A new proof of variance-optimal hedging in incomplete time discrete markets Part I: Valuation Model

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A new proof of variance-optimal hedging in incomplete time discrete markets Part I: Valuation Model

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This article presents a new proof for the existence and uniqueness of a variance-optimal trading strategy to hedge square integrable claims in incomplete and time discrete markets provided the underlying price process fulfills the mean-variance-tradeoff condition (nondegeneracy condition) of Martin Schweizer. The variance-optimal hedging is described as the minimum problem of an energy functional and the existence and uniqueness of the variance-optimal trading strategy is done by the theorem of Lax-Milgram.

Introduction The fundamental theorem of asset pricing states, that if the market is free of arbitrage, then there exists a martingale measure equivalent to the given physical probability measure (see [18] and the literature therein). And if the market is complete, i.e. the payoff of every claim can be replicated by a self-financing trading strategy of the underlying risky securities and a risk free bank account, then the martingale measure is also unique and the fair price of every claim is the expected value under the martingale measure. As an example, the market underlying the option pricing model of Black and Scholes is free of arbitrage and complete and therefore every claim can be priced uniquely. Now completeness of a market is a very restrictive assumption and is not achieved in almost all cases. So choosing an equivalent martingale measure is still possible but there is no unique choice. Hence a change of measure does not solve the problem of hedging and pricing derivatives in incomplete markets.

Based on the work of many authors, in 1994 M. Schäl published a solution of the variance-optimale hedging problem in his article *On quadratic cost criteria for option hedging* (see [19] and the literature therein). His solution is based on the assumption that the price process $\{S_t^*; t = 0, 1, \ldots, T\}$[2] defined on the filtered probability space $(\Omega, \{\mathcal{F}_t; t = 0, 1, \ldots, T\}, \mathcal{F}, P)$ fulfills the so called deterministic mean-variance-tradeoff. This means that the standardized drift coefficient μ_t^*/σ_t^* with $\mu_t^* = E[\Delta S_t^* | \mathcal{F}_{t-1}]$ and $\sigma_t^* = Var[\Delta S_t^* | \mathcal{F}_{t-1}]$ of the underlying price process is bounded and deterministic, i.e. there exists $M > 0$ with $|\mu_t^*(\omega)|/\sigma_t^*(\omega) \leq M$ and $|\mu_t^*(\omega)|/\sigma_t^*(\omega)$ is not a random variable for all $t = 0, 1, \ldots, T$.

In 1995 M. Schweizer published a complete solution of the variance-optimal hedging problem in his article *Variance-optimal hedging in discrete time* (see [20]). His solution is based on the *nondegeneracy condition* which nowadays is called the mean-variance-tradeoff (see the definition below). The idea is as follows: Let $c \in \mathbb{R}$ be any real number and $\{\vartheta_t; t = 1, 2, \ldots, T\}$ any trading strategy, i.e. the amount of capital invested in the risky security at time $t-1$ is given by ϑ_t and let $\vartheta_t \Delta S_t^*$ be square integrable for

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[2] The precise definition of the price process and the other variables will given belove.

all t. Then the profit and loss $c + \sum_{t=1}^{T} \vartheta_t \Delta S_t^*$ for a given $c \in \mathbb{R}$ and trading strategy $\{\vartheta_t; t = 1, 2, \ldots, T\}$ is a square integrable random variable. Let $G(\Theta) \subseteq \mathcal{L}^2(\Omega, \mathcal{F}, P)$ be the subspace of all such profits and losses. M. Schweizer proves that $G(\Theta)$ is a closed subspace of $\mathcal{L}^2(\Omega, \mathcal{F}, P)$, provided the price process fulfills the mean-variance-tradeoff respectively the nondegeneracy condition. The existence and uniqueness of the optimal trading strategy, i.e. minimizing the variance of the profit and loss, follows by the projection theorem of Hilbert spaces. Furthermore he solves the projection equation (see [23]) and gives an explicit formula for the optimal trading strategy and the initial investment. Finally he gives a counterexample for the case that the price process does not satisfy the nondegeneracy condition.

In the last two decades many articles were published concerning the problem of variance-optimal hedging in incomplete time discrete markets. These articles range from the calculation of the variance-optimal trading strategy for specified price processes to the transfer of variance-optimal hedging to other markets such as the electricity market. See for example [2, 3, 4, 5, 6, 9, 11, 13, 14, 15, 16, 17, 22] and the literature therein. A very interesting current article is *Best-estimate claims reserving in incomplete markets*, written by S. Happ, M. Merz and M.V. Wüthrich (see [17]), due to the connection with article 75 of the solvency II directive [1], which states:

a) assets shall be valued at the amount for which they could be exchanged between knowledgeable willing parties in an arm's length transaction;

b) liabilities shall be valued at the amount for which they could be translated, or settled, between knowledgeable willing parties in an arm's length transaction.

To value any cash flow based on article 75 of the directive the first step is that the parties reach an agreement about the investment universe[3]. Then they can calculate hedging portfolios and the profit and loss process for any given initial investment and trading strategy. However the amount of profit of one party is equivalent to the amount of loss of the counterparty and vice versa. Therefore both parties search for trading strategies which minimize their own expected loss. This is only possible if they search for trading strategies which minimizing the expected profit and loss of both parties simultaneously. In the sequel we will show that the trade-off between the two parties can be characterized as a minimum problem of an energy functional and that the existence and uniqueness of the minimum of the energy functional can be given by the theorem of Lax-Milgram provided the underlying price process fulfills the mean-variance-tradeoff of Martin Schweizer.

The rest of the paper is organized as follows. The first paragraph describes the investment universe and the definition of the mean-variance-tradeoff. The second paragraph describes the profit and loss process and mean-self-financing trading strategies as one possible generalisation of self-financing trading strategies and formulates the optimisa-

[3] It is easy to see that for the investment universe it is enough to consider one risky asset and a bank account.

tion problem as a minimum problem of an energy functional. The next two paragraphs present a proof for the existence and uniqueness of the minimum of the energy functional by the theorem of Lax-Milgram and an explicit calculation of the optimal trading strategy and the initial investment. The last paragraph describes the calculation of the shareholder value as an application of the valuation model and gives an interpretation of the Economic Value Added (EVA).

The investment universe Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\{\mathcal{F}_t; t = 0, 1, \ldots, T\}$ and w.l.o.g. $\mathcal{F}_0 = \{\emptyset, \Omega\}$. We assume that all stochastic processes are square integrable and adapted to the filtration.

In order to build portfolios for hedging claims there need to be given a risk free asset and risky assets. The risk free asset is a predictable *bank account* process $\{A_t; t = 0, 1, \ldots, T\}$ with $A_0 = 1$ and $A_t > 1$ P-a.s. for all t. The risky assets are given by *price processes* $\{S_{i,t}; t = 0, 1, \ldots, T\}$ for $i = 1, 2, \ldots, n$. One can combine the price processes to a n-dimensional vector-valued process $\{(S_{1,t}, S_{2,t}, \ldots, S_{n,t})'; t = 0, 1, 2, \ldots, T\}$, where $'$ is the transposition of vectors. For every $\omega \in \Omega$ the price $S_{i,t}(\omega) \in \mathbb{R}$ is called the market price of the i-th risky asset at time t in state ω. A *trading strategy* is a predictable stochastic processes $\{(\vartheta_{1,t}, \vartheta_{2,t}, \ldots, \vartheta_{n,t}, \eta_t); t = 1, 2, \ldots, T\}$ with $\vartheta_{i,t}, \eta_t \in \mathcal{L}^0(\Omega, \mathcal{F}_{t-1}, P)$ for $i = 1, 2, \ldots, n$ and $t = 1, 2, \ldots, T$. $\vartheta_{i,t}$ is the proportion of capital which is invested in the i-th security at time $t-1$ and η_t is the proportion which is invested in the risk-free security at the same time. We consider only trading strategies with $\vartheta_{i,t}\Delta S_{i,t}, \eta_t \Delta A_t \in \mathcal{L}^2(\Omega, \mathcal{F}, P)$ for all $(i,t) \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, T\}$.

Let $(\vartheta_{1,t}, \vartheta_{2,t}, \ldots, \vartheta_{n,T}, \eta_t)$ be a trading strategy with portfolio value $V_{t-1} = \vartheta_{1,t}S_{1,t-1} + \cdots + \vartheta_{n,t}S_{n,t-1} + \eta_t A_{t-1}$. Then $\vartheta_{1,t}S_{1,t-1} + \cdots + \vartheta_{n,t}S_{n,t-1}$ is the risky invested capital with value $\vartheta_{1,t}S_{1,t} + \cdots + \vartheta_{n,t}S_{n,t}$ at time t. It is assumed that $\vartheta_{1,t}S_{1,t-1} + \cdots + \vartheta_{n,t}S_{n,t-1} \neq 0$. Then $\vartheta_t = \vartheta_{1,t} + \vartheta_{2,t} + \cdots + \vartheta_{n,t} \neq 0$ and with $\alpha_{i,t} = \vartheta_{i,t}/\vartheta_t$ one gets $V_{t-1} = \vartheta_t(\alpha_{1,t}S_{1,t-1} + \cdots + \alpha_{n,t}S_{n,t-1}) + \eta_t A_{t-1}$. The sum $\alpha_{1,t}S_{1,t-1} + \cdots + \alpha_{n,t}S_{n,t-1}$ is the price process of a risky asset. Denoting this by $\{S_t; t = 0, 1, \ldots, T\}$ with $S_T = \alpha_{1,T}S_{1,T} + \cdots + \alpha_{n,T}S_{n,T}$ one gets $V_{t-1} = \vartheta_t S_{t-1} + \eta_t A_{t-1}$. This shows that it is sufficient to build hedge portfolios with only one risk free and one risky asset $\{S_t; t = 0, 1, 2, \ldots, T\}$.

By discounting S_t with A_t one gets the discounted price process $\{S_t^*; t = 0, 1, 2, \ldots, T\}$. This process is square integrable because of $A_t(\omega) \geq 1$ for all $(t, \omega) \in \{1, 2, \ldots, T\} \times \Omega$.

Definition 1 *The price process $\{S_t^*; t = 0, 1, 2, \ldots, T\}$ satisfies the mean-variance-tradeoff (MVT) if there is a real number $\delta \in [0, 1)$, so that*

$$\frac{(E[\Delta S_t^* | \mathcal{F}_{t-1}](\omega))^2}{E[(\Delta S_t^*)^2 | \mathcal{F}_{t-1}](\omega)} \leq \delta$$

is valid for all $(t, \omega) \in \{1, 2, \ldots, T\} \times \Omega$.

Substituting $\delta = C/(1+C)$ for some $C > 0$ the definition of MVT is the same as $(\mu_t^*)^2 \le C(\sigma_t^*)^2$, where $\mu_t^* = E[\Delta S_t^* | \mathcal{F}_{t-1}]$ is the conditional expectation and $(\sigma_t^*)^2 = \mathrm{Var}[\Delta S_t^* | \mathcal{F}_{t-1}]$ is the conditional variance of ΔS_t^*. Adding to $E[(\Delta S_t^*)^2 | \mathcal{F}_{t-1}]$ the equation $(E[\Delta S_t^* | \mathcal{F}_{t-1}])^2 - (E[\Delta S_t^* | \mathcal{F}_{t-1}])^2 = 0$ one gets the inequality $E[(\Delta S_t^*)^2 | \mathcal{F}_{t-1}] \le (1+C)(\sigma_t^*)^2$. In this paper we only consider price processes which satisfy $P(\sigma_t^* = 0) = 0$ P-a.s. for $t = 1, 2, \dots, T$.

The profit and loss process Let $\{Z_t^*; t = 1, 2, \dots, T\}$ be a square integrable discounted payment process or cash flow. For a given trading strategy $\{(\vartheta_t, \eta_t); t = 1, 2, \dots, T\}$ we can hedge the payments Z_t^* as good as possible by the portfolio process $\{V_t^*; t = 0, 1, 2, \dots, T\}$ defined by

$$V_t^* = \begin{cases} \vartheta_{t+1} S_t^* + \eta_{t+1} & , t = 0, 1, \dots, T-1 \\ 0 & , t = T \end{cases}$$

with initial investment V_0^*. At time t we have the value $\vartheta_t S_t^* + \eta_t$ of the portfolio to pay Z_t^* and to buy the portfolio V_t^* for the next period. This gives the profit and loss

$$g_t^* = \vartheta_t S_t^* + \eta_t - V_t^* - Z_t^*$$

With the definitions $\vartheta_0 = \eta_0 = 0$ and $Z_0^* = 0$ one has the profit and loss for the initial investment $g_0^* = -V_0^*$. And with $\eta_t = V_{t-1}^* - \vartheta_t S_{t-1}^*$ we have

$$\begin{aligned} g_0^* &= -V_0^* \\ g_t^* &= \vartheta_t \Delta S_t^* - \Delta V_t^* - Z_t^*, t = 1, 2, \dots, T \end{aligned}$$

The *accumulated profit and loss process* is given by

$$\begin{aligned} G_0^* &= -V_0^* \\ G_t^* &= g_0^* + g_1^* + \cdots + g_t^*, t = 1, 2, \dots, T \end{aligned}$$

and with g_t^* and g_0^* defined as above we have

$$\begin{aligned} G_t^* &= \sum_{j=1}^t \vartheta_j \Delta S_j^* - V_0^* - \sum_{j=1}^t \Delta V_j^* - \sum_{j=1}^t Z_j^* \\ &= \sum_{j=1}^t \vartheta_j \Delta S_j^* - V_t^* - \sum_{j=1}^t Z_j^* \end{aligned}$$

If $\{(\vartheta_t, \eta_t); t = 1, 2, \dots, T\}$ is a self-financing trading strategy then we have $g_t^* = 0$ for $t = 1, 2, \dots, T$ and one gets

$$\sum_{t=1}^T Z_t^* = \sum_{t=1}^T \vartheta_t \Delta S_t^*$$

On the left hand-side of the equation we have the square integrable random variable $H_T^* \in \mathcal{L}^2(\Omega, \mathcal{F}_t, P)$ and on the right hand-side we have a discrete stochastic integral of the price process $\{S_t^*; t = 0, 1, \ldots, T\}$. One can show that this equation is not true for all square integrable random variable $X_T \in \mathcal{L}^2(\Omega, \mathcal{F}_T, P)$ (see [20] for a proof). This means that the hedge of H_T^* with a self-financing trading strategy is not always possible. Therefore we consider *mean-self-financing* trading strategies. Such trading strategies have the property

$$E[\Delta G_t^* | \mathcal{F}_{t-1}] = 0$$

for $t = 1, 2, \ldots, T$ (see [19]). For a mean-self-financing trading strategy the accumulated profit and loss process is a martingale.

Let $\{\vartheta_t; t = 1, 2, \ldots, T\}$ be an arbitrary trading strategy. Given the martingale property of the accumulated profit and loss process one can proof that the portfolio process is given by

$$V_t^* = \sum_{t=t+1}^{T} E\big[Z_j^* - \vartheta_j \Delta S_j^* | \mathcal{F}_t\big], t = 0, 1, \ldots, T$$

with the empty sum equal to 0. Now let $E[G_T^*] = -V_0^*$ be and define the portfolio process by the given formula. Then the accumulated profit and loss process is a martingale. This proves that a mean-self-financing trading strategy is not uniquely defined by the martingale property of the accumulated profit and loss process. For the uniqueness we need a second condition. The difference between a mean-self-financing trading strategy and a self-financing trading strategy is minimal, if $E[\sum_{t=1}^{T} \{\Delta G_t^*\}^2]$ is minimized. Therefore we consider the optimisation problem

$$E[\sum_{t=1}^{T} \{\Delta G_t^*\}^2] \to \text{Min!}$$

With the tower property of conditional expectations and the martingale property of the accumulated profit and loss process we have $E[\Delta G_s^* \Delta G_t^*] = 0$ for $s \neq t$ P-a.s. So we can replace the sum $\sum_{t=1}^{T} \{\Delta G_t^*\}^2$ by the square $(G_T^* - G_0)^2$. Therefore we have $E[G_T^*] = G_0^*$ and we see that $E[(G_T^* - G_0^*)^2]$ is equal to the variance of the random variable $G_T^* - G_0^*$. If we set $\vartheta_0 = V_0^*$ and $H_T^* = \sum_{t=1}^{T} Z_t^*$ then the difference $G_T^* - G_0^*$ is

$$G_T^* - G_0^* = \vartheta_0 + \sum_{t=1}^{T} \vartheta_t \Delta S_t^* - H_T^*$$

We define $\vartheta = (\vartheta_0, \vartheta_1, \ldots, \vartheta_T)$ where $\vartheta_0 \in \mathbb{R}$ is the initial investment and $\{\vartheta_t; t = 1, 2, \ldots, T\}$ is a trading strategy with $\vartheta_t \Delta S_t^* \in \mathcal{L}^2(\Omega, \mathcal{F}_T, P)$ for $t = 1, 2, \ldots, T$. We are searching for such ϑ minimizing the term

$$E[(G_T^* - G_0)^2] = E[(\vartheta_0 + \sum_{t=1}^{T} \vartheta_t \Delta S_t^* - H_T^*)^2]$$

With the definition $A_T(\vartheta) = \vartheta_0 + \sum_{t=1}^T \vartheta_t \Delta S_t^*$ this reads

$$\begin{aligned} E[(G_T^* - G_0)^2] &= E[(\vartheta_0 + \sum_{t=1}^T \vartheta_t \Delta S_t^* - H_T^*)^2] \\ &= E[A_T^2(\vartheta)] - 2E[A_T(\vartheta)H_T^*] + E[(H_T^*)^2] \end{aligned}$$

and because $E[(H_T^*)^2]$ being a real number we can also search for ϑ that minimizes the so called *energy functional*

$$J(\vartheta) = E[A_T^2(\vartheta)] - 2E[A_T(\vartheta)H_T^*]$$

If ϑ^* minimizes the energy functional $J(\vartheta)$ we call ϑ^* the optimal mean-self-financing trading strategy.

The minimum of the energy functional Energy functionals often occur in physics and technology related to the solutions of partial differential equations, where the first term of the energy functional is given by a continuous and coercive bilinear form and the second term by a linear, continuous functional each defined on a suitable Hilbert space. For an energy functional given by a continuous and coercive bilinear form and a continuous linear functional the existence and uniqueness of a solution for the minimum of the energy functional follows from the theorem of Lax-Milgram. Therefore the essential step for solving the minimum problem of the above energy functional is to construct a suitable Hilbert space so that $a(\varphi, \eta) = E[A_T(\varphi)A_T(\eta)]$ is bilinear, continuous and coercive and $b(\varphi) = E[A_T(\varphi)H_T^*]$ is linear and continuous.

Let $\mathcal{H}_\sigma$ be the space of all $(T+1)$-tuples $\vartheta = (\vartheta_0, \vartheta_1, \ldots, \vartheta_T)$ with the following properties:

1. $\vartheta_0 \in \mathbb{R}$
2. $\{\vartheta_t; t = 1, 2, \ldots, T\}$ is a predictable and adapted stochastic process with $\vartheta_t \in \mathcal{L}^0(\Omega, \mathcal{F}_{t-1}, P)$ for $t = 1, 2, \ldots, T$
3. For $t = 1, 2, \ldots, T$ the random variable $\vartheta_t \sigma_t^*$ is square integrable, i.e. $\vartheta_t \sigma_t^* \in \mathcal{L}^2(\Omega, \mathcal{F}_{t-1}, P)$ where σ_t^* is the square root of $Var[\Delta S_t^* | \mathcal{F}_{t-1}]$

If $\alpha, \beta \in \mathcal{H}_\sigma$ and if $\alpha_0 = \beta_0$ and $\alpha_t = \beta_t$ P-a.s. for $t = 1, 2, \ldots, T$ then α, β are identified with each other. It is easy to see that $\mathcal{H}_\sigma$ is a linear space. Given $\alpha, \beta \in \mathcal{H}_\sigma$ then

$$(\alpha, \beta) = \alpha_0 \beta_0 + \sum_{j=1}^T E[\alpha_j \beta_j (\sigma_j^*)^2] \tag{1}$$

defines a weighted inner product on $\mathcal{H}_\sigma$ [4]. We write $\|\cdot\|_\sigma$ for the norm given by the inner product on $\mathcal{H}_\sigma$.

[4] We have assumed that $P(\sigma_t^* = 0) = 0$ P-f.s. for $t = 1, 2, \ldots, T$. Without this assumption one must also demand that ϑ_t is equal to 0 on the set $\{\sigma_t^* = 0\}$.

Theorem 1 *With the weighted inner product (1) defined as above, $\mathcal{H}_\sigma$ is a Hilbert space.*

Proof: Let $(\vartheta_n)_{n\geq 1}$ be a Cauchy sequence in $\mathcal{H}_\sigma$. Then for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ with $\|\vartheta_n - \vartheta_m\|_\sigma \leq \epsilon$ for all $n, m \geq n_0$. By the definition of the weighted inner product $(\vartheta_{0,n})_{n\geq 1}$ is a Cauchy sequence in $\mathbb{R}$ and is therefore convergent to an element $\vartheta_0 \in \mathbb{R}$. For every $t = 1, 2, \ldots, T$ we have that $(\vartheta_{t,n}\sigma_t^*)_{n\geq 1}$ is a Cauchy sequence in $\mathcal{L}^2(\Omega, \mathcal{F}_{t-1}, P)$ and is therefore convergent to an element $\alpha_t \in \mathcal{L}^2(\Omega, \mathcal{F}_{t-1}, P)$ by the completeness of the space $\mathcal{L}^2(\Omega, \mathcal{F}_{t-1}, P)$. With the definition $\vartheta_t = \alpha_t(\sigma_t^*)^{-1}$ we have $\vartheta_t\sigma_t^* \in \mathcal{L}^2(\Omega, \mathcal{F}_{t-1}, P)$, $\vartheta = (\vartheta_0, \vartheta_1, \ldots, \vartheta_T) \in \mathcal{H}_\sigma$ and $\vartheta_n \to \vartheta$ for $n \to \infty$. ❑

Theorem 2 *If the price process $\{S_t^*; t = 0, 1, 2, \ldots, T\}$ fulfills the MVT then the map*

$$\begin{aligned} A_T : \mathcal{H}_\sigma &\to \mathcal{L}^2(\Omega, \mathcal{F}, P) \\ \vartheta &\mapsto \vartheta_0 + \sum_{j=1}^{T} \vartheta_j \Delta S_j^* \end{aligned}$$

is linear, continuous and injective and $A_T(\mathcal{H}_\sigma)$ is a closed subspace of $\mathcal{L}^2(\Omega, \mathcal{F}, P)$[5]*.*

Proof:
(1) The linearity of A_T is clear. For $\vartheta \in \mathcal{H}_\sigma$ we have

$$\begin{aligned} E[(\vartheta_t \Delta S_t^*)^2] &= E[\vartheta_t^2 E[(\Delta S_t^*)^2 | \mathcal{F}_{t-1}]] \\ &\leq (1 + C) E[\vartheta_t^2 (\sigma_t^*)^2] \end{aligned}$$

by the predictability of ϑ_t and because the price process fulfills the MVT. Moreover $\vartheta_t\sigma_t^*$ is square integrable by the definition of $\mathcal{H}_\sigma$ and this shows that $A_T(\vartheta)$ is square integrable. With the Minkonwski inequality and the definition of A_T we get $\|A_T(\vartheta)\|_{\mathcal{L}^2} \leq |\vartheta_0| + \sum_{j=1}^{T} \|\vartheta_j \Delta S_j^*\|_{\mathcal{L}^2}$ in the $\mathcal{L}^2$ norm. With the inequality shown above and because of $E[\vartheta_t^2(\sigma_t^*)^2] \leq \|\vartheta\|_\sigma$ for $t = 0, 1, \ldots, T$ we have $\|A_T(\vartheta)\|_{\mathcal{L}^2} \leq \sqrt{(T+1)(1+C)}\|\vartheta\|_\sigma$ so A_T is continuous.

(2) To show that A_T is injective let $A_T(\vartheta) = 0$. By the definition of $A_T(\vartheta)$ it is easy to see that $A_T(\vartheta) = A_{T-1}(\vartheta) + \vartheta_T \Delta S_T^*$. If we multiply with $\Delta S_T^* - \mu_T^*$ and calculate the conditional expectation, given $\mathcal{F}_{T-1}$, and use the assumption $A_T(\vartheta) = 0$ and the $\mathcal{F}_{T-1}$-measurability of $A_{T-1}(\vartheta)$ it follows that $\vartheta_T(\sigma_T^*)^2 = 0$. Moreover σ_T^* is P-a.s. not equal 0, so we get $\vartheta_T = 0$ P-f.s. Now we have $A_{T-1}(\vartheta) = 0$ and the result that $\vartheta = 0$ P-a.s. follows by backward induction.

(3) Finally we will show that $A_T(\mathcal{H}_\sigma)$ is a closed subspace of $\mathcal{L}^2(\Omega, \mathcal{F}, P)$. Let $u \in \overline{A_T(\mathcal{H}_\sigma)}$, so there exists a sequence $u_n \in A_T(\mathcal{H}_\sigma)$ with $\|u_n - u\|_{L^2} \to 0$ for $n \to \infty$. For every $u_n \in A_T(\mathcal{H}_\sigma)$ there exists a $\vartheta_n \in \mathcal{H}_\sigma$ with $A_T(\vartheta_n) = u_n$. Moreover the sequence

[5] Remember that all equations are only true P-a.s.

$(\vartheta_n)_{n\geq 1}$ is a Cauchy sequence in $\mathcal{H}_\sigma$. To proof this, let $\epsilon > 0$. Then there exists $n_0 \in \mathbb{N}$, so that $\|u_n - u_m\|_{\mathcal{L}^2} \leq \epsilon$ is true for all $n, m \geq n_0$. Now we have $|\vartheta_{0,n} - \vartheta_{0,m}| \leq \epsilon$ and $\|(\vartheta_{t,n} - \vartheta_{t,m})\Delta S_t^*\|_{\mathcal{L}^2} \leq \epsilon$ for $t = 1, 2, \ldots, T$ and with the inequality $(\sigma_t^*)^2 \leq E[(\Delta S_t^*)^2|\mathcal{F}_{t-1}]$ we see that $(\vartheta_n)_{n\geq 1}$ is a Cauchy sequence in $\mathcal{H}_\sigma$. By the completeness of $\mathcal{H}_\sigma$ there exists $\vartheta \in \mathcal{H}_\sigma$ so that $(\vartheta_n)_{n\geq 1}$ is convergent to ϑ and because of the continuity of A_T it follows that $A_T(\vartheta) = u$. This proves that $A_T(\mathcal{H}_\sigma)$ is closed in $\mathcal{L}^2(\Omega, \mathcal{F}, P)$. ❑

From the last theorem it follows that A_T is a continuous, linear and bijective map from the Banachspace $\mathcal{H}_\sigma$ to the Banachspace $A_T(\mathcal{H}_\sigma)$. With the open mapping theorem it follows that A_T^{-1} is a linear and continuous map (see [8]), i.e. there exists $k > 0$ with the property $\|A_T^{-1}(u)\|_\sigma \leq k\|u\|_{\mathcal{L}^2}$ for all $u \in A_T(\mathcal{H}_\sigma)$. But this is the same as $\|A_T(\vartheta)\|_{\mathcal{L}^2} \geq m\|\vartheta\|_\sigma$ for a real number $m > 0$ and all $\vartheta \in \mathcal{H}_\sigma$. Now let $\mathcal{M} \subseteq \mathcal{H}_\sigma$ be a nonempty, convex and closed subset of $\mathcal{H}_\sigma$. Then the best least-square approximation of $x \in \mathcal{H}_\sigma$ is given by $u \in \mathcal{M}$ so that $\|x - u\|_\sigma = \inf_{v\in\mathcal{M}} \|x - v\|_\sigma$ is true. The best least-square approximation defines a continuous map $P_\mathcal{M} : \mathcal{H}_\sigma \to \mathcal{M}$ with the property $\|P_\mathcal{M}(x) - P_\mathcal{M}(y)\|_\sigma \leq \|x - y\|_\sigma$ for all $x, y \in \mathcal{H}_\sigma$. Let $x \in \mathcal{H}_\sigma$ then $P_\mathcal{M}(x) = u$ also fulfills the variational inequality $(x - u, v - u) \leq 0 \ \forall v \in \mathcal{M}$.

Theorem 3 *If $\mathcal{M} \subseteq \mathcal{H}_\sigma$ is a nonempty, convex and closed subset of $\mathcal{H}_\sigma$ and if the price process fulfills the MVT then for every square integrable and adapted payment process $\{Z_t^*; t = 1, 2, \ldots, T\}$ there exists a unique $\vartheta^* \in \mathcal{M}$ with the property that $J(\vartheta^*)$ is the minimum of the energy functional.*

Proof: To proof the theorem we muss show that $E[A_T(\varphi)A_T(\eta)]$ is a symmetric, bounded and coercive bilinear form and $E[A_T(\varphi)H_T^*]$ is a bounded linear functional. With the definition

$$\begin{aligned} a : \mathcal{H}_\sigma \times \mathcal{H}_\sigma &\to \mathbb{R} \\ (\varphi, \eta) \quad &\mapsto E[A_T(\varphi)A_T(\eta)] \end{aligned}$$

we have a symmetric bilinear form. By the continuity of A_T and the Cauchy-Schwarz inequality $a(\varphi, \eta)$ is bounded. And because A_T^{-1} is continuous there exists a real number $m > 0$ so that $\|A_T(\varphi)\|_{\mathcal{L}^2} \geq m\|\varphi\|_\sigma$ for alle $\varphi \in \mathcal{H}_\sigma$. This proves $a(\varphi, \varphi) \geq m^2\|\varphi\|^2_{\mathcal{H}_\sigma}$ for all $\varphi \in \mathcal{H}_\sigma$, so a is coercive. Let

$$\begin{aligned} b : \mathcal{H}_\sigma &\to \mathbb{R} \\ \varphi \quad &\mapsto E[A_T(\varphi)H_T^*] \end{aligned}$$

Then $b(\varphi)$ is a continuous linear functional on $\mathcal{H}_\sigma$. This follows from the Cauchy-Schwarz inequality and the continuity of A_T and because $H_T^* = \sum_{t=1}^T Z_t^*$ is a square integrable. ❑

For $\mathcal{M} = \mathcal{H}_\sigma$ the minimum $\vartheta^* \in \mathcal{H}_\sigma$ is the solution of the equation $a(\vartheta^*, \vartheta) = b(\vartheta) \ \forall \vartheta \in \mathcal{H}_\sigma$ [7] and if we use the definition for $A_T(\vartheta^*)$ we have

$$E\Big[\Big(\vartheta_0^* + \sum_{t=1}^T \vartheta_t^*\Delta S_t^* - H_T^*\Big)A_T(\vartheta)\Big] = 0 \ \forall \vartheta \in \mathcal{H}_\sigma$$

In his paper [20] M. Schweizer proves this equation directly using the projection theorem of Hilbert spaces [23] by showing that $G_T(\Theta)$ defined by

1. $G_T(\Theta) \subseteq \mathcal{L}^2(\Omega, \mathcal{F}, P)$,
2. $u \in G_T(\Theta)$ if and only if there is a predictable stochastic process $\{\vartheta_t; t = 1, 2, \ldots, T\}$ with $\vartheta_t \Delta S_t^*$ square integrable for all t and $u = \sum_{t=1}^{T} \vartheta_t \Delta S_t^*$,

is closed in $\mathcal{L}^2(\Omega, \mathcal{F}, P)$ if the price process fulfills the MVT. With this approach the structure of the problem remains hidden. That is the reason why the results of the present paper are not only a new proof for the variance-optimal hedging in incomplete and time discrete markets but also a slight generalization.

The existence and uniqueness for the minimum problem of the energy functional is often proven using the Banach fixed point theorem. Since this also is an algorithm by which the optimal trading strategy - at least in principle - can be calculated in a numerical way, this will be formulated as a theorem. Previously we gave another discription for $a(\varphi, \psi)$ and $b(\psi)$. By the theorem of Ritz there exists a linear and continuous map $\mathbf{A} : \mathcal{H}_\sigma \to \mathcal{H}_\sigma$ with $a(\varphi, \psi) = (\mathbf{A}(\varphi), \psi)$ for all $\varphi, \psi \in \mathcal{H}_\sigma$. Furthermore there exists $\mathbf{f} \in \mathcal{H}_\sigma$ with $b(\psi) = (\mathbf{f}, \psi)$ for all $\psi \in \mathcal{H}_\sigma$ [7]. Hence the optimal trading strategy is the solution of the equation

$$\mathbf{A}(\vartheta^*) = \mathbf{f} \tag{2}$$

which we solve in the next section [12]. With the definitions $\Delta S_0^* = 1$, $\sigma_0^* = 1$ and $\mathcal{F}_{-1} = \{\emptyset, \Omega\}$ one gets

$$\begin{aligned} b(\psi) &= E[A_T(\psi) H_T^*] \\ &= \sum_{t=0}^{T} E[\psi_t \Delta S_t^* H_T^*] \\ &= \sum_{t=0}^{T} E[\psi_t \frac{E[\Delta S_t^* H_T^* | \mathcal{F}_{t-1}]}{(\sigma_t^*)^2} (\sigma_t^*)^2] \\ &= (\psi, \mathbf{f}) \end{aligned}$$

i.e.

$$\mathbf{f} = \left(E[H_T^*], \frac{E[\Delta S_1^* H_T^* | \mathcal{F}_0]}{(\sigma_1^*)^2}, \ldots, \frac{E[\Delta S_T^* H_T^* | \mathcal{F}_{T-1}]}{(\sigma_T^*)^2} \right)$$

In the same way one showes

$$\mathbf{A}(\varphi) = \left(E[A_T(\varphi)], \frac{E[\Delta S_1^* A_T(\varphi) | \mathcal{F}_0]}{(\sigma_1^*)^2}, \ldots, \frac{E[\Delta S_T^* A_T(\varphi) | \mathcal{F}_{T-1}]}{(\sigma_T^*)^2} \right)$$

Now we proof the following theorem.

Theorem 4 *If $\mathcal{M} \subseteq \mathcal{H}_\sigma$ is a nonempty, convex and closed subset of $\mathcal{H}_\sigma$ and the price process fulfills the MVT then the unique defined trading strategy $\vartheta^* \in \mathcal{M}$ given by Theorem (3) is the fixpoint of the map $\Phi : \mathcal{M} \to \mathcal{M}$ defined by*

$$\Phi(\vartheta) = P_{\mathcal{M}}(\lambda \mathbf{f} - \lambda \mathbf{A}(\vartheta) + \vartheta) \tag{3}$$

with $\lambda > 0$ arbitrary. Starting with any trading strategy $\vartheta \in \mathcal{M}$ and with a suitably chosen $\lambda > 0$ one can calculate ϑ^* at specified accuracy by the fixpoint theorem of Banach.

Proof: By theorem (3) there exists an optimal trading strategy ϑ^* which fulfills the inequality $a(\vartheta^*, \vartheta - \vartheta^*) \geq b(\vartheta - \vartheta^*)$ for all $\vartheta \in \mathcal{M}$. With $\mathbf{A}$ and f defined as above we have $(\mathbf{A}(\vartheta^*), \vartheta - \vartheta^*) \geq (\mathbf{f}, \vartheta - \vartheta^*)$. Multiplication with an arbitrary real number $\lambda > 0$ and subtracting the term on the left hand-side to the right hand-side gives

$$\Big(\lambda \mathbf{f} - \lambda \mathbf{A}(\vartheta^*), \vartheta - \vartheta^*\Big) \leq 0 \ \forall \vartheta \in \mathcal{M}$$

Now expand the first slot in the inner product with $\vartheta^* - \vartheta^*$ to get

$$\Big(\big\{\lambda \mathbf{f} - \lambda \mathbf{A}(\vartheta^*) + \vartheta^*\big\} - \vartheta^*, \vartheta - \vartheta^*\Big) \leq 0 \ \forall \vartheta \in \mathcal{M}$$

This means ϑ^* is the best least-square approximation to $\lambda \mathbf{f} - \lambda \mathbf{A}(\vartheta^*) + \vartheta^* \in \mathcal{H}_\sigma$ and this is the same as $\vartheta^* = P_{\mathcal{M}}(\lambda \mathbf{f} - \lambda \mathbf{A}(\vartheta^*) + \vartheta^*)$. Therefore ϑ^* is a fixpoint of $\Phi(\vartheta)$. By the continuity of the projection $P_{\mathcal{M}}$ we have

$$\begin{aligned}
\|\Phi(\varphi) - \Phi(\eta)\|^2_{\mathcal{H}_\sigma} &\leq \|\lambda \mathbf{f} - \lambda \mathbf{A}(\varphi) + \varphi - (\lambda \mathbf{f} - \lambda \mathbf{A}(\eta) + \eta)\|^2_{\mathcal{H}_\sigma} \\
&= \|\varphi - \eta - \lambda(\mathbf{A}(\varphi) - \mathbf{A}(\eta))\|^2_{\mathcal{H}_\sigma} \\
&= \|\varphi - \eta\|^2_{\mathcal{H}_\sigma} - 2\lambda(\varphi - \eta, \mathbf{A}(\varphi - \eta)) + \lambda^2 \|\mathbf{A}(\varphi - \eta)\|^2_{\mathcal{H}_\sigma} \\
&\leq (1 - 2\lambda m^2 + \lambda^2 M^2)\|\varphi - \eta\|^2_{\mathcal{H}_\sigma}
\end{aligned}$$

The last inequality is given by the continuity of $\|\mathbf{A}(\varphi)\|_{\mathcal{H}_\sigma} \leq M\|\varphi\|_{\mathcal{H}_\sigma}$ and the coercivity of $a(\varphi, \psi)$, i.e. $(\mathbf{A}(\varphi), \varphi) \geq m^2 \|\varphi\|^2_{\mathcal{H}_\sigma}$. If we choose $0 < \lambda < 2\frac{m^2}{M^2}$ we have $(1 - 2\lambda m^2 + \lambda^2 M^2) < 1$ and therefore $\Phi(\vartheta)$ is a contraction mapping. ❏

If we know the projection map $P_{\mathcal{M}}$ then we can start with an arbitrary $\vartheta_0 \in \mathcal{M}$ to calculate the trading strategies $\vartheta_{n+1} = P_{\mathcal{M}}(\lambda \mathbf{f} - \lambda \mathbf{A}(\vartheta_n) + \vartheta_n)$ for $n = 0, 1, \dots$ iteratively. The number of iterations result from the desired accuracy of the approximation concerning the optimal trading strategy ϑ^*. At first glance this looks like a very easy possibility to calculate the optimal trading strategy. However we have to calculate $\mathbf{A}(\vartheta_n)$ and $P_{\mathcal{M}}(\lambda \mathbf{f} - \lambda \mathbf{A}(\vartheta_n) + \vartheta_n)$ in every calculation step. And this can be massy because of the complex structure of the map $\mathbf{A}$ and the projection map $P_{\mathcal{M}}$. Additionally the convergence speed to the fixpoint can be very slow for small values of λ. So we must calculate a lot of iterations to approximate the optimal trading strategy in order to achieve acceptable accuracy. In the case of $\mathcal{M} = \mathcal{H}_\sigma$ one can solve the equation (2) explicitly. This will be done in the next section.

Calculation of ϑ^* in the case $\mathcal{M} = \mathcal{H}_\sigma$ In this section we solve the equation

$$\mathbf{A}(\vartheta^*) = \mathbf{f}$$

in the case of $\mathcal{M} = \mathcal{H}_\sigma$. To do this, let $\vartheta^* = (\vartheta_0^*, \vartheta_1^*, \ldots, \vartheta_T^*)'$ be the optimal trading strategy. With the definition of $\mathbf{A}(\vartheta^*)$ and $\mathbf{f}$ one gets the following system of equations

$$\begin{array}{rcl} E[A_T(\vartheta^*)] & = & E[H_T^*] \\ E[A_T(\vartheta^*)\Delta S_1^*|\mathcal{F}_0] & = & E[H_T^*\Delta S_1^*|\mathcal{F}_0] \\ & \vdots & \\ E[A_T(\vartheta^*)\Delta S_T^*|\mathcal{F}_{T-1}] & = & E[H_T^*\Delta S_T^*|\mathcal{F}_{T-1}] \end{array} \tag{4}$$

For ease of description we define the following random variables

$$\begin{aligned} M_t &= \prod_{j=t}^{T}(1-\alpha_j\Delta S_j^*) \\ \alpha_t &= \frac{E[\Delta S_t^* M_{t+1}|\mathcal{F}_{t-1}]}{E[(\Delta S_t^*)^2 M_{t+1}|\mathcal{F}_{t-1}]}, t = 1, 2, \ldots, T \end{aligned}$$

where the empty product is equal to one. We have to show that $\Delta S_t^* M_{t+1}$ is square integrable for $t = 1, 2, \ldots, T$. With the MVT and the definition of M_T and α_T one gets the equation

$$E[M_T^2|\mathcal{F}_{T-1}] = E[M_T|\mathcal{F}_{T-1}] \in [0,1] \tag{5}$$

Now we make the assumption $E[M_{t+1}^2|\mathcal{F}_t] = E[M_{t+1}|\mathcal{F}_t] \in [0,1]$. Then the equation and restriction follows by backward induction for M_t. To proof this let

$$\begin{aligned} E[(\Delta S_t^* M_{t+1})^2|\mathcal{F}_{t-1}] &= E[(\Delta S_t^*)^2 M_{t+1}^2|\mathcal{F}_{t-1}] \\ &= E[(\Delta S_t^*)^2 E[M_{t+1}^2|\mathcal{F}_t]|\mathcal{F}_{t-1}] \\ &\leq E[(\Delta S_t^*)^2|\mathcal{F}_{t-1}] \end{aligned}$$

where we have used $E[M_{t+1}^2|\mathcal{F}_t] \leq 1$. This proves that $\Delta S_t^* M_{t+1}$ is square integrable und therefore α_t is well defined. Now we have

$$\begin{aligned} E[M_t\Delta S_t^*|\mathcal{F}_{t-1}] &= E[(1-\alpha_t\Delta S_t^*)\Delta S_t^* M_{t+1}|\mathcal{F}_{t-1}] \\ &= E[\Delta S_t^* M_{t+1}|\mathcal{F}_{t-1}] - \alpha_t E[(\Delta S_t^*)^2 M_{t+1}|\mathcal{F}_{t-1}] = 0 \end{aligned} \tag{6}$$

and[6]

$$\begin{aligned} E[M_t^2|\mathcal{F}_{t-1}] &= E[(1-\alpha_t\Delta S_t^*)^2 M_{t+1}^2|\mathcal{F}_{t-1}] \\ &= E[(1-\alpha_t\Delta S_t^*)^2 E[M_{t+1}^2|\mathcal{F}_t]|\mathcal{F}_{t-1}] \\ &= E[(1-\alpha_t\Delta S_t^*)^2 E[M_{t+1}|\mathcal{F}_t]|\mathcal{F}_{t-1}] \\ &= E[(1-\alpha_t\Delta S_t^*)^2 M_{t+1}|\mathcal{F}_{t-1}] \\ &= E[(1-\alpha_t\Delta S_t^*)M_t|\mathcal{F}_{t-1}] = E[M_t|\mathcal{F}_{t-1}] \end{aligned}$$

[6] Therefore $\{M_1 S_t^*; t = 0, 1, \ldots, T\}$ is a martingale.

The random variable M_t^2 is P-a.s. not negative, i.e. we have the inequality $E[M_t^2|\mathcal{F}_{t-1}] = E[M_t|\mathcal{F}_{t-1}] \geq 0$. The other inequality results from

$$\begin{aligned} E[M_t|\mathcal{F}_{t-1}] &= E[(1-\alpha_t \Delta S_t^*)M_{t+1}|\mathcal{F}_{t-1}] \\ &= E[M_{t+1}|\mathcal{F}_{t-1}] - \alpha_t E[\Delta S_t^* M_{t+1}|\mathcal{F}_{t-1}] \\ &\leq 1 - \frac{(E[\Delta S_t^* M_{t+1}|\mathcal{F}_{t-1}])^2}{E[(\Delta S_t^* M_{t+1})^2|\mathcal{F}_{t-1}]} \\ &= \frac{\mathrm{Var}[\Delta S_t^* M_{t+1}|\mathcal{F}_{t-1}]}{E[(\Delta S_t^* M_{t+1})^2|\mathcal{F}_{t-1}]} \leq 1 \end{aligned}$$

Theorem 5 *If the price process fulfills the MVT then the solution of (4) is given for every square integrable and adapted payment process $\{Z_t^*; t = 1,2,\dots,T\}$ by $\vartheta^* = (\vartheta_0^*, \vartheta_1^*, \dots, \vartheta_T^*)$ with the t-th component*

$$\vartheta_t^* = \frac{E[H_T^* \Delta S_t^* M_{t+1}|\mathcal{F}_{t-1}]}{E[(\Delta S_t^*)^2 M_{t+1}|\mathcal{F}_{t-1}]} - A_{t-1}(\vartheta^*)\alpha_t$$

and with $A_{t-1}(\vartheta^*) = \vartheta_0^* + \sum_{j=1}^{t-1} \vartheta_j^* \Delta S_j^*$ for $t = 0,1,\dots,T$.

Proof: If one inserts $A_T(\vartheta^*) = A_{T-1}(\vartheta^*) + \vartheta_T^* \Delta S_T^*$ in the last equation of (4) and use the $\mathcal{F}_{T-1}$-measurability of ϑ_T^* we have

$$\vartheta_T^* = \frac{E[H_T^* \Delta S_T^*|\mathcal{F}_{T-1}]}{E[(\Delta S_T^*)^2|\mathcal{F}_{T-1}]} - A_{T-1}(\vartheta^*)\alpha_T$$

which is the desired formula for $t = T$. If we insert ϑ_T^* in the equations

$$E[A_{T-1}(\vartheta^*)\Delta S_t^*|\mathcal{F}_{t-1}] + E[\vartheta_T^* \Delta S_T^* \Delta S_t^*|\mathcal{F}_{t-1}] = E[H_T^* \Delta S_t^*|\mathcal{F}_{t-1}]$$

for $t = 1,2,\dots,T-1$ we have

$$\begin{aligned} E[A_{T-1}(\vartheta^*)\Delta S_t^*|\mathcal{F}_{t-1}] &+ E\Big[\Delta S_t^* \Delta S_T^* \frac{E[\Delta S_T^* H_T^*|\mathcal{F}_{T-1}]}{E[(\Delta S_T^*)^2|\mathcal{F}_{T-1}]}\Big|\mathcal{F}_{t-1}\Big] \\ &- E[\Delta S_t^* \Delta S_T^* A_{T-1}(\vartheta^*)\alpha_T|\mathcal{F}_{t-1}] \\ &= E[\Delta S_t^* H_T^*|\mathcal{F}_{t-1}] \end{aligned} \tag{7}$$

Now we transform the second expectation to

$$
\begin{aligned}
E\Big[\Delta S_t^*\Delta S_T^*\frac{E[\Delta S_T^* H_T^*|\mathcal{F}_{T-1}]}{E[(\Delta S_T^*)^2|\mathcal{F}_{T-1}]}\Big|\mathcal{F}_{t-1}\Big] &= E\Big[E\Big[\Delta S_t^*\Delta S_T^*\frac{E[\Delta S_T^* H_T^*|\mathcal{F}_{T-1}]}{E[(\Delta S_T^*)^2|\mathcal{F}_{T-1}]}\Big|\mathcal{F}_{T-1}\Big]\Big|\mathcal{F}_{t-1}\Big] \\
&= E\Big[\Delta S_t^* E[\Delta S_T^*|\mathcal{F}_{T-1}]\frac{E[\Delta S_T^* H_T^*|\mathcal{F}_{T-1}]}{E[(\Delta S_T^*)^2|\mathcal{F}_{T-1}]}\Big|\mathcal{F}_{t-1}\Big] \\
&= E\Big[\Delta S_t^* E[\Delta S_T^* H_T^*|\mathcal{F}_{T-1}]\frac{E[\Delta S_T^*|\mathcal{F}_{T-1}]}{E[(\Delta S_T^*)^2|\mathcal{F}_{T-1}]}\Big|\mathcal{F}_{t-1}\Big] \\
&= E\Big[\Delta S_t^* E[\Delta S_T^* H_T^*|\mathcal{F}_{T-1}]\alpha_T\Big|\mathcal{F}_{t-1}\Big] \\
&= E\Big[E[\Delta S_t^*\Delta S_T^* H_T^*\alpha_T|\mathcal{F}_{T-1}]\Big|\mathcal{F}_{t-1}\Big] \\
&= E[\Delta S_t^*\Delta S_T^* H_T^*\alpha_T|\mathcal{F}_{t-1}]
\end{aligned}
$$

and and by inserting in the last expectation in (7) we have

$$
\begin{aligned}
E[A_{T-1}(\vartheta^*)\Delta S_t^*|\mathcal{F}_{t-1}] &+ E[\Delta S_t^*\Delta S_T^* H_T^*\alpha_T|\mathcal{F}_{t-1}] \\
&- E[\Delta S_t^*\Delta S_T^* A_{T-1}(\vartheta^*)\alpha_T|\mathcal{F}_{t-1}] \\
&= E[\Delta S_t^* H_T^*|\mathcal{F}_{t-1}]
\end{aligned}
$$

With the definition of M_T we have

$$
E[A_{T-1}(\vartheta^*)\Delta S_t^* M_T|\mathcal{F}_{t-1}] = E[\Delta S_t^* H_T^* M_T|\mathcal{F}_{t-1}]
$$

for $t = 1, 2, \ldots, T-1$. It is easy to see that the same is true for $t = 0$, i.e. we can transform the equation $E[A_T(\vartheta^*)] = E[H_T^*]$ to $E[A_{T-1}(\vartheta^*)M_T] = E[H_T^* M_T]$. We see that the „transformed“ system of equations is reduced by one dimension. Now we assume by backward induction, that the formula is true for ϑ_k^* for $k = t+1, t+2, \ldots, T$ and that the „reduced“ system of equations is given by

$$
\begin{array}{ccc}
E[A_t(\vartheta^*)M_{t+1}] & = & E[H_T^* M_{t+1}] \\
E[A_t(\vartheta^*)\Delta S_1^* M_{t+1}|\mathcal{F}_0] & = & E[H_T^*\Delta S_1^* M_{t+1}|\mathcal{F}_0] \\
 & \vdots & \\
E[A_t(\vartheta^*)\Delta S_t^* M_{t+1}|\mathcal{F}_{t-1}] & = & E[H_T^*\Delta S_t^* M_{t+1}|\mathcal{F}_{t-1}]
\end{array}
\tag{8}
$$

Then we use the decomposition $A_t(\vartheta^*) = A_{t-1}(\vartheta^*) + \vartheta_t^*\Delta S_t^*$ and solve the equation for ϑ_t^*. With the same calculation as before and the definition of $M_t = (1 - \alpha_t\Delta S_t^*)M_{t+1}$ we get the system of equations

$$
\begin{aligned}
E[A_{t-1}(\vartheta^*)M_t] &= E[H_T^* M_t] \\
E[A_{t-1}(\vartheta^*)\Delta S_1^* M_t|\mathcal{F}_0] &= E[H_T^*\Delta S_1^* M_t|\mathcal{F}_0] \\
&\vdots \\
E[A_{t-1}(\vartheta^*)\Delta S_{t-1}^* M_t|\mathcal{F}_{t-2}] &= E[H_T^*\Delta S_{t-1}^* M_t|\mathcal{F}_{t-2}]
\end{aligned}
$$

for $k = t$. ❑

Although the system of equations could be solved simply, the calculation of the components of the trading strategy is rather complicated. Therefore a direct calculation of the trading strategy is extremely difficult or the calculation is only possible if one makes restrictive assumptions (see for example [5]). To this extent it is necessary to develop suitable methods for the numerical calculation of the trading strategy. This will be done in a following article called *calculation model*.

Initial investment From theorem 5 we get for $t = 0$ the initial investment

$$V_0^* = \vartheta_0^* = \frac{E[H_T^* M_1]}{E[M_1]} \tag{9}$$

We have proved the equation $E[M_1^2] = E[M_1] \in [0,1]$ and this means that $E[M_1] = 0$ if and only if $M_1 = 0$ P-a.s. Although we cannot exclude the case $M_1 = 0$ for the rest of this article we will assume that $M_1 \neq 0$ P-a.s. In this case we have the change of measure $dQ = \frac{M_1}{E[M_1]} dP$. But it is possible that the probability of the event $\{M < 0\}$ is greater as null so this change of measure is in general a change to a signed measure (see [21] for more information about signed measures).

With the optimal trading strategy ϑ^* the portfolio process is given by

$$V_t^* = \sum_{t=t+1}^{T} E\big[Z_j^* - \vartheta_j^* \Delta S_j^* | \mathcal{F}_t\big], t = 0, 1, \ldots, T \tag{10}$$

where the empty sum is equal to zero by convention. We proof another representation for the portfolio process. To do this we need the definition.

Definition 2 *Let $\{Z_t^*; t = 1, 2, \ldots, T\}$ be a payment process and $\vartheta \in \mathcal{H}_\sigma$. Then the portfolio process (10) is nondegenerated if $P(V_t^* = 0) = 0$ P-a.s. for $t = 0, 1, \ldots, T-1$.*

Now we proof the following representation for the undiscounted portfolio process:

Theorem 6 *With the assumptions of theorem (3) let $\{Z_t; t = 1, 2, \ldots, T\}$ be a payment process and $\vartheta^* \in \mathcal{H}_\sigma$ the optimal trading strategy. If the portfolio process is nondegenerated then V_t has the representation*

$$V_t = E\left[\sum_{j=t+1}^{T} \frac{Z_j}{\prod_{i=t+1}^{j}(1+k_i)} \middle| \mathcal{F}_t\right], t = 0, 1, \ldots, T-1$$

where $\{k_t; t = 1, 2, \ldots, T\}$ is a predictable return process.

Proof: The profit and loss process is a martingale, so we have for $t = 0, 1, \ldots, T-1$ the equation

$$V_t^* + E[\vartheta_{t+1}^* \Delta S_{t+1}^* | \mathcal{F}_t] = E[Z_{t+1}^* + V_{t+1}^* | \mathcal{F}_t]$$

and with the following definition of the return process

$$\begin{aligned} R_{t+1}^P &= \frac{\vartheta_{t+1}^* \Delta S_{t+1} + \eta_{t+1} \Delta A_{t+1}}{V_t} \\ k_{t+1} &= E[R_{t+1}^P | \mathcal{F}_t] \end{aligned}$$

we have[7]

$$V_t^* \frac{1 + k_{t+1}}{1 + r_{t+1}} = E[Z_{t+1}^* + V_{t+1}^* | \mathcal{F}_t]$$

Since $\frac{1+k_{t+1}}{1+r_{t+1}}$ and A_t are $\mathcal{F}_t$-measurable we get

$$V_t = E\left[\frac{Z_{t+1}}{1 + k_{t+1}} + \frac{V_{t+1}}{1 + k_{t+1}} \middle| \mathcal{F}_t \right]$$

With $V_T = 0$ we have

$$V_{T-1} = E\left[\frac{Z_T}{1 + k_T} \middle| \mathcal{F}_{T-1} \right]$$

We assume by backward induction that the portfolio value at time $t+1$ is given by

$$V_{t+1} = E\left[\sum_{j=t+2}^{T} \frac{Z_j}{\prod_{i=t+2}^{j} (1 + k_i)} \middle| \mathcal{F}_{t+1} \right]$$

Finally we insert V_{t+1} in

$$V_t = E\left[\frac{Z_{t+1}}{(1 + k_{t+1})} + \frac{V_{t+1}}{(1 + k_{t+1})} \middle| \mathcal{F}_t \right]$$

and find for the portfolio value at time t the equation

$$V_t = E\left[\sum_{j=t+1}^{T} \frac{Z_j}{\prod_{i=t+1}^{j} (1 + k_i)} \middle| \mathcal{F}_t \right]$$

which proves the theorem. ❑

[7] For this equation we need that the portfolio process to be nondegenerated.

Theorem 7 *For $t = 1, 2, \ldots, T$ the return process is given by*

$$k_t = r_t + \beta_t(E[R_t^M | \mathcal{F}_{t-1}] - r_t)$$
$$\beta_t = \frac{Cov[R_t^P, R_t^M | \mathcal{F}_{t-1}]}{Var[R_t^M | \mathcal{F}_{t-1}]}$$

where $\{R_t^M; t = 1, 2, \ldots, T\}$ is the market return process.

Proof: It is easy to see that

$$\frac{\vartheta_t^* \Delta S_t^*}{V_{t-1}^*} = \frac{1}{1 + r_t}(R_t^P - r_t)$$

With the definition $R_t^M = \frac{\Delta S_t}{S_{t-1}}$ we have the equation

$$\frac{\Delta S_t^*}{S_{t-1}^*} = \frac{1}{1 + r_t}(R_t^M - r_t)$$

With $\beta_t = \frac{\vartheta_t^* S_{t-1}^*}{V_{t-1}^*}$ we get

$$\frac{\beta_t}{1 + r_t}(R_t^M - r_t) = \frac{1}{1 + r_t}(R_t^P - r_t)$$

i.e.

$$R_t^P = r_t + \beta_t(R_t^M - r_t)$$

and if we calculate the conditional expectation, given $\mathcal{F}_{t-1}$, we get the desired representation for k_t

$$k_t = r_t + \beta_t(E[R_t^M | \mathcal{F}_{t-1}] - r_t)$$

To proof that

$$\beta_t = \frac{Cov[R_t^P, R_t^M | \mathcal{F}_{t-1}]}{Var[R_t^M | \mathcal{F}_{t-1}]}$$

we only have to calculate the conditional covariance $Cov[R_t^P, R_t^M | \mathcal{F}_{t-1}]$. ❏

Applications As an application we calculate the shareholder value [10] of a firm and give an interpretation of the Economic Value Added (EVA). We assume that the investment universe is given and that the *dividend payment process* $\{D_t; t = 1, 2, \ldots, T\}$ of a firm is square integrable. Furthermore we assume that D_T is the *terminal value*, i.e. D_T is the value after the liquidation of the firm at time T.

There exists an optimal trading strategy $\vartheta^* \in \mathcal{H}_\sigma$ such that the value of the firm is given by

$$U_t = E\left[\sum_{j=t+1}^{T} \frac{D_j}{\prod_{i=t+1}^{j}(1+k_i)}\middle|\mathcal{F}_t\right]$$

for $t = 0, 1, \ldots, T-1$. Recall that the shareholder value of the firm U_T at time T is equal to zero as a result of the liquidation of the firm. We call the predictable return process $\{k_t; t = 1, 2, \ldots, T\}$ the *cost of capital process.*

With the definition $K(t) = (1+k_1)(1+k_2)\cdots(1+k_t)$ we have the slightly simpler formula

$$U_t = E\left[\sum_{j=t+1}^{T} D_j \frac{K(t)}{K(j)}\middle|\mathcal{F}_t\right], t = 0, 1, \ldots, T-1$$

Let EK_t for $t = 0, 1, \ldots, T$ be the shareholders' equity where $EK_T = 0$ results from the liquidation of the firm at time T. We assume for every business year, running from $t-1$ to t for $t = 1, 2, \ldots, T$, that the change of the shareholders' equity is given by the *clean surplus relation*

$$\Delta EK_t = P\&L_t - D_t$$

where $P\&L_t$ is the profit and loss of the t-th business year. If we insert the dividend $D_t = P\&L_t - \Delta EK_t$ in the formula for the shareholder value and use the equation $EK_T = 0$ then we get

$$U_t = EK_t + \sum_{j=t+1}^{T} E\left[(GuV_j - k_j EK_{j-1})\frac{K(t)}{K(j)}\middle|\mathcal{F}_t\right] \tag{11}$$

Now the *market value added* is defined by the equation $MVA_t = U_t - EK_t$. With equation (11) we have the following representation:

$$\begin{aligned}
MVA_t &= \frac{1}{1+k_{t+1}} E[(GuV_{t+1} - k_{t+1}EK_t)|\mathcal{F}_t] \\
&+ \frac{1}{1+k_{t+1}} E\left[\sum_{j=t+2}^{T}(GuV_j - k_j EK_{j-1})\frac{K(t+1)}{K(j)}\middle|\mathcal{F}_t\right] \\
&= \frac{1}{1+k_{t+1}} E[(GuV_{t+1} - k_{t+1}EK_t)|\mathcal{F}_t] \\
&+ \frac{1}{1+k_{t+1}} E\left[E\left[\sum_{j=t+2}^{T}(GuV_j - k_j EK_{j-1})\frac{K(t+1)}{K(j)}\middle|\mathcal{F}_{t+1}\right]\middle|\mathcal{F}_t\right] \\
&= \frac{1}{1+k_{t+1}} E[(GuV_{t+1} - k_{t+1}EK_t)|\mathcal{F}_t] + \frac{1}{1+k_{t+1}} E[MVA_{t+1}|\mathcal{F}_t]
\end{aligned}$$

The second equation is true by the tower property of the conditional expectation and the third equation by the definition of the market value added. After that divide both sides by $K(t)$ and use the $\mathcal{F}_t$-measurability of $K(t+1)$ to get

$$E[\Delta\widetilde{MVA}_{t+1}|\mathcal{F}_t] = -\frac{E[GuV_{t+1} - k_{t+1}EK_t|\mathcal{F}_t]}{K(t+1)}$$

The tilde-symbol is used to indicate discounting with the cost of capital process. Based on this equation we have the following interpretation of the economic value added. By the EVA concept the management has the task to earn the cost of capital $k_t EK_{t-1}$ for all business years $t = 1, 2, \ldots, T$. The best thing the management can do is to fulfill this requirement in mean, i.e.

$$E[GuV_{t+1} - k_{t+1}EK_t|\mathcal{F}_t] = 0$$

In this case the discounted market value added $\{\widetilde{MVA}_t; t = 1, 2, \ldots, T\}$ is a martingale, i.e.

$$E[\Delta\widetilde{MVA}_{t+1}|\mathcal{F}_t] = 0$$

and this equation can be transformed to

$$E[\Delta\tilde{U}_{t+1}|\mathcal{F}_t] = E[\Delta\widetilde{EK}_{t+1}|\mathcal{F}_t]$$

or

$$E[\tilde{U}_{t+1}|\mathcal{F}_t] = \tilde{U}_t + E[\Delta\widetilde{P\&L}_{t+1}|\mathcal{F}_t] - E[\tilde{D}_{t+1}|\mathcal{F}_t]$$

If the management fulfills the concept of EVA in the described meaning, then the shareholder value is increased by the expected profit and loss and reduced by the expected paid dividend.

Concluding remarks In this article we have developed a new proof of the twenty years old theorem of Martin Schweizer for the variance-optimal hedging in incomplete time discrete markets. New to this proof is the definition of the Hilbert space of all possible trading strategies. Based on this space the hedging-problem is reduced to the minimum problem of an energy functional. Moreover there exists an old and rich theory for the solution of this minimum problem. If one accepts that the knowledgeable willing parties of article 75 search for trading strategies which minimize the expected profit and loss of both parties simultaneously, the presented valuation model is a nice, easy and complete solution of this valuation problem. Still an open question is how the parties can calculate the optimal trading strategy for a given investment universe and cash flow. This will be done in an article to be published called *calculation model.*

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