product provider typically invests the guarantee fee in some derivative security (or hedge portfolio) on the considered fund to hedge the guarantee. Since we focus on analyses from a client's perspective, this is not further investigated in this paper.

⁷ We ignore default risk in our model.

equity fund). Whenever new contributions enter the contract, the allocation in riskless and risky asset for the whole investment portfolio is determined as follows:

$$\begin{aligned} riskless_t &= \min(A_t, F_t) \\ risky_t &= A_t - riskless_t \end{aligned}$$

where $F_t := G_t \cdot \frac{p(t,T)}{(1-\gamma)^{T-t}}$ defines the so-called floor. Note, in case $A_t < F_t$ above constraint ensures that at most the currently available amount A_t is invested in the riskless asset and the product can be “underhedged”, i.e. the product provider’s loss is only realized at the contract’s maturity.⁸

2.2.3 Client-individual constant proportion portfolio insurance (iCPPI)

In this product, the well-known CPPI-algorithm⁹ is applied on a client-individual basis. In theory, the asset allocation of CPPI-products is rebalanced continuously according to some given rule. In practice however, such re-allocations can only be applied at certain trading dates. At each such rebalancing time t (in our numerical as typically also in practice: daily) the provider determines the asset allocation for each client’s individual account by

$$\begin{aligned} risky_t &= \max(0, \min(A_t, m \cdot (A_t - F_t))) \\ riskless_t &= A_t - risky_t \end{aligned}$$

where m denotes the multiplier¹⁰. Hence, m times the so called cushion $(A_t - F_t)$ is invested in the risky asset but (due to the borrowing constraint that is typically included in old age provision contracts) no more than A_t .¹¹

Obviously in practice – when continuous rebalancing is impossible – the product provider faces two sources of risk within a CPPI structure: First, the risky asset might lose more than $\frac{1}{m}$ during one period (this risk is often referred to as gap risk or overnight risk). Second, the floor might have changed within one period due to interest rate fluctuations. Since we

⁸ $F_t > A_t$ is possible due to charges and (in the considered iCPPI product below) also due to fluctuations in interest rates and equity.

⁹ Cf. Black and Perold (1992).

¹⁰ Obviously, for a multiplier of 1, the zero plus underlying product and the iCPPI-product coincide.

¹¹ For an analytical treatment of CPPI strategies without borrowing constraint compare e.g. Black and Perold (1992). As a consequence of the borrowing constraint, the analytical tractability is lost. We still chose to analyze the products actually offered in the market since products without a borrowing constraint may behave significantly different than products with borrowing constraint.

perform analyses from a client's perspective, we do not investigate how the product provider deals with these risks¹² and assume a flat additional charge on the risky asset within the iCPPI product instead.¹³

2.3 Inflation-linked products

Our numerical analysis in Section 4 will show that the considered products, especially products with nominal guarantees, come with a significant downside risk in real term. Therefore, in Section 5 we will introduce some modifications of the considered products that might be suitable to reduce inflation risk. Basically¹⁴, on the one hand we change the calculation of the floor F_t by taking into account the inflation accrued so far and some estimate for future inflation and on the other hand we analyze the impact of using inflation-linked zero-bonds instead of "standard" zero-bonds as risk-free asset.

3 Financial model

We start with an introduction of the (real-world) asset model used for our analysis where we add inflation risk to the model introduced in Graf et al. (2012) who consider a slightly modified version of the Heston model (cf. Heston (1993)) for stock markets and the Cox-Ingersoll-Ross model (cf. Cox et al. (1985)) for interest rate markets. We add inflation by means of the Vasiček-model (cf. Vasiček (1977)) which is typically used for modeling the term structure of interest rates. As already mentioned, this approach coincides with the economic scenario generator as introduced by Ahlgrim et al. (2005).

Let $(\Omega, \mathcal{F}, \mathfrak{F}, \mathcal{P})$ be a filtered probability space equipped with the natural filtration $\mathfrak{F} = (\mathcal{F}_t)_t = \left(\sigma(W^S(z), W^V(z), W^r(z), W^i(z), z \leq t) \right)_t$ generated by (correlated) \mathcal{P} – Brownian Motions $W^S(t), W^V(t), W^r(t)$ and $W^i(t)$. Further, let $r(t)$ denote the (nominal) short-rate, $i(t)$ the annualized rate of inflation and $S(t)$ the equity's spot price at time t . The \mathcal{P} – dynamics of the asset model are then given by

$$\begin{aligned} dr(t) &= \kappa_r(\theta_r - r(t))dt + \sigma_r\sqrt{r(t)}dW^r(t) \\ di(t) &= \kappa_i(\theta_i - i(t))dt + \sigma_i dW^i(t) \\ dS(t) &= S(t) \left((r(t) + \lambda_S)dt + \sqrt{V(t)}dW^S(t) \right) \end{aligned}$$

¹² Cf. Graf (2012) and references therein for more details.

¹³ Note, there also exist products for old age provision where CPPI algorithms are applied in mutual funds instead.

¹⁴ Details and respective formulae are provided in Section 5.

$$dV(t) = \kappa_V(\theta_V - V(t))dt + \sigma_V dW^V(t)$$

where λ_S denotes the equity risk premium. Further, the (instantaneous) correlation of the underlying Brownian Motions¹⁵ is given as

$$\Sigma = \begin{pmatrix} 1 & \rho_{ir} & \rho_{is} & \rho_{iv} \\ \rho_{ri} & 1 & \rho_{rs} & \rho_{rv} \\ \rho_{si} & \rho_{sr} & 1 & \rho_{sv} \\ \rho_{vi} & \rho_{vr} & \rho_{vs} & 1 \end{pmatrix}.$$

Graf et al. (2012) show how a potential pricing measure Q (i.e. a risk-neutral probability measure) can be introduced in this setting. Within this setting, zero-bond prices $p(t, T + t)$ with time to maturity T at time t are given by $p(t, T + t) = A(T) \exp(-B(T)r(t))$ with $A(T) =$

$$\left[\frac{2 \cdot h \cdot \exp(\kappa_r + h) \cdot \frac{T}{2}}{(\kappa_r + h) \cdot (\exp(h \cdot T) - 1) + 2 \cdot h} \right]^{\frac{2\kappa_r \theta_r}{\sigma_r^2}}, B(T) = \frac{2 \cdot (\exp(h \cdot T) - 1)}{(\kappa_r + h)(\exp(h \cdot T) - 1) + 2 \cdot h} \text{ where } h = \sqrt{\kappa_r^2 + 2 \cdot \sigma_r^2}.$$

For the sake of consistency to our previous work, interest rate and equity parameters are directly adopted from Graf et al. (2012) and summarized in Table 1.

κ_r	θ_r	σ_r	$r(0)$	κ_V	θ_V	σ_V	$V(0)$	ρ_V	λ_S
20%	4.5%	7.5%	4.5%	475%	$(22\%)^2$	55%	$(22\%)^2$	-57%	3%

Table 1: Capital market parameters (without inflation)

Within this work, the parameters for the inflation model have been developed by applying a maximum likelihood approach as e.g. proposed by Sørensen (1997) to data of the German Consumer Price Index (CPI) provided by the Deutsche Bundesbank.¹⁶ The Index exists since the monetary reform in Germany in 1948 and is available for monthly and annual data. As already analyzed e.g. in Ahlgrim et al. (2005), disruptions in monthly data might lead to overstated mean reversion strength and volatility parameters of the inflation rate process. Hence, we derive the parameters by considering annual averages of the respective CPI. Due to the rather high volatility of the annual CPI in the first years after World War II, we exclude these observations and concentrate on the time series reflecting the period from 1952 to 2010. Results of the Maximum-likelihood estimates are summarized in Table 2.

¹⁵ Note, that the processes $r(t)$, $S(t)$ and $i(t)$ are potentially correlated differently than their Brownian motions.

¹⁶ The considered time series can be found under: http://www.bundesbank.de/statistik/%20statistik_zeitreihen.php and signature UJFB99.

κ_i	θ_i	σ_i	$i(0)$
20%	2%	1%	2%

Table 2: Inflation rate parameters

The correlations were estimated using a discretization approach where we obtained $\rho_{ir} = \rho_{ri} = 33\%$ and $\rho_{is} = \rho_{si} = -15\%$. For consistency reasons with the results derived in Graf et al. (2012) we assume $\rho_{sr} = \rho_{rs} = 0$. The Variance process is assumed to be independent of the interest rate and the inflation rate.

We now define the consumer price index $I(t)$ at time t as $I(t) = e^{\int_0^t i(u) du}$ with $I(0) = 1$. Further, an inflation-linked zero-bond issued at time s with time to maturity T is defined as derivative on the inflation rate paying the relative increase $\frac{I(T)}{I(s)}$ of the underlying price index on a notional of 1. We denote its price at time t by $p_{I,s}(t, T)$.

In our numerical analyses, we approximate $p_{I,s}(t, T)$ using a Vasiček-type interest rate process and the inflation rate process as introduced above. This allows for a quick pricing and hence for a broad quantitative simulation study outlined in more detail in Appendix A.

4 Inflation adjusted risk-return profiles of the standard products

In this section we analyze the impact of inflation risk on risk-return profiles of the products introduced in Section 2. We analyze nominal and real internal rate of returns where real returns are set as $\text{IRR}_{\text{Real}} = \frac{1 + \text{IRR}_{\text{Nominal}}}{1 + \text{InflRate}} - 1$ where InflRate gives the (annualized) inflation rate on the premiums invested.¹⁷ Section 4.1 first treats the case of a single contribution to the products whereas Section 4.2 then gives some results for the case of regular contributions.

Throughout the quantitative sections, we assume a term to maturity of $T = 30$ years and use $\beta = 5\%$, $\gamma = 0.5\% \text{ p.a.}$, $c = 1.3\% \text{ p.a.}$ The static option based products' guarantee fee is taken from Graf et al. (2012) as $g = 0.43\% \text{ p.a.}$ for a single premium and $g = 0.81\% \text{ p.a.}$ for regular contributions. Further, for the iCPPI product we set $m = 4$ and apply an additional crash- protection fee of $0.2\% \text{ p.a.}$

¹⁷ This definition is in line with most academic papers treating inflation risk and is e.g. introduced in Ibbotson and Sinquefeld (1976) (an academic outlet of the Ibbotson investment reports).

4.1 Single contribution

We start with analyzing a single contribution to the products introduced in Section 2. Figure 1 gives the empirical frequency distributions of internal rates of return applying 50,000 trajectories.

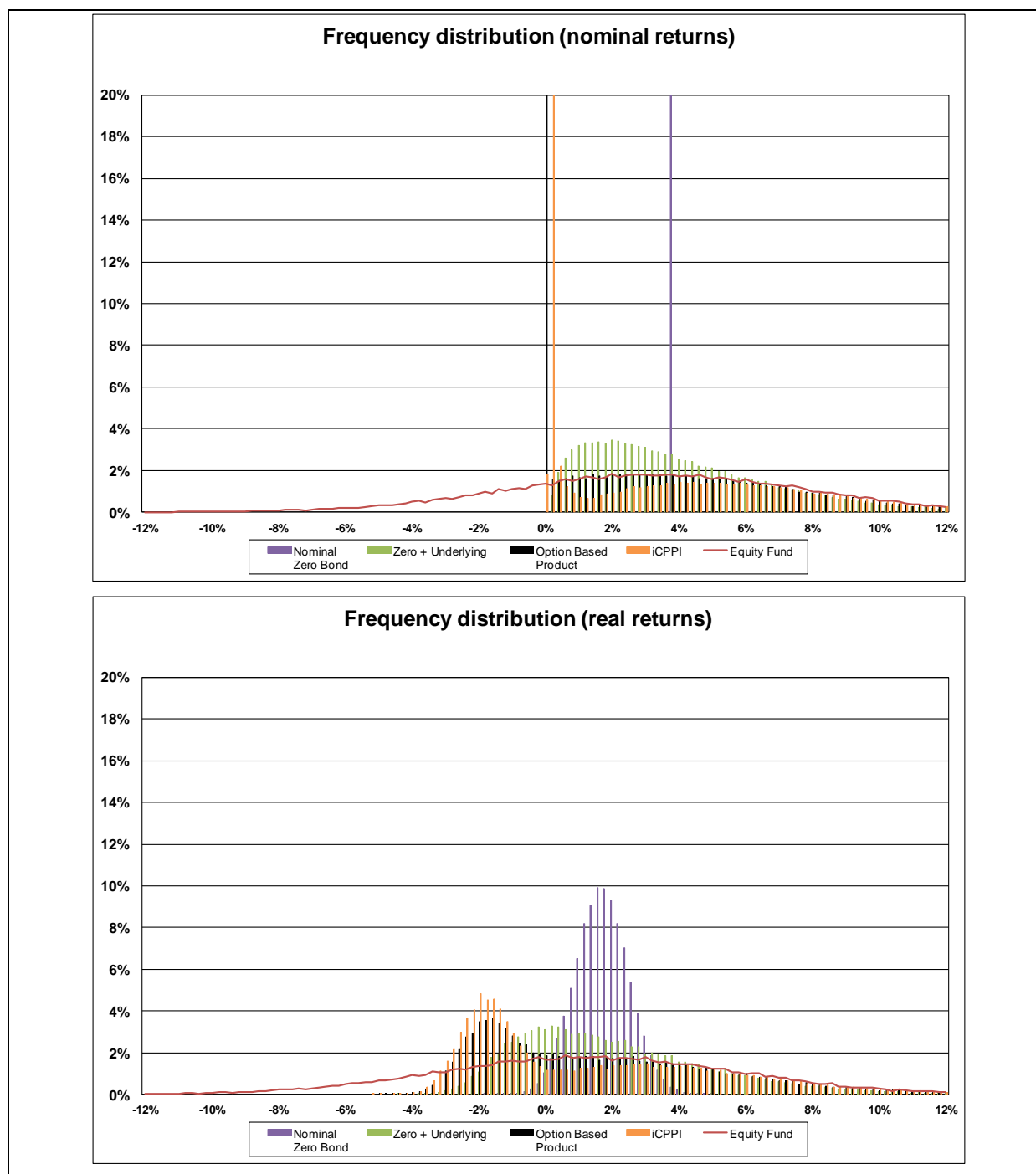


Figure 1: Empirical nominal (upper) and real (lower) rates of return - single contribution

Next, Table 3 and Table 4 summarize some key statistics derived from the empirical nominal and real returns respectively.

	Nominal Zero Bond	Zero + Underlying	iCPPI	Option Based Product	Equity Fund
5%	3.68%	0.58%	0.00%	0.00%	-4.08%
25%	3.68%	1.82%	0.00%	0.00%	0.26%
Median	3.68%	3.39%	2.07%	2.80%	3.23%
75%	3.68%	5.57%	5.89%	5.81%	6.26%
95%	3.68%	9.29%	10.30%	10.07%	10.54%
Expected return	3.68%	5.31%	5.61%	5.57%	5.93%
P(IRR<0%)	0.00%	0.00%	0.00%	0.00%	23.25%
P(IRR<2%)	0.00%	28.20%	49.69%	42.89%	39.14%
Expected shortfall	0.00%	0.00%	0.00%	0.00%	46.56%
CTE 95	3.68%	0.37%	0.00%	0.00%	-5.63%

Table 3: Key figures nominal returns – single premium

	Nominal Zero Bond	Zero + Underlying	iCPPI	Option Based Product	Equity Fund
5%	0.29%	-1.79%	-2.91%	-2.75%	-6.14%
25%	1.08%	-0.23%	-1.86%	-1.49%	-1.80%
Median	1.62%	1.41%	0.01%	0.77%	1.20%
75%	2.16%	3.59%	3.85%	3.78%	4.22%
95%	2.96%	7.31%	8.27%	8.05%	8.51%
Expected return	1.72%	3.32%	3.61%	3.57%	3.93%
P(IRR<0%)	2.10%	28.66%	49.93%	42.98%	39.48%
Expected shortfall	7.96%	25.73%	40.96%	37.68%	52.27%
CTE 95	-0.01%	-2.27%	-3.26%	-3.14%	-7.69%

Table 4: Key figures real returns – single premium

First, it is worthwhile noting that due to the chosen modeling approach, estimated nominal returns of the considered products are similar to the nominal returns observed by Graf et al. (2012): The iCPPI product and the option based product manage to combine guarantees with potential for rather high returns. However, at the same time they show quite a lot of probability mass at a nominal return of 0% or slightly above. The zero plus underlying product shows a somewhat more balanced risk-return profile. The zero bond investment, on a nominal basis does not show any randomness at all.

When analyzing the same products' real returns, the impact of inflation risk becomes visible: The zero bond investment (obviously) shows some volatility in real returns. While the nominal return was fixed at 3.68%, real returns deviate between 0.29% and 2.96% in 90% of the observed cases. With a probability of 2.1%, the real return even becomes negative.

An approximation often used in practice for assessing real rates of return is subtracting some assumed long-term average for the rate of inflation (e.g. 2% *p.a.*) from the nominal returns.

Comparing the results in Table 3 and Table 4, this approximation actually may deliver appropriate results when certain key figures such as expected returns are considered, but may also massively fail when e.g. risk measures are analyzed. This can be seen by comparing e.g. the probability that nominal returns fall below 2% with the probability of a negative real return. Further, we will show in Section 5 that even for the key figures where the approximation might work well for the above products, the approximation completely fails when applied to products where the investment is linked to inflation.

Considering products equipped with nominal investment guarantees, we find a significant exposure to inflation risk. The products with the highest nominal upside potential come with a significant point mass at a nominal rate of return of 0% *p.a.*, i.e. generally losses in real terms. However, note that the empirical distribution of real rates of return does not yield a point mass at -2% *p.a.* which would be implied by the above approximation, but reveals the random characteristics of inflation.

Summarizing, considering inflation in risk-return profiles uncovers important characteristics of the considered products and especially challenges the perceived "safety" of products issued with nominal guarantees.

4.2 Regular contributions

We also performed analyses for all considered products with regular monthly premium payment. The difference in nominal risk return profiles between products with single and regular premiums is explained in detail in Graf et al. (2012). These differences remain in place (e.g. the point mass at a 0% nominal return is higher for regular premiums than for a single premium). This results in some of the effects in real rates described above being even more pronounced for regular premium contracts. However, we observe no structurally different effects and hence refrain from a detailed description of the results.

However, note that investing in the considered fixed income asset on a regular basis results in a non-degenerated distribution of (both, nominal and real) internal rates of return.

5 Inflation-Linked Product Designs

Section 4 showed that the considered products including products with nominal guarantees come with a significant downside risk in real terms. Hence, in this section we introduce possible product modifications that might be suitable to reduce inflation risk. Investing in an inflation-linked zero-bond as introduced in Section 3 may eliminate inflation risk if real interest rates exceed the product provider's charges, however does not provide any

additional upside potential. Nevertheless, we include this type of investment in the following analyses.

Corrigan et al. (2011) propose ideas on issuing “Variable Annuity type” guarantees providing inflation risk protection by some form of an additional option on the mutual fund investment as introduced above. Further, they derive fair (i.e. risk-neutral) prices of these types of guarantees. Their results indicate that a complete option-based inflation risk protection may come at a rather high cost. Hence, full coverage of inflation risk by means of an option seems very expensive resulting in products with rather limited upside potential. To the best of our knowledge no research has been done with respect to modifying the products introduced above in order to cope with inflation risk. However, Fulli-Lemaire (2012) introduce some hybrid trading strategy taking into account inflation estimates and analyze their potential using historical backtesting and block bootstrapping techniques.

In what follows, we focus on modifying the iCPPI and the zero plus underlying products in order to provide some reduction (not necessarily a complete elimination) of inflation risk. Our modifications are based on two basic ideas: 1) incorporating some estimate for future inflation in the guarantee basis which then impacts the calculation of the floor F_t and hence results in a different asset allocation as compared to the original products and 2) using a different asset than nominal zero-bonds as a “risk-free asset”, e.g. an inflation-linked zero-bond, and calculate the floor accordingly (see below). Similar to Section 2, we state the necessary formulae for both single and regular contributions, however concentrate on quantitative analyses of a single contribution and only briefly comment on the results for regular contributions where additional insights are gained.

Note that vis-a-vis the client, the modified products come *without* any nominal or real investment guarantee. Nevertheless, we still use the terms guarantee basis and floor as introduced in Section 2.

5.1 Modification of the nominal guarantee basis G_t

Our first approach is a modification of the nominal guarantee basis G_t by allowing for inflation. At time t the modified guarantee basis G_t consists of the realized inflation on the premiums until time t (which is a \mathcal{F}_t -measurable random variable determined by $I(t)$ as introduced in Section 3) and some estimate $\tilde{i}(t)$ for the future rate of inflation within $(t, T]$.

We set

$$G_t = P \cdot \frac{I(t)}{I(0)} \cdot (1 + \tilde{i}(t))^{T-t} \quad \text{when a single premium is considered and}$$

$$G_t = \begin{cases} \left(\sum_{i=0}^{[t \cdot 12]} P \cdot \frac{I(t)}{I(\frac{i}{12})} \right) (1 + \tilde{i}(t))^{T-t}, & t < T \\ \sum_{i=0}^{T \cdot 12 - 1} P \cdot \frac{I(T)}{I(\frac{i}{12})}, & t = T \end{cases} \quad \text{when regular contributions are in place.}$$

We consider two different approaches for estimating $\tilde{i}(t)$: In the *historic* approach, we use the realized rate of inflation until time t as an estimate for the future rate of inflation as well. In the *market* approach, we use the market's expectation on the future rate of inflation until maturity T .

In the historic approach we obviously obtain $\tilde{i}(t) = \left(\frac{I(t)}{I(0)} \right)^{\frac{1}{t}} - 1$ for the single premium case and $\tilde{i}(t) = IRR \left((P, \dots, P), \sum_{i=0}^{[t \cdot 12]} P \cdot \frac{I(t)}{I(\frac{i}{12})} \right)$ in the regular premium case where $IRR(\mathbf{x}, y)$ gives the internal rate of return of a vector of contributions \mathbf{x} and a corresponding benefit y .

In the market approach, we instead use the market's expectation on the future rate of inflation implied by inflation-linked derivatives (e.g. inflation-linked bonds or inflation swaps). According to Corrigan et al. (2011) and e.g. a report by Kerkhof (2005), the market for zero-coupon inflation swaps is amongst the most liquid inflation-linked derivatives markets. In our model setup, the inflation swap issued at time t delivers the relative increase $\frac{I(T)}{I(t)}$ at maturity T against the fixed rate K on a notional of 1. No arbitrage arguments then yield (cf. e.g. Mercurio (2005)) the fair swap rate K to fulfil $(1 + K)^{T-t} = \frac{p_{I,t}(t, T)}{p(t, T)}$.¹⁸ Hence, the fair swap rate is (without any model assumption) uniquely determined by the prices of nominal and inflation-linked zero-bonds. Further note that the swap rate K in general not only shows the market's expectation for the future rate of inflation but also includes various additional risk premiums such as e.g. liquidity or credit risk. Hence, these additional risk premiums may be filtered out to obtain a "best estimate" of the market's inflation expectation as e.g. done in Schulz and Stapf (2009). Since we do not model any additional risk factors, we extract the market's inflation expectation directly from K and set $\tilde{i}(t) = K$.¹⁹

Of course, the historic approach is easier implemented in practice, since no additional data is required. However, if significant changes in the (future) rate of inflation occur, e.g. due to a

¹⁸ Note, the swap rate K is usually directly quoted by brokers, e.g. on Bloomberg.

¹⁹ Within the Appendix we show how prices $p_{I,t}(t, T)$ of the inflation-linked zero-bond are computed in our work and how we are hence able to derive an estimate for K .

change in the monetary policy, the historic approach only reacts with a certain time-lag whereas the market approach should be able to pick up this change rather quickly.

Based on the modified guarantee basis, the allocation mechanisms for the iCPPI and the zero plus underlying remain unchanged as described in Section 2. In particular, in this approach, the products still use nominal zero-bonds as riskless assets.

5.2 Use of inflation-linked zero-bonds instead of nominal zero-bonds

Neglecting liquidity issues that may arise in a practical application, in our second approach, we use inflation-linked bonds as “safe assets” in the iCPPI and zero plus underlying products introduced in Section 2. For ease of notation, we assume that the product provider solely invests in the inflation-linked zero-bond issued at time 0.

Similar with Section 2, let G_t denote the guarantee base at time t . However, here we identify G_t as the number of units that have to be invested in the considered inflation-linked zero-bond in order to provide the inflation protection for the invested premium(s). If a premium P is contributed to the contract at time t , $P \cdot \frac{I(0)}{I(t)}$ units of above inflation-linked zero-bond deliver $\left(P \cdot \frac{I(0)}{I(t)}\right) \left(\frac{I(T)}{I(0)} \cdot (1 - \gamma)^{T-t}\right) = P \cdot \frac{I(T)}{I(t)} \cdot (1 - \gamma)^{T-t}$ which exactly gives the premium P after allowing for inflation. Hence, we set $G_0 = G_t = P \cdot \frac{I(0)}{I(t)} \forall t$ for single premium contracts. For regular premium contracts, we set $G_0 = P \cdot \frac{I(0)}{I(t)}$ and $G_t = G_{t-\frac{1}{12}} + P \cdot \frac{I(0)}{I(t)}$ at every monthly premium payment date t . The floor F_t is consequently calculated as $F_t = G_t \cdot \frac{p_{I,0}(t,T)}{(1-\gamma)^{T-t}}$. The asset allocation of the zero plus underlying and the iCPPI products then follow exactly the formulae introduced in Section 2, but using the floor F_t introduced in this section and the inflation-linked zero-bond as riskless asset.

Note, that this product is somehow similar to the product introduced in Section 5.1 applying a market-based approach, since we obtain

$$\begin{aligned} F_t &= G_t \cdot \frac{p(t,T)}{(1-\gamma)^{T-t}} = P \cdot \frac{I(t)}{I(0)} (1 + \tilde{i}(t))^{T-t} \cdot \frac{p(t,T)}{(1-\gamma)^{T-t}} = P \cdot \frac{I(t)}{I(0)} \cdot \frac{p_{I,t}(t,T)}{p(t,T)} \cdot \frac{p(t,T)}{(1-\gamma)^{T-t}} \\ &= P \cdot \frac{I(t)}{I(0)} \cdot p_{I,t}(t,T) \cdot \frac{1}{(1-\gamma)^{T-t}} = P \cdot p_{I,0}(t,T) \cdot \frac{1}{(1-\gamma)^{T-t}} \\ &= G_t \cdot p_{I,0}(t,T) = F_t \end{aligned}$$

when a single contribution is considered. Similar derivations are possible for regular contributions. Hence, both products allocate the same amount of (nominal) money into the riskless asset but assume a different riskless asset.

5.3 Results for single premium contracts

Figure 2 and Figure 3 show the frequency distribution of the return of the modified products on a nominal and real basis, respectively.

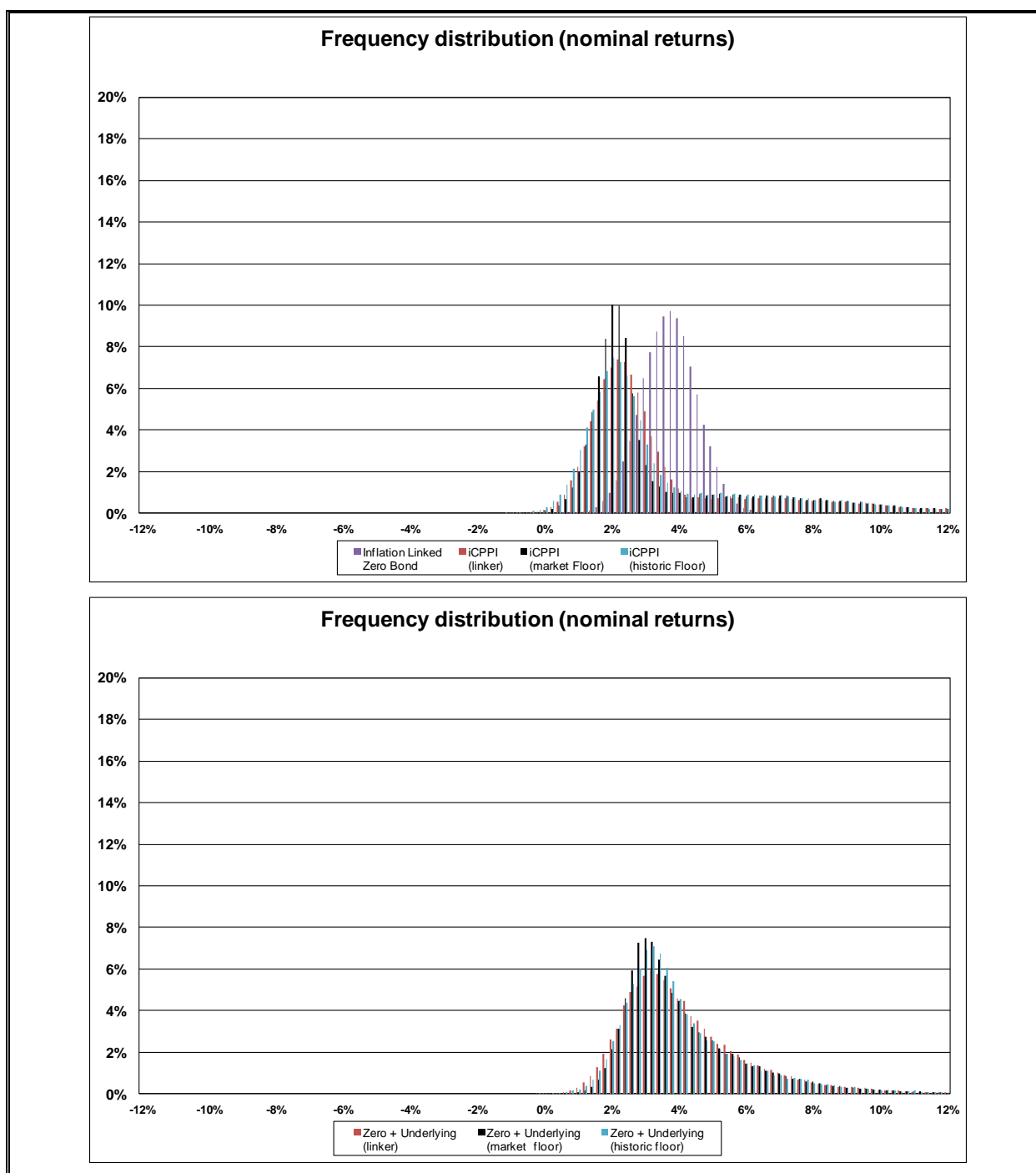


Figure 2: Empirical nominal rates of return – iCPPI / inflation-linked zero (upper) and zero plus underlying (lower)

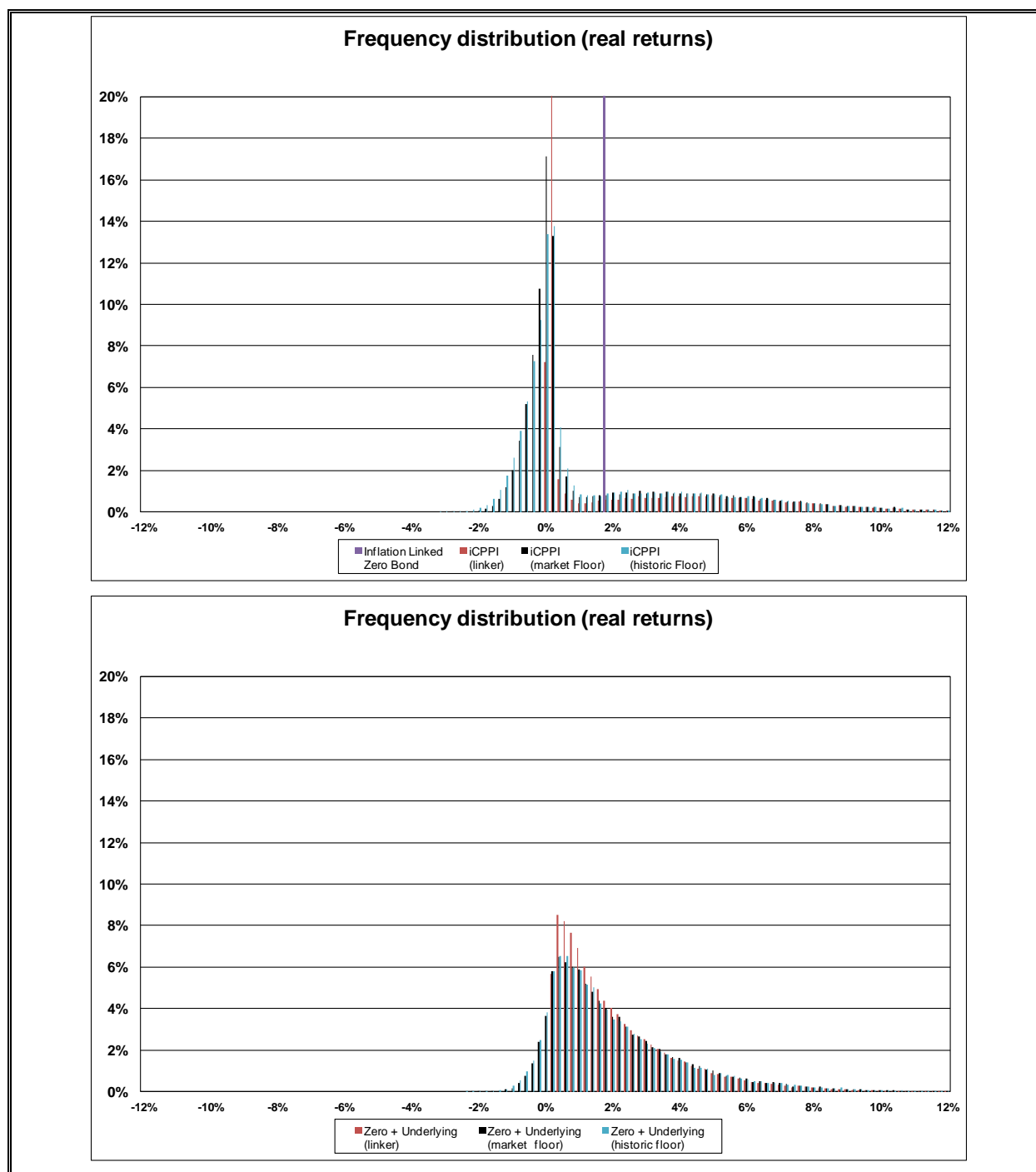


Figure 3: Empirical real rates of return – iCPPI / inflation-linked zero (upper) and zero plus underlying (lower)

Further, Table 5 and Table 6 summarize some key statistics of these empirical distributions.

	Inflation Linked Zero Bond	Zero + Underlying (historic floor)	iCPPI (historic Floor)	Zero + Underlying (market floor)	iCPPI (market Floor)	Zero + Underlying (linker)	iCPPI (linker)
5%	2.32%	1.87%	0.75%	2.03%	1.03%	1.78%	0.93%
25%	3.11%	2.78%	1.64%	2.78%	1.74%	2.79%	1.79%
Median	3.66%	3.51%	2.34%	3.51%	2.25%	3.67%	2.49%

75%	4.22%	4.78%	4.42%	4.83%	4.38%	4.97%	3.83%
95%	5.04%	7.83%	9.78%	7.82%	9.83%	7.82%	9.77%
Expected return	7.35%	4.67%	5.21%	4.68%	5.23%	4.74%	5.18%
P(IRR<0%)	0.00%	0.01%	0.62%	0.00%	0.15%	0.01%	0.27%
P(IRR<2%)	2.09%	6.77%	37.95%	4.66%	37.74%	7.85%	32.31%
Expected shortfall	0.00%	4.06%	7.63%	0.00%	5.51%	4.59%	6.85%
CTE 95	1.99%	1.51%	0.40%	1.73%	0.72%	1.41%	0.60%

**Table 5: Key figures nominal returns – single premium
(Redesigned Products)**

	Inflation Linked Zero Bond	Zero + Underlying (historic floor)	iCPPI (historic Floor)	Zero + Underlying (market floor)	iCPPI (market Floor)	Zero + Underlying (linker)	iCPPI (linker)
5%	1.61%	-0.27%	-1.13%	-0.22%	-0.95%	0.18%	0.00%
25%	1.61%	0.49%	-0.36%	0.52%	-0.31%	0.67%	0.00%
Median	1.61%	1.37%	0.03%	1.43%	0.01%	1.46%	0.00%
75%	1.61%	2.87%	2.60%	2.93%	2.61%	2.84%	1.59%
95%	1.61%	6.12%	7.88%	6.05%	7.89%	5.79%	7.81%
Expected return	1.61%	2.75%	3.26%	2.74%	3.27%	2.67%	3.17%
P(IRR<0%)	0.00%	9.80%	45.89%	8.91%	48.37%	0.00%	0.00%
P(IRR<0.01%)	0.00%	10.04%	47.14%	9.19%	50.44%	0.02%	60.76%
Expected shortfall	0.00%	10.03%	13.81%	9.33%	11.51%	0.00%	0.00%
CTE 95	1.61%	-0.59%	-1.46%	-0.52%	-1.24%	0.11%	0.00%

**Table 6: Key figures real returns – single premium
(Redesigned Products)**

First, note that the approximation often used by practitioners to assess real returns by means of nominal returns less some estimate for the long-term rate of inflation now completely fails as an approximation for the probability distribution of real returns. While for the products analyzed in Section 4 the approximation delivered appropriate results at least for expected values, a deterministic correction for inflation is obviously not suitable for path-dependent inflation-linked products. An analogy between inflation-linked and nominal zero-bond is further identified by observing a degenerated probability distribution in real terms and a non-degenerated probability distribution in nominal terms for the inflation-linked zero-bond returns and vice versa for the nominal zero-bond.

Now, we have a closer look at the products resulting from the first approach i.e. the products with a modified guarantee basis (where still nominal zero-bonds are used as a risk-free asset). These products are denoted by “historic floor” and “market floor” (depending on the used estimate $\tilde{i}(t)$) in the figures and tables. Compared to the products with nominal guarantees from Section 4, the probability and the extent of negative real returns are reduced significantly (cf. Table 4 and Table 6), however real losses are still possible. This is mainly due to the fact that the “guarantee” G_t may not be sufficient whenever realized inflation deviates from estimated inflation.

Comparing the historic and market approach, we find that the market approach may deliver better results than the historic approach especially when lower percentiles of the real returns' distributions are used as a risk measure. Hence, the extent of real losses (if they occur) is generally lower when a market based approach is implemented instead of a pure historic estimate, since the market based approach generally picks up changes in the future rate of inflation more quickly than the historic approach does.²⁰

We now look at the second approach for modification, i.e. products where inflation-linked zero-bonds instead of nominal zero-bonds are used as a safe asset. These two products are denoted by "linker" in the figures and tables. Although nominal losses are possible (cf. Table 5), the risk of negative real returns is significantly reduced: Considering the zero plus underlying product, there is no risk of negative real returns in the single premium case²¹ whereas for the iCPPI product, real losses are possible due to gap events which are however not observed in our simulation study. Further, the probability distributions of real rates for the modified products and the corresponding probability distributions of nominal rates for the "standard" products (cf. Figure 1 and Figure 2) have a similar shape.

Both products still offer some upside potential. Clearly, the iCPPI product comes with more fluctuation compared to the zero plus underlying product. In the 95th percentile, for example, the iCPPI versions even under real returns achieve an IRR of almost 8%. However the cost of providing a very high upside potential in good capital market scenarios goes hand in hand with a very high probability of approximately 60% of "just" getting the inflated premiums back. In contrast, the zero plus underlying delivers a more moderate distribution of real returns which results in less upside potential but also a lower probability of very low or even zero real returns. Hence, considering these products and products 'in between these two products' seems a promising way of creating different inflation-protected products for clients with different degrees of risk aversion.

Finally note that, although in our simulation study no negative real returns were observed, there is no guarantee embedded in the products. If e.g. a massive increase in the expected future rate of inflation (and thus massive increase in the inflation-linked zero-bonds' price) occurs simultaneously with a dramatic loss in equity, especially the introduced iCPPI product may not be able to buy the required inflation-linked zero-bonds after that event and hence real losses are possible.

²⁰ In a model with "regime switches" in the monetary policy this effect might even be more pronounced.

²¹ Obviously this only holds if $A_0 \geq F_0$.

5.4 Regular premium

We performed the same analyses for regular premium payments and here provide a summary of the most interesting results. The zero plus underlying and the iCPPI product based on a historic floor or market floor products don't show any additional (i.e. non-expected) effects in the case of regular premiums payments and we hence refrain from a detailed discussion of the corresponding results.

The products using an inflation-linked bond, however, show some interesting additional effects that are shown in Table 7.

	Zero + Underlying (linker)	iCPPI (linker)
5%	0.23%	-0.04%
25%	0.73%	-0.01%
Median	1.44%	0.05%
75%	2.67%	2.01%
95%	5.46%	8.50%
Expected return	2.33%	2.95%
P(IRR<0%)	0.07%	31.43%
P(IRR<0.01%)	0.12%	39.02%
Expected shortfall	0.33%	0.33%
CTE 95	0.14%	-0.06%

**Table 7: Key figures real returns – regular premium
(Redesigned Products)**

In contrast to the single premium case (see above) both products now give a positive probability of negative real returns. While for the zero plus underlying product this is very unlikely, for the iCPPI product in about one third of the cases a negative real return is observed. The extent of negative real returns, however, is very small in both cases. This for example can be seen by comparing the CTE 95 or the expected shortfall. Thus, even though a guarantee of nonnegative real returns is not possible, the two products provide a rather effective protection against inflation also in the regular premium case.

6 Sensitivity Analyses

In this section we analyze sensitivities of our results with respect to capital market assumptions for the products analyzed in Sections 4 and 5. In particular, we vary the underlying correlation structure and the level of nominal and model-induced real interest rates. We concentrate on single premium contracts and real returns. Note that, within this section we only comment on the most important effects and show a comprehensive set of sensitivity results in Appendix B.

6.1 Sensitivity with respect to the correlation structure

At first we stress the assumed correlation matrix of the considered asset classes' random innovations in the following. Remember, the correlation matrix used in the previous analyses was

$$\Sigma = \begin{pmatrix} 1 & \rho_{ir} & \rho_{is} & \rho_{iv} \\ \rho_{ri} & 1 & \rho_{rs} & \rho_{rv} \\ \rho_{si} & \rho_{sr} & 1 & \rho_{sv} \\ \rho_{vi} & \rho_{vr} & \rho_{vs} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 33\% & -15\% & 0 \\ 33\% & 1 & 0 & 0 \\ -15\% & 0 & 1 & -57\% \\ 0 & 0 & -57\% & 1 \end{pmatrix}$$

Sensitivity I

We first assume except for the correlation between stock returns and their volatility all correlations are zero, i.e.

$$\Sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -57\% \\ 0 & 0 & -57\% & 1 \end{pmatrix}.$$

Sensitivity II

Next, we analyze the effect of extreme values of ρ_{ri} by setting them to $\pm 100\%$. Since the resulting "correlation matrix" may not be positive semi-definite by construction, we apply the approach introduced by Higham (2002) and adjust the correlation matrix accordingly.

Hence, sensitivity IIa) assuming an extreme positive correlation results in the following correlation matrix:

$$\Sigma = \begin{pmatrix} 1 & 98.50\% & -14.68\% & 0.18\% \\ 98.50\% & 1 & -0.31\% & -0.17\% \\ -14.68\% & -0.31\% & 1 & -56.96\% \\ 0.18\% & -0.17\% & -56.96\% & 1 \end{pmatrix}$$

Sensitivity IIb) assuming an extreme negative correlation is accordingly summarized by

$$\Sigma = \begin{pmatrix} 1 & -98.50\% & -14.68\% & 0.18\% \\ -98.50\% & 1 & 0.31\% & 0.17\% \\ -14.68\% & 0.31\% & 1 & -56.96\% \\ 0.18\% & 0.17\% & -56.96\% & 1 \end{pmatrix}$$

Results:

First note that the distribution of real investment returns remains unchanged for the nominal zero-bond.²² Further, Table 8 in Appendix B shows that the assumption of a zero correlation does not change the results too much and hence we only comment on the extreme positive and negative correlation assumptions in the following.

The equity fund investment and the option based product provide a pure investment in the considered equity process whose performance is influenced by the interest rates and hence due to the correlation also related to the rate of inflation. We observe that when a positive correlation of interest rates and the rate of inflation is assumed, the products' lower respectively upper tails (roughly approximated by the 5th and 95th percentile) increase and decrease respectively, i.e. we observe less variability. For a negative correlation, we observe the opposite effect. A similar effect occurs with the considered hybrid products, on the one side due to the same effect in the equity part and on the other side due to the fact that when the inflation rate increases, the product's floor – as a function of the considered interest rates – potentially decreases (increases) when correlation is positive (negative) (cf. Table 9) leaving the products with higher (lower) equity share.

For the modified products as introduced in Section 5, it is worthwhile noting that, although the payout of the inflation-linked zero-bond does not change, we observe a different deterministic real return of the considered investment product when compared to the base-case. This is due to the inflation-linked bond's different price at $t = 0$. Further, the modified guarantee products are generally more “volatile” when negative correlations of interest rates and inflation rates are assumed, leaving the products with a potentially higher upside at the cost of a more pronounced downside as well (cf. Table 9).

6.2 Sensitivity with respect to the level of nominal interest rates and rate of inflation

We now stress the level of nominal interest rates and inflation by analyzing the following sensitivities. Again, all other assumptions remain unchanged, in particular the “base case” correlation matrix as introduced in Section 3 is applied.

²² Note, this effect is only observed for a single contribution and changes when regular contributions are in place.

Sensitivity III

We assume an increase (decrease) of both, the start level and the long term average of interest rates and inflation, i.e. $\theta_r = r(0) = 5.5\%$, $\theta_i = i(0) = 3\%$ in sensitivity IIIa) and $\theta_r = r(0) = 3.5\%$, $\theta_i = i(0) = 1\%$ in sensitivity IIIb).

Results:

Since in this sensitivity analysis, the level of interest rates and inflation is stressed by the same amount, the level of real rates remains more or less unchanged. Therefore, the real probability distributions of non-hybrid standard products (i.e. pure equity and fixed income investment) are very similar to the base case. However, when considering hybrid products, this observation changes (tremendously). When the level of interest rates is increased (decreased), the equity exposure of the considered products increases (decreases) as well which generally generates more (less) volatility in the products. Further, from a real point of view, the nominal guarantee's value decreases (increases) when the level of inflation increases (decreases). These effects therefore increase the product's downside (in terms of real returns) even when interest rates are higher (cf. Table 10).

In contrast, the modified products as introduced in Section 5 are generally only influenced by the level of real interest rates and hence their probability distributions remain largely unchanged when sensitivity III is applied (cf. Table 10) when real rates stay at a similar level.

Sensitivity IV

Finally we investigate an increase (decrease) of the level of real interest rates by 1% by assuming $\theta_r = r(0) = 5.5\%$ in sensitivity IVa) and $\theta_r = r(0) = 3.5\%$ in sensitivity IVb) while leaving $\theta_i = i(0) = 2\%$ unchanged.

Results:

When real rates increase (decrease) the effects on the standard products are exactly as expected: We observe a positive (negative) shift of the resulting return distributions (cf. Table 11). Increasing the level of real interest rates decreases the price of an inflation-linked zero-bond and hence increases the cushion of all considered modified guarantee products (and vice versa). Therefore, the results for these products are (obviously) worse when real interest rates tend to be low. Additionally, path-dependant products suffer more than path-independent products since the former essentially result in very skewed return distributions (cf. Table 11).

7 Conclusion and Outlook

This paper shows the impact of inflation risk on the risk-return profiles of various old age provision products. After extending the model introduced in Graf et al. (2012) by taking inflation risk into account we derived the risk-return profiles for various common standard old age provision products assuming e.g. different portfolio insurance strategies. We found that most products – including products that are often considered particularly safe by practitioners and regulators due to nominal guarantees – bear a significant inflation risk and especially found that information about a product's return distribution in real terms (that is not revealed by information provided to clients and their financial advisors so far) is relevant for sustainable financial planning. Finally, we have proposed product modifications that may reduce or even remove the risk of negative real investment returns while still allowing for some upside potential.

Of course, this work is only a first starting point in this area of research. Future research could include an assessment of model risk, in particular taking into account the impact of the recent monetary policy by the various central banks due to the debt crisis. Further, it seems worthwhile studying whether the inflation-linked derivatives market is liquid enough for institutional investors to actually implement the proposed product modifications in practice.

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A Pricing of inflation-linked zero-bonds

In this section we briefly sketch the implemented pricing of inflation-linked zero-bonds used in the products in Section 5. Note that in the considered asset model introduced in Section 3 nominal interest rates follow a Cox-Ingersoll-Ross model and rates of inflation follow a Vasiček model. To the best of our knowledge, no closed form solution for pricing inflation-linked zero-bonds is available in this setup. Hence, we assume the following approximation for pricing:²³ We assume the rate of inflation and the nominal interest rates to follow correlated Vasiček processes:

$$\begin{aligned} dr(t) &= \tilde{\kappa}_r (\tilde{\theta}_r - r(t)) dt + \tilde{\sigma}_r dW^r(t) \\ di(t) &= \kappa_i (\theta_i - i(t)) dt + \sigma_i dW^i(t) \end{aligned}$$

with $dW^i(t)dW^r(t) = \rho_{ri}$. This model is “consistent” with the original model (cf. Section 3) if we set $\tilde{\kappa}_r = \kappa_r$, $\tilde{\theta}_r = \theta_r$ and $\tilde{\sigma}_r = \sigma_r \sqrt{\theta_r}$. By no-arbitrage arguments, the price at time t of an inflation-linked zero-bond $p_{I,s}(t, T)$ issued at time s ($s \leq t$) with time-to-maturity T is then derived as

$$\begin{aligned} p_{I,s}(t, T) &= \mathbb{E}_{\mathcal{Q}} \left[\exp \left(- \int_t^T r(u) du \right) \exp \left(\int_s^T i(u) du \right) \middle| \mathcal{F}_t \right] \\ &= \exp \left(\int_s^t i(u) du \right) \mathbb{E}_{\mathcal{Q}} \left[\exp \left(- \int_t^T r(u) du \right) \exp \left(\int_t^T i(u) du \right) \middle| \mathcal{F}_t \right] \end{aligned}$$

Since $(i(t))_{t \geq 0}$ and $(r(t))_{t \geq 0}$ follow a normal distribution, $\int_t^T r(u) du$ and $\int_t^T i(u) du$ are normally distributed as well. Hence, the price $p_{I,s}(t, T)$ is finally computed as the expectation of a log-normal distributed random variable.

We now briefly sketch the (not complicated though rather tedious) derivation of the distribution of $\left(- \int_t^T r(u) du + \int_t^T i(u) du \right)$ which is derived from the joint multivariate normal distribution of $\left(\int_t^T r(u) du, \int_t^T i(u) du \right)$ which is itself completely determined by the expectation, variance and covariance of $\int_t^T r(u) du$ and $\int_t^T i(u) du$ respectively.

²³ This approximation may also be identified as incorporating some model risk in our analyses when inflation-linked derivatives are priced.

Expectation and variance of the processes are derived using the identity $r(s) = e^{-\tilde{\kappa}_r(s-t)}r(t) + \tilde{\theta}_r(1 - e^{-\tilde{\kappa}_r(s-t)}) + \tilde{\sigma}e^{-\tilde{\kappa}_rs} \int_t^s e^{\tilde{\kappa}_ru} dW^r(u)$ for $s > t$ and the fact that $\int_s^t f(u) dW^r(u) \sim \mathcal{N}\left(0, \int_s^t f^2(u) du\right)$.

Finally, the covariance of the considered processes is calculated as follows

$$\begin{aligned} & \mathbb{Cov}_Q \left[\int_t^T r(s) ds, \int_t^T i(s) ds \right] \\ &= \mathbb{Cov}_Q \left[\tilde{\sigma}_r e^{-\tilde{\kappa}_rs} \int_t^T \left(\int_t^s e^{\tilde{\kappa}_ru} dW^r(u) \right) ds, \sigma_i e^{-\kappa_i s} \int_t^T \left(\int_t^s e^{\kappa_i u} dW^i(u) ds \right) \right] \\ &= \tilde{\sigma}_r \sigma_i \left(\int_t^T \int_t^T \mathbb{Cov}_Q \left[e^{-\tilde{\kappa}_rs} \int_t^s e^{\tilde{\kappa}_ru} dW^r(u), e^{-\kappa_i z} \int_t^z e^{\kappa_i u} dW^i(u) \right] dz ds \right) \\ &= \tilde{\sigma}_r \sigma_i \rho_{ri} \int_t^T \int_t^T \left(\int_t^{s \wedge z} e^{-\tilde{\kappa}_r(s-u)} e^{-\kappa_i(z-u)} \right) dz ds \end{aligned}$$

which can then be computed analytically as well.

B Sensitivity Analyses – Results

This section gives details with respect to the results explained in Section 6. We display key statistics of the observed real returns considering a single premium investment only.

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	Nominal Zero Bond	Zero + Underlying	iCPPI	Option Based Product	Equity Fund	Inflation Linked Zero Bond	Zero + Underlying (historic floor)	iCPPI (historic Floor)	Zero + Underlying (market floor)	iCPPI (market Floor)	Zero + Underlying (linker)	iCPPI (linker)
5%	0.29%	-1.75%	-2.89%	-2.72%	-6.10%	1.51%	-0.25%	-1.12%	-0.17%	-0.89%	0.18%	0.00%
25%	1.08%	-0.22%	-1.83%	-1.46%	-1.78%	1.51%	0.51%	-0.35%	0.56%	-0.27%	0.64%	0.00%
Median	1.62%	1.40%	0.01%	0.76%	1.19%	1.51%	1.38%	0.04%	1.45%	0.02%	1.40%	0.00%
75%	2.16%	3.56%	3.81%	3.75%	4.19%	1.51%	2.86%	2.62%	2.91%	2.59%	2.72%	1.31%
95%	2.96%	7.27%	8.24%	8.02%	8.48%	1.51%	6.08%	7.84%	5.96%	7.83%	5.62%	7.72%
Expected return	1.72%	3.31%	3.59%	3.55%	3.91%	1.51%	2.74%	3.25%	2.72%	3.25%	2.57%	3.11%

Table 8: Sensitivity I

	Nominal Zero Bond	Zero + Underlying	iCPPI	Option Based Product	Equity Fund	Inflation Linked Zero Bond	Zero + Underlying (historic floor)	iCPPI (historic Floor)	Zero + Underlying (market floor)	iCPPI (market Floor)	Zero + Underlying (linker)	iCPPI (linker)
5%	0.29%	-1.58%	-2.75%	-2.56%	-5.93%	1.81%	-0.15%	-1.11%	-0.13%	-1.00%	0.21%	0.00%
	(0.29%)	(-2.13%)	(-3.10%)	(-3.01%)	(-6.57%)	(1.21%)	(-0.39%)	(-1.12%)	(-0.26%)	(-0.82%)	(0.13%)	(0.00%)
25%	1.08%	-0.13%	-1.74%	-1.36%	-1.71%	1.81%	0.60%	-0.32%	0.60%	-0.33%	0.74%	0.00%
	(1.08%)	(-0.47%)	(-2.10%)	(-1.78%)	(-2.04%)	(1.21%)	(0.31%)	(-0.40%)	(0.40%)	(-0.26%)	(0.51%)	(0.00%)
Median	1.62%	1.44%	0.05%	0.80%	1.23%	1.81%	1.46%	0.05%	1.50%	0.01%	1.58%	0.00%
	(1.62%)	(1.34%)	(-0.11%)	(0.72%)	(1.15%)	(1.21%)	(1.20%)	(0.01%)	(0.31%)	(0.00%)	(1.19%)	(0.00%)
75%	2.16%	3.50%	3.75%	3.69%	4.14%	1.81%	2.85%	2.74%	2.91%	2.74%	2.96%	2.26%
	(2.16%)	(3.76%)	(4.00%)	(3.96%)	(4.40%)	(1.21%)	(2.91%)	(2.60%)	(2.93%)	(2.58%)	(2.51%)	(0.17%)
95%	2.96%	7.10%	8.06%	7.86%	8.32%	1.81%	5.80%	7.71%	5.85%	7.75%	5.89%	7.73%
	(2.96%)	(7.87%)	(8.85%)	(8.63%)	(9.10%)	(1.21%)	(6.76%)	(8.39%)	(6.46%)	(8.30%)	(5.60%)	(8.01%)
Expected return	1.72%	3.20%	3.47%	3.43%	3.78%	1.81%	2.65%	3.16%	2.68%	3.20%	2.76%	3.17%
	(1.72%)	(3.67%)	(4.05%)	(3.99%)	(4.36%)	(1.21%)	(3.06%)	(3.65%)	(2.92%)	(3.60%)	(2.49%)	(3.27%)

Table 9: Sensitivity II a (b)

	Nominal Zero Bond	Zero + Underlying	iCPPI	Option Based Product	Equity Fund	Inflation Linked Zero Bond	Zero + Underlying (historic floor)	iCPPI (historic Floor)	Zero + Underlying (market floor)	iCPPI (market Floor)	Zero + Underlying (linker)	iCPPI (linker)
5%	0.25% (0.34%)	-2.44% (-1.12%)	-3.73% (-2.06%)	-3.58% (-1.89%)	-6.18% (-6.11%)	1.56% (1.65%)	-0.30% (-0.24%)	-1.16% (-1.10%)	-0.24% (-0.20%)	-0.96% (-0.94%)	0.18% (0.19%)	0.00% (0.00%)
25%	1.04% (1.13%)	-0.59% (-0.19%)	-2.48% (-1.12%)	-2.00% (-0.80%)	-1.82% (-1.78%)	1.56% (1.65%)	0.46% (0.51%)	-0.38% (-0.34%)	0.50% (0.54%)	-0.32% (-0.30%)	0.65% (0.68%)	0.00% (0.00%)
Median	1.57% (1.67%)	1.33% (1.51%)	0.67% (-0.16%)	0.77% (-0.82%)	1.20% (1.21%)	1.56% (1.65%)	1.34% (1.40%)	0.03% (0.04%)	1.41% (1.45%)	0.00% (0.01%)	1.43% (1.49%)	0.00% (0.00%)
75%	2.12% (2.21%)	3.76% (3.36%)	3.98% (3.43%)	3.80% (3.76%)	4.24% (4.20%)	1.56% (1.65%)	2.84% (2.90%)	2.55% (2.65%)	2.89% (2.96%)	2.53% (2.67%)	2.81% (2.86%)	1.39% (1.73%)
95%	2.91% (3.01%)	7.71% (6.77%)	8.36% (8.12%)	8.12% (8.00%)	8.58% (8.46%)	1.56% (1.65%)	6.13% (6.12%)	7.93% (7.83%)	6.04% (6.07%)	7.93% (7.86%)	5.78% (5.80%)	7.83% (7.78%)
Expected return	1.67% (1.77%)	3.52% (3.08%)	3.72% (3.45%)	3.58% (3.59%)	3.98% (3.88%)	1.56% (1.65%)	2.74% (2.75%)	3.29% (3.23%)	2.73% (2.75%)	3.30% (3.25%)	2.66% (2.68%)	3.19% (3.15%)

Table 10: Sensitivity III a (b)

	Nominal Zero Bond	Zero + Underlying	iCPPI	Option Based Product	Equity Fund	Inflation Linked Zero Bond	Zero + Underlying (historic floor)	iCPPI (historic Floor)	Zero + Underlying (market floor)	iCPPI (market Floor)	Zero + Underlying (linker)	iCPPI (linker)
5%	1.25% (-0.66%)	-1.46% (-2.11%)	-2.76% (-3.03%)	-2.61% (-2.87%)	-5.24% (-7.04%)	2.59% (0.64%)	0.09% (-0.82%)	-0.90% (-1.38%)	0.10% (-0.74%)	-0.79% (-1.12%)	0.34% (0.06%)	0.00% (0.00%)
25%	2.05% (0.13%)	0.41% (-0.81%)	-1.50% (-2.10%)	-1.01% (-1.78%)	-0.83% (-2.76%)	2.59% (0.64%)	1.05% (-0.07%)	-0.15% (-0.63%)	1.07% (-0.01%)	-0.15% (-0.47%)	1.17% (0.24%)	0.00% (0.00%)
Median	2.59% (0.66%)	2.35% (0.50%)	1.68% (-1.15)	1.78% (-0.18%)	2.21% (0.21%)	2.59% (0.64%)	2.33% (0.48%)	0.33% (-0.15%)	2.37% (0.53%)	0.23% (-0.10%)	2.39% (0.57%)	0.03% (0.03%)
75%	3.14% (1.20%)	4.80% (2.33%)	5.02% (2.40%)	4.84% (2.73%)	5.29% (3.16%)	2.59% (0.64%)	4.28% (1.42%)	4.52% (0.26%)	4.32% (1.45%)	4.57% (0.22%)	4.28% (1.23%)	4.40% (0.00%)
95%	3.95% (1.98%)	8.79% (5.71%)	9.45% (7.05%)	9.21% (6.92%)	9.67% (7.38%)	2.59% (0.64%)	7.98% (3.96%)	9.30% (6.03%)	7.95% (3.75%)	9.34% (5.85%)	7.81% (3.06%)	9.31% (5.19%)
Expected return	2.69% (0.75%)	4.56% (2.05%)	4.76% (2.42%)	4.62% (2.56%)	5.02% (2.84%)	2.59% (0.64%)	4.13% (1.32%)	4.57% (1.83%)	4.13% (1.27%)	4.59% (1.79%)	4.09% (1.13%)	4.54% (1.55%)

Table 11: Sensitivity IV a (b)