Nonparametric estimation of the kernel function of symmetric stable moving average random functions

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Abstract

We use the empirical normalized (smoothed) periodogram of a $S\alpha S$ moving average random function to estimate its kernel function from high frequency observation data. The weak consistency of the estimator is shown. A simulation study of the performance of the estimates rounds up the paper.

Keywords: Stable random field, Moving average random field, Self-normalized periodogram, High frequency observations

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1 Introduction

We consider the problem of estimation of a kernel $f : \mathbb{R}^d \to \mathbb{R}$ from observations of the $S\alpha S$ stationary random function (*moving average)*

$$X(t) = \int_{\mathbb{R}^d} f(t-s)\Lambda(ds), \quad t \in \mathbb{R}^d, \tag{1}$$

where Λ is a $S\alpha S$ random measure with independent increments and Lebesgue control measure, $0 < \alpha < 2$, $f \in L^\alpha(\mathbb{R}^d)$. Let $\hat{f}$ be the Fourier transform of f, and let $\hat{f}^{-1}$ be its inverse, whenever these exist. We additionally assume that

(F1) f is *positive semidefinite.*

It follows from [17, 6.2.1] that f is *even* (or *symmetric*), i.e. $f(t) = f(-t)$ for all $t \in \mathbb{R}^d$. Assumptions (F2′)–(F3′) on f introduced in Section 3 imply, in particular, that f is uniformly continuous and bounded and that $f \in L^1(\mathbb{R}^d)$. Under the condition $f \in C(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ it can easily be shown by the Bochner-Khintchine theorem, see e.g. [17, 6.2.3] or [1, p. 54], that (F1) is equivalent to $\hat{f}(\lambda) \geq 0$ for all $\lambda \in \mathbb{R}^d$, i.e. f being of *positive type*. In order to show that f being of positive type implies (F1) one also has to use the inversion formula for Fourier transforms which holds almost everywhere (for short, a.e.) on $\mathbb{R}^d$ by [1, p. 17–18, Corollary 2 and Theorem 2] or by [17, 3.1.10 and 3.1.15].

For simplicity, we set $d = 1$ in the sequel, whereas all our results stay valid with obvious modifications also in the case of general d, cf. Remark 5. For $d = 1$, the class of stochastic processes (1) includes, e.g., stable CARMA processes which are popular in econometric and financial applications, cf. [3].

Notice that the spectral representation (1) of X for $0 < \alpha \leq 2$ is in general not unique, hence the inverse problem of identifying f out of X is in general ill–posed. However, it is shown in [14, Example 3.2] for $0 < \alpha < 2$ that two functions $f_1, f_2 \in L^\alpha(\mathbb{R})$ fulfilling (1) are connected by $f_2(t) = \pm f_1(t+t_0)$ for almost all $t \in \mathbb{R}$ and for some fixed $t_0 \in \mathbb{R}$. Assuming (F1), it can be easily shown that $f_1 = f_2$ a.e. on $\mathbb{R}$, i.e., f is determined uniquely a.e. on $\mathbb{R}$. In the Gaussian case $\alpha = 2$, the existence of the so called *canonical kernel* can be shown for a centered purely nondeterministic mean square continuous X, see [7, Theorem 3.4]. The uniqueness of f can not be guaranteed. However, under some additional assumptions f is unique which can be shown directly by the following covariance–based approach.

Let X in (1) be an infinitely divisible moving average with finite second moments, i.e., Λ be an infinitely divisible independently scattered random measure with Lebesgue control measure, $\mathsf{E}\,[\,\Lambda^2(B)\,] < \infty$ for any bounded Borel set $B \subset \mathbb{R}$, and $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then the covariance function of X is given by

$$C(t) = \mathrm{Cov}\,(X(0), X(t)) = \int_{\mathbb{R}} f(t-s) f(-s)\, ds, \quad t \in \mathbb{R}$$

By assumption (F1) we get $\hat{C} = \hat{f}^2$, and hence the relation

$$f = \pm \widehat{\sqrt{\hat{C}}}^{-1} \text{ a.e. on } \mathbb{R} \tag{2}$$

proves the uniqueness of f under the assumptions (F1), (F2′) and (F3′) in the Gaussian case.

Our aim is to provide a non-parametric estimator for the function f. We assume that the observations are taken at the points $\{t_{k,n}, k = 1, \dots, n\}$, where $t_{k,n} = k\Delta_n$, $k \in \mathbb{Z}$, $\Delta_n \to 0$, $n \to \infty$, and $n\Delta_n \to \infty, n \to \infty$. In other words, we have high frequency observations, and the observation horizon expands to the whole $\mathbb{R}_+$. It is worth to mention that under low frequency observations, it is in general not possible to identify f in a unique way. Indeed, let $\Delta_n = \Delta$ be constant. Define for any $h \in L^\alpha[-\Delta/2, \Delta/2]$ with $\|h\|_\alpha = 1$ the process $X_h(t) = \int_{-\Delta/2}^{\Delta/2} h(t-s)\Lambda(ds)$. Then the observations $\{X_h(t_{k,n}), k = 1, \dots, n\}$ are iid SαS with scale parameter 1, so their distribution does not depend on h. Why the observation interval should expand indefinitely, is less obvious. In the Gaussian case, on any finite interval $[0, t]$ it is possible to construct stationary processes such that the corresponding probability measures on $C[0, t]$ are different but the processes have the same distribution. Therefore, one is not able to identify the kernel function (not even the distribution) from observations of the process on a finite interval. However, to the best of our knowledge, there are no such results in the stable case. It still seems quite unlikely that a consistent estimation of f is possible from observations from a fixed finite interval, but we do not have a mathematically strict argument here.

Relation (2) can be used to build a strongly consistent estimator of a symmetric piecewise constant compact supported f if smoothed spectral density estimates are used (cf. e.g. [8, § 3.3]). The same problem for random process (1) with square integrable random measure Λ and causal f, i.e., $\operatorname{supp} f \subseteq \mathbb{R}_+$, was treated in [2]. There a non-parametric estimator for the kernel function f was proposed and its consistency was shown under CARMA assumptions. The estimator made use of the Wold expansion of the sampled process X.

In [13], a moving average time series with innovations belonging to the domain of attraction of the stable law was considered. For X being α-stable, $1 < \alpha < 2$, the parametric estimation of f via a minimum contrast method for the first–order madogram of X is performed in [9]. A non–parametric estimator of a piecewise constant symmetric f based on the covariation of X was proposed in [11]. However, this procedure is defined recursively and thus errors made at one step influence all following steps.

Here we extend the ideas of the paper [13] and use the empirical (properly normalized) periodogram of the random function X to estimate the symmetric uniformly continuous kernel function f of positive type satisfying some additional assumptions if the stability index $\alpha \in (0, 2)$ is known. The paper is organized as follows. After introducing the notation and the normed smoothed periodogram in Section 2, the weak consistency of the kernel estimation is stated in Section 3. There, Theorems 1 and 2 treat the cases of compact and unbounded support of f, respectively. The consistency of the estimation of the L^2–norm of f is treated in Corollary 1. For the ease of reading, proofs are moved to Appendices A (Theorems 1 and 2) and B (auxiliary lemmata). A simulation study shows the good performance of estimation for $d = 1, 2$ in Section 4. There, the scope of applicability of this estimation method is studied empirically. The estimator performs well also for skewed stable, symmetric infinitely divisible and for Gaussian Λ, whereas it fails to work with some skewed non–stable Λ. We conclude with a summary and conjectures (Section 5).

2 Preliminaries

We use the following notation: $a_n = o_P(b_n)$, $n \to \infty$, means $a_n/b_n \xrightarrow{P} 0$, $n \to \infty$; $a_n \overset{P}{\sim} b_n$, $n \to \infty$, means $a_n/b_n \xrightarrow{P} 1$, $n \to \infty$; we write $a_n = O_P(b_n)$, $n \to \infty$, if the sequence $\{a_n/b_n, n \geq 1\}$ is bounded in probability. The symbol C will denote a generic constant, the value of which is not important.

To estimate the function f in (1), we use the *self-normalized (empirical) periodogram* of X, defined as

$$I_{n,X}(\lambda) = \frac{\left|\sum_{j=1}^{n} X(t_{j,n}) e^{i t_{j,n} \lambda}\right|^2}{\sum_{j=1}^{n} X(t_{j,n})^2}. \tag{3}$$

It is known [4, Theorem 2.11] that $\Delta_n \cdot I_{n,X}(\lambda)$ converges to a random limit as $n \to \infty$, and so it can not be a consistent estimator of any deterministic quantity of interest. Thus, following [5] we define its smoothed version. Let $\{m_n, n \geq 1\}$ be a sequence of positive integers such that $m_n \to \infty$ and $m_n = o(n)$, $n \to \infty$. Consider a sequence of filters $\{W_n(m), |m| \leq m_n, n \geq 1\}$ satisfying

(W1) $W_n(m) \geq 0$;

(W2) $\sum_{|m| \leq m_n} W_n(m) = 1$;

(W3) $\max_{|m| \leq m_n} W_n(m) \to 0$, $n \to \infty$;

(W4) $\sum_{|m| \leq m_n} m^2 W_n(m) = o\big((n\Delta_n)^2\big)$, $n \to \infty$.

In the following we will denote $W_n^* = \max_{|m| \leq m_n} W_n(m)$, $W_n^{(2)} = \sum_{|m| \leq m_n} m^2 W_n(m)$.

Denote $\nu_n(m, \lambda) = \lambda + m/(n\Delta_n)$, $m = -m_n, \dots, m_n$. Then a smoothed periodogram is defined as

$$I_{n,X}^s(\lambda) = \sum_{|m| \leq m_n} W_n(m) I_{n,X}(\nu_n(m, \lambda)). \tag{4}$$

3 Main results

For the sake of brevity, define the normalized function $g(t) = f(t)/\|f\|_2$, where $\|f\|_2 = \sqrt{\int_{\mathbb{R}} f(x)^2\,dx}$ is the L^2–norm of f whenever it is finite; the Fourier transform of g is

$$\hat{g}(\lambda) = \int_{\mathbb{R}} g(t)e^{-i\lambda t}dt, \quad \lambda \in \mathbb{R},$$

whenever it exists. First, we estimate g and $\|f\|_2$ separately. If $\tilde{g}$ and $\widetilde{\|f\|}_2$ are their weakly consistent estimators, then $\tilde{f} = \widetilde{\|f\|}_2 \cdot \tilde{g}$ is a weakly consistent estimator of f.

For the estimation of g we need a number sequence $\{a_n, n \geq 1\}$ with the following properties:

(A1) $a_n \to \infty$, $n \to \infty$;

(A2) $a_n^2 W_n^* \to 0$, $n \to \infty$;

(A3) $a_n^{3/4} = o((n\Delta_n)^{1/\alpha})$, $n \to \infty$;

(A4) $a_n^2 \Delta_n \to 0$, $n \to \infty$;

(A5) $a_n^2 W_n^{(2)} = o\big((n\Delta_n)^2\big)$, $n \to \infty$.

Remark 1. From (W2) and (W2) it is clear that $\limsup_{n\to\infty} W_n^{(2)} > 0$. Therefore, (A5) implies that $a_n = o(n\Delta_n)$, $n \to \infty$ (this will be used in the future). In particular, (A3) follows from (A5) for $\alpha \leq \frac{4}{3}$. Besides this, the assumptions are rather independent.

Introduce the estimator

$$\tilde{g}(t) = \frac{1}{2\pi} \int_{[-a_n,a_n]} \sqrt{\Delta_n I_{n,X}^s(\lambda)}e^{it\lambda}\,d\lambda, \quad t \in \mathbb{R}, \tag{5}$$

of g.

Let f satisfy (F1). Further assumptions depend on whether f is compactly supported or not. In the case of compact support, we assume

(F2) $a_n \omega_f(\Delta_n) \to 0$, $n \to \infty$,

where $\omega_f(\Delta_n) = \sup_{|t-s|<\Delta_n} |f(t) - f(s)|$ is the modulus of continuity of f. Clearly, assumption (F2) implies the uniform continuity of f. Hence, f is bounded, and then $f \in L^p(\mathbb{R})$ for all $p \in (0, \infty]$. In the case of non-compact support, we assume (additionally to (F1)) that for some $a > \max\{2, 1/\alpha\}$

(F2′) $a_n \omega_f(\Delta_n)^{1-1/a} \to 0$, $n \to \infty$;

(F3′) $f(t) = O(|t|^{-a})$, $|t| \to \infty$;

(F4′) $a_n^{3/4} = o\big(\omega_f(\Delta_n)^{1/(a\alpha)}(n\Delta_n)^{1/\alpha}\big)$, $n \to \infty$.

It follows from (F2′) and (F3′) that f is uniformly continuous and bounded, $f \in L^p(\mathbb{R})$ for $p \in (\frac{1}{a}, \infty] \ni \alpha, 1, 2$, so $\hat{f}$ is bounded too, moreover, it is square integrable.

Explicit examples of kernels and corresponding sequences satisfying the above assumptions are given in Section 4.

Theorem 1. *Let f be compactly supported and* (F2), (A1)–(A5), and (W1)–(W4) *be satisfied.*

(i) The following convergence in probability holds:

$$a_n \cdot \int_{-a_n}^{a_n} (\Delta_n I^s_{n,X}(\lambda) - |\hat{g}(\lambda)|^2)^2 d\lambda \xrightarrow{P} 0, \quad n \to \infty. \tag{6}$$

(ii) If additionally (F1) *is true then* $\|\tilde{g} - g\|_2 \xrightarrow{P} 0, \quad n \to \infty.$

Remark 2. The assumptions (F1) and (F2) are needed in order to reconstruct f from the absolute value of its Fourier transform.

Remark 3. Carefully examining the proof, we can bound the rate of convergence in (6) by

$$O_P\big(a_n^2 \omega_f(\Delta_n)^2 + a_n^2 W_n^* + a_n^2 W_n^{(2)}(n\Delta_n)^{-2} + a_n^4 \Delta_n^2\big), \quad n \to \infty.$$

Remark 4. Using [3, Lemma 2.3] it can be shown that in $CARMA(p,q)$ models $|\hat{g}(\lambda)|^2$ coincides with the power transfer function if $p > q+1$. Thus Theorem 1(i) shows that $\Delta_n I^s_{n,X}(\lambda)$ is a weakly consistent estimator for the power transfer function. However, this is already known [5, Theorem 1] under the weaker assumption $p > q$.

Theorem 2. *The assertion of Theorem* 1 *holds true also under the assumptions* (F1), (F2′)–(F4′), (A1)–(A5), (W1)–(W4).

Taking into account the evident relation $\|f\|_2 = \|f\|_\alpha / \|g\|_\alpha$, the estimation of the norm $\|f\|_2$ is reduced to the estimation of $\|f\|_\alpha = \sigma_{X(0)}$, the scale parameter of $X(0)$ (see [15, Property 3.2.2]), and $\|g\|_\alpha$. In the literature, there is a number of estimators of scale available, see [18, Chapter 4], [19, Chapter 9]. We consider two scale parameter estimators: moment-based and quantile-based.

To construct a moment-based estimator, we use the relation $\sigma^p_{X(0)} = c(p,\alpha) E\,|X(0)|^p$ (cf. [15, Property 1.2.17]) for $X(0) \sim S_\alpha(\sigma_{X(0)}, 0, 0)$, where $0 < p < \alpha$ and

$$c(p,\alpha) = \frac{p \int_0^\infty u^{-1-p} \sin^2 u \, du}{2^{p-1}\Gamma(1-p/\alpha)} = \begin{cases} \frac{\Gamma(2-p)}{(1-p)\Gamma(1-p/\alpha)} \cos(\pi p/2), & p \neq 1, \\ \frac{\pi}{2\Gamma(1-\alpha^{-1})}, & p = 1, \end{cases}$$

to get an estimate

$$\widetilde{\sigma}_m = \left(\frac{c(p,\alpha)}{n} \sum_{k=1}^n |X(t_{k,n})|^p \right)^{1/p}. \tag{7}$$

Obviously, $\widetilde{\sigma}_m$ is sensitive to outliers, i.e., not robust. This can be fixed by considering quantile estimators.

The quantile estimator is based on the observation that the quantiles of $X(0)$ are equal to those of $S_\alpha(1,0,0)$, multiplied by $\sigma_{X(0)}$. Taking different quantile levels, this can be used to construct a variety of estimators. The most popular choice is quartiles, so that the correspondent estimator is

$$\widetilde{\sigma}_q = \frac{\widetilde{x}_{3/4;n} - \widetilde{x}_{1/4;n}}{x_{3/4} - x_{1/4}}, \tag{8}$$

where $\widetilde{x}_{1/4;n}$ and $\widetilde{x}_{3/4;n}$ are, respectively, the lower and upper empirical quartiles of the sample $\{X(t_{k,n}), k = 1, \ldots, n\}$; $x_{1/4}$ and $x_{3/4}$ are, respectively, the lower and upper quartiles of $S_\alpha(1,0,0)$.

Since it is not our main concern here, we will only sketch the proof of consistency of these estimators and the respective rates of convergence. For simplicity, assume that there is a positive number T such that for any $n \geq 1$, $N_n = T/\Delta_n$ and $k_n = n/N_n = n\Delta_n/T$ are integers. Then

$$\frac{1}{n}\sum_{k=1}^{n} |X(t_{k,n})|^p = \frac{1}{k_n}\sum_{j=1}^{k_n} \sum_{k:(j-1)T<t_{k,n}\leq jT} \frac{\Delta_n}{T}|X(t_{k,n})|^p$$
$$\approx \frac{1}{k_n}\sum_{j=1}^{k_n} \frac{1}{T}\int_{(j-1)T}^{jT} |X(s)|^p ds \to \frac{1}{T}\mathsf{E}\left[\int_{(j-1)T}^{jT} |X(s)|^p ds\right] = E[|X(0)|^p], \quad n\to\infty,$$

where we have used the stationarity of X and the ergodic theorem. The $\approx$ sign can be justified under some extra regularity assumptions, but we will not go into detail. There are two useful observations from the above heuristic writing. First, the rate of convergence of $\widetilde{\sigma}_m$ is the same as that of

$$\widehat{\sigma}_m = \left(\frac{c(p,\alpha)}{n}\sum_{j=1}^{k_n} |X(jT)|^p\right)^{1/p}.$$

Thus, extra effort needed to compute $\widetilde{\sigma}_m$ is not justified: as n grows to infinity, the ratio of numbers of terms involved in it and in $\widehat{\sigma}_m$ grows to infinity as well, while the precision is only improved by some constant.

Secondly, for $\operatorname{supp} f = [-T,T]$, the sequence involved in the computation of $\widehat{\sigma}_m$ is 2-dependent. This means that the rate of convergence of $\widehat{\sigma}_m$ (and that of $\widetilde{\sigma}_m$, thanks to the above heuristics) towards $\sigma_{X(0)}$ is given by the central limit theorem, i.e. is equal to $O_P(k_n^{1/2}) = O_P((n\Delta_n)^{1/2})$.

The consistency of $\widetilde{\sigma}_q$ can be shown in a standard way. Namely, the consistency of empirical quantiles follows, through the Glivenko–Cantelli argument, from that of the empirical cumulative distribution function

$$\widetilde{F}_n(x) = \frac{1}{n}\sum_{k=1}^{n} \mathbb{1}_{\{X(t_{k,n})<x\}},$$

which is, in turn, is justified exactly as the consistency of empirical moments. Moreover, one can define a "low-frequency" version of the estimator:

$$\widehat{\sigma}_q = \frac{\widehat{x}_{3/4;n} - \widehat{x}_{1/4;n}}{x_{3/4} - x_{1/4}},$$

where $\widehat{x}_{1/4;n}$ and $\widehat{x}_{3/4;n}$ are, respectively, the lower and upper empirical quartiles of $\{X(jT), j = 1,\dots,k_n\}$. As above, it has the same accuracy as $\widetilde{\sigma}_q$, but requires less computational effort.

Now let us turn to the estimation of $\|g\|_\alpha$. In the case where f is supported by $[-T,T]$ (and T is known a priori), one can use the estimator

$$\widetilde{\|g\|}_{\alpha,T} = \left(\int_{-T}^{T} |\tilde{g}(t)|^\alpha \, dt\right)^{1/\alpha}.$$

In the case of unbounded support, we need a number sequence $\{b_n, n\geq 1\}$ such that

(B1) $b_n \to \infty$, $n \to \infty$;

(B2) $b_n^{2/\alpha-1} a_n^2 W_n^* \to 0$, $n \to \infty$;

(B3) $b_n^{2/\alpha-1} a_n = o((n\Delta_n)^{1/\alpha})$, $n \to \infty$;

(B4) $b_n^{2/\alpha-1} a_n^4 \Delta_n^2 \to 0$, $n \to \infty$;

(B5) $b_n^{2/\alpha-1} a_n^2 W_n^{(2)} = o\big((n\Delta_n)^2\big)$, $n \to \infty$;

(B6) $b_n^{2/\alpha-1} a_n^2 \omega_f(\Delta_n)^{2-2/a} \to 0$, $n \to \infty$;

(B7) $b_n^{2/\alpha-1} \int_{\{\lambda:|\lambda|>a_n\}} \hat{g}(\lambda)^2 d\lambda \to 0$, $n \to \infty$.

With this at hand, an estimator for $\|g\|_\alpha$ is constructed as

$$\widetilde{\|g\|}_{\alpha,b_n} = \left(\int_{-b_n}^{b_n} |\tilde{g}(t)|^\alpha \, dt \right)^{1/\alpha}. \tag{9}$$

Theorem 3. *1. Let f be supported by $[-T,T]$ and the assumptions of Theorem 1 hold. Then*

$$\widetilde{\|g\|}_{\alpha,T} \xrightarrow{P} \|g\|_\alpha, \quad n \to \infty.$$

2. Under the assumptions of Theorem 1 and (B1)–(B7)*,*

$$\widetilde{\|g\|}_{\alpha,b_n} \xrightarrow{P} \|g\|_\alpha, \quad n \to \infty.$$

Introduce a plug–in estimator $\widetilde{\|f\|}_2 = \widetilde{\sigma}_{X(0)} / \widetilde{\|g\|}_\alpha$ of $\|f\|_2$ where $\widetilde{\sigma}_{X(0)}$ is a scale estimator of $X(0)$ (e.g., $\widetilde{\sigma}_m$, $\widehat{\sigma}_m$, $\widetilde{\sigma}_q$, or $\widehat{\sigma}_q$) and $\widetilde{\|g\|}_\alpha$ is any of the estimators $\widetilde{\|g\|}_{\alpha,T}$ and $\widetilde{\|g\|}_{\alpha,b_n}$ corresponding to the case of compact or non–compact support of f.

Corollary 1. *Let $\widetilde{\sigma}_{X(0)}$ be any weakly consistent estimator of scale of $X(0)$. Under the assumptions of Theorems 1 and 3 for compact–supported f (or Theorems 2 and 3, otherwise) it holds*

$$\widetilde{\|f\|}_2 \xrightarrow{P} \|f\|_2, \quad n \to \infty.$$

Remark 5. The above results stay true also for the case of estimation of the kernel function $f : \mathbb{R}^d \to \mathbb{R}$ of a stationary random field

$$X(t) = \int_{\mathbb{R}^d} f(t-s)\Lambda(ds), \quad t \in \mathbb{R}^d, \tag{10}$$

where Λ is a homogeneous $S\alpha S$ independently scattered random measure on $\mathbb{R}^d$. Let $(\Delta_n)_{n\in\mathbb{N}}$, $(m_n)_{n\in\mathbb{N}}$ and $(a_n)_{n\in\mathbb{N}}$ be real-valued sequences mit $\Delta_n \to 0$, $n\Delta_n \to \infty$, $m_n \to \infty$ and $m_n = o(n)$ as $n \to \infty$. Let $\{W_n(m) \mid n \in \mathbb{N}, m \in \{-m_n, \dots, m_n\}^d\}$ be a sequence of filters. Denote by $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^d$. Additionally to (A1) and (W1) above, assume that the following regularity conditions are fulfilled:

(W2) $\sum_{m\in\{-m_n,\dots,m_n\}^d} W_n(m) = 1$;

(W3) $W_n^* := \max_{m\in\{-m_n,\dots,m_n\}^d} W_n(m) \to 0$, $n \to \infty$;

(W4) $W_n^{(2)} := \sum_{m\in\{-m_n,\dots,m_n\}^d} W_n(m)\|m\|^2 = o\big((n\Delta_n)^2\big)$, $n\to\infty$;

(A2) $a_n^{2d} W_n^* \to \infty$, $n\to\infty$;

(A3) $a_n^{3d/4} = o((n\Delta_n)^{1/\alpha})$, $n\to\infty$;

(A4) $a_n^{d+1}\Delta_n \to 0$, $n\to\infty$;

(A5) $a_n^{2d} W_n^{(2)} = o\big((n\Delta_n)^2\big)$, $n\to\infty$;

Moreover, assume that the function f satisfies (F1) and that it either has compact support and fulfills

(F2) $a_n^d \omega_f(\Delta_n) \to 0$, $n\to\infty$,

where $\omega_f(\Delta_n) = \sup_{\|t-s\|<\Delta_n} |f(t)-f(s)|$ is the modulus of continuity of f, or that there is some $a > \max\{d+1, d/\alpha\}$ such that f fulfills

(F2′) $a_n^d \omega_f(\Delta_n)^{\frac{1}{d}-\frac{1}{a}} \to 0$, $n\to\infty$;

(F3′) $f(t) = O\left(\|t\|^{-a}\right)$, $\|t\|\to\infty$;

(F4′) $a_n^{3d/4} = o(\omega_f(\Delta_n)^{1/(a\alpha)}(n\Delta_n)^{1/\alpha})$, $n\to\infty$.

Put

$$I_{n,X}(\lambda) = \frac{\left|\sum_{j\in\{1,\dots,n\}^d} X(t_{j,n}) e^{i\langle t_{j,n},\lambda\rangle}\right|^2}{\sum_{j\in\{1,\dots,n\}^d} X(t_{j,n})^2}, \quad \lambda\in\mathbb{R}^d,$$

where $t_{j,n} = (j_1\Delta_n,\dots,j_d\Delta_n)$ for $j=(j_1,\dots,j_d)\in\mathbb{R}^d$ and $n\in\mathbb{N}$, and

$$I_{n,X}^s(\lambda) = \sum_{m\in\{-m_n,\dots,m_n\}^d} W_n(m) I_{n,X}\big(\lambda + \frac{m}{n\Delta_n}\big).$$

Then for the estimator

$$\tilde{g}(t) := \frac{1}{(2\pi)^d}\int_{[-a_n,a_n]^d} \sqrt{\Delta_n I_{n,X}^s(\lambda)} e^{i\langle t,\lambda\rangle}\, d\lambda, \quad t\in\mathbb{R}^d,$$

the assertions of Theorem 1 and Theorem 2 hold.

4 Simulation study

In this section, we study the performance and the applicability range of the above estimation method empirically, i.e., by estimating f from each of $M=100$ Monte Carlo simulations of the trajectories of X. First, dwell on the particular choice of the weights W_n and sequences $\{\Delta_n\}$, $\{m_n\}$, and $\{a_n\}$.

Assumptions (W1)–(W4) and (A1)–(A5) are evidently satisfied e.g. for

- uniform weights $W_n(m) = \frac{1}{2m_n+1}$,
- $\Delta_n = n^{-\delta}$, $\delta\in(0,1)$,
- $m_n = n^\gamma$, $\gamma\in(0,1-\delta)$,

- $a_n = \log n$.

Assumptions (F1)–(F2) hold for all positive semidefinite compact supported Lipschitz continuous kernels f. For all Lipschitz continuous functions (F2′) holds. Assumption (F3′) is valid whenever f decays at infinity rapidly enough, e.g., for $f(t) = e^{-|t|}$, while (F4′) holds for all non-constant functions f provided $\delta < \frac{a}{a+1}$, since then $\omega_f(\Delta) \geq c \cdot \Delta$ for an appropriate constant $c > 0$ and sufficiently small $\Delta > 0$.

Now let us comment on the choice of Λ. Although the consistency of the estimator of f was proven only for $S\alpha S$ integrators Λ, $\alpha \in (0, 2)$ (see Figures 1, 2 (left)), it seems to work well also for Gaussian ($\alpha = 2$, cf. Figure 2 (right)) and skewed Λ with different values of stability index $\alpha \in (0, 2)$ and skewness intensity $\beta \in [-1, 1]$, cf. Figure 3. Numerical experiments with non–stable infinitely divisible integrators Λ show however that symmetry is an important assumption that can not be omitted there. We tested random measures whose marginal distributions are Γ-distributed or defined by (11) and (12) and saw that estimation method for f does not work (cf. Figure 5) but it works well for symmetric infinitely divisible measures Λ with or without a finite second moment, compare Figure 4. In the infinitely divisible case, we have chosen the infinitely divisible Λ with Lévy density

$$h(x) = \begin{cases} c_1 \frac{|\log x|}{|x|^{p_1}}, & x > \varepsilon, \\ c_2 \frac{|\log(-x)|}{|x|^{p_2}}, & x < -\varepsilon, \\ 0, & |x| \leq \varepsilon \end{cases} \tag{11}$$

for some $\varepsilon \geq 0$, $c_1, c_2 > 0$, $p_1, p_2 > 0$. In more detail, we choose Λ such that for any bounded Borel set $B \subset \mathbb{R}$ we have $\Lambda(B) = \xi(|B|)$ in distribution where $|B|$ is the Lebesgue measure of B and $\xi = \{\xi(t),\ t \geq 0\}$ is the Lévy process given by

$$\xi(t) = \int_0^t \int_{\mathbb{R}} x Q(dx, ds) - t \int_{|x|<1} x h(x)\, dx, \quad t \geq 0, \tag{12}$$

cf. [16, Theorem 19.2]. Here Q is a random Poisson measure on $\mathbb{R}_+ \times \mathbb{R}$ with intensity measure $\nu(A, B) = |A| \int_B h(x)\, dx$ for any bounded Borel subset $A \times B \subset \mathbb{R}_+ \times \mathbb{R}$. If $p_1, p_2 \in (0, 3)$ then Λ is not square integrable, cf. [16, Corollary 25.8]. Λ is symmetric iff h is symmetric, i.e., $c_1 = c_2$ and $p_1 = p_2$, cf. [16, Exercise 18.1]. It is known that the distribution of Λ is completely determined by the law of $\xi(1)$. In the case of Γ–distributed Λ, we set $\Lambda(B) \sim \Gamma(1, |B|)$ for any bounded Borel subset B where a random variable $Y \sim \Gamma(\lambda, p)$ has the density

$$p(x) = \frac{\lambda^p x^{p-1}}{\Gamma(p)} e^{-\lambda x} \mathbf{1}_{\{x \geq 0\}}.$$

For a Gaussian measure Λ, we put $\Lambda(B) \sim N(0, |B|)$ for a bounded Borel subset B.

To simulate the realizations of X, we used the algorithms given in [10]. For infinitely divisible Λ, the Lévy–Ito representation (12) was used to generate $\xi(1)$. There ε was chosen to be positive in order to avoid exremely high jumps. For $d = 1$, we set $n = 1000$, $m_n = n^{1/4}$, $a_n = 20$, $\Delta = 2T/N = 0.01$, $T = 20$ with uniform weights as above. As kernel functions for our simulations, we have chosen the triangular, the spherical and the exponential kernels

$$f(t) = c(1 - |t|)\mathbf{1}_{[-1,1]}(t), \tag{13}$$

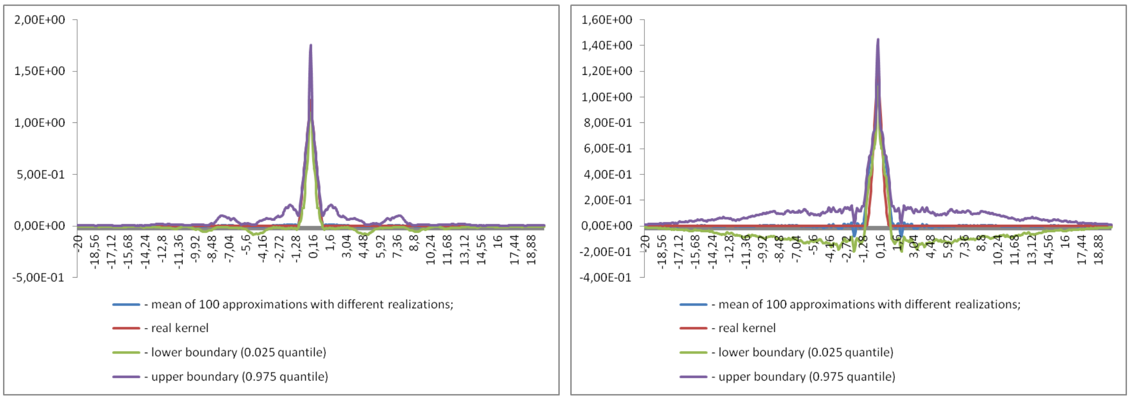

Figure 1: Estimation results for $S\alpha S$ X with triangular kernel (13), $\alpha = 0.3$ (left) and with spherical kernel (14), $\alpha = 1.7$ (right)

$$f(t) = c \cdot (1 - 1.5|t| + 0.5|t|^3)\mathbf{1}_{[-1,1]}(t), \tag{14}$$

$$f(t) = c\exp(-|t|) \tag{15}$$

with constants $c > 0$ chosen such that $\|f\|_2 = 1$. These kernels f satisfy conditions (F1)–(F2) and (F1), (F2′)–(F4′), respectively. Indeed, assumption (F1) holds since all these functions are valid covariance functions which are positive semidefinite. One can check that their Fourier transforms are non–negative also directly, compare [12, Table 4, p. 245]. (F2) and (F2′) follow from Lipschitz continuity of the functions (13)–(15).

The following numerical results have been obtained for $\alpha = 2,\ 1.7,\ 1.3,\ 1,\ 0.7,\ 0.3$. In Figures 1 – 5, we concentrated on the estimation of function $g = f$ (which is equivalent to setting $\|f\|_2 = 1$). Each figure contains the graph of the real kernel function f used to simulate X, the mean of 100 estimates of f and their $(0.025, 0.975)$–quantile envelope, i.e. the region containing 95% of all estimated curves of f.

If the norm of f is unknown and has to be estimated separately as described in Section 3 then the volatility of the estimates of f increases, compare Figure 6. There, the constant c in (15) was chosen to be 2.5, and the estimator of scale is the quantile estimator (8). Not surprisingly, the performance of the estimators of the norm $\widetilde{\|f\|_2}$ gets better with increasing α. Usually, the estimator (8) outperforms the moment estimator (7) which justifies its choice in the estimation procedure for f. But even for the quantile estimator, its empirical standard deviation is much higher for small $\alpha \in (0, 1)$. This is the reason why the empirical mean of M estimated values of f in Figure 6 (left) for $\alpha = 0.3$ had to be substituted by the empirical median which is robust to outliers. Numerical experiments with different sampling mesh values Δ_n show that the estimation of f performs well for $\Delta_n \in (0, 0.1]$ (high frequency framework).

In order to evaluate the performance of the estimator when $d = 2$, we examined a (symmetric) field with $\alpha = 1.8$ and kernel

$$f(t) = \frac{1}{2\pi} e^{-\|t\|_2^2/2}, \tag{16}$$

on a grid with $n = 1000$ points in each dimension and grid distance $\Delta_n = T^2/(2n) \approx 0.0024$. For computational reasons the kernel was restricted to $t \in [-T, T]^2$, where $T = 2.2$. As parameters for the estimator we used uniform weights $W_n(m) = 1/(2m_n + 1)^2$, $m_n = \lfloor \sqrt[8]{n} \rfloor = 2$ and $a_n = \log(n) - 4.5 \approx 2.4$. Since the computation time is much higher

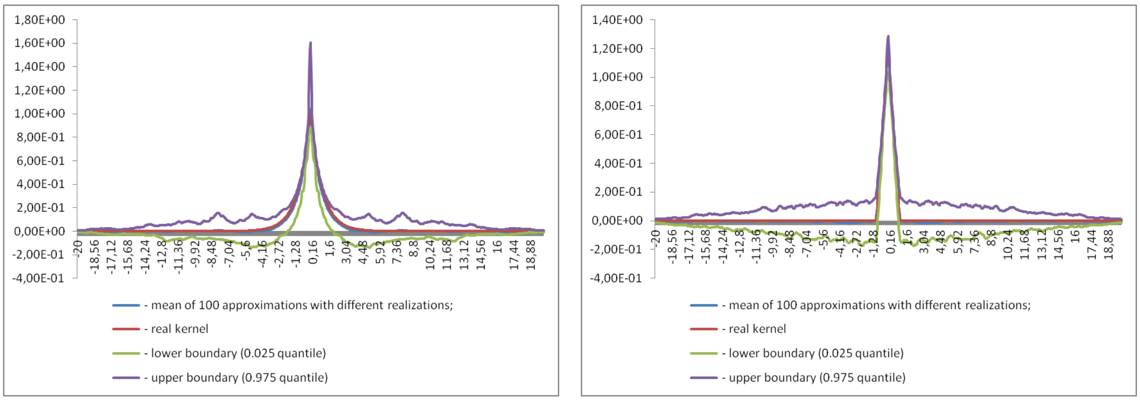

Figure 2: Estimation results for $S\alpha S$ X with exponential kernel (15), $\alpha = 0.7$ (left) and with triangular kernel (13), $\alpha = 2$ (Gaussian case, right)

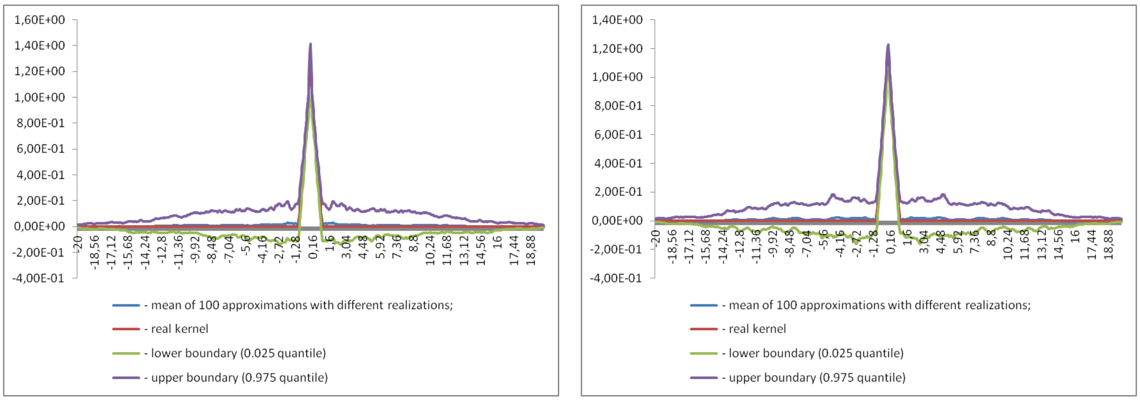

Figure 3: Estimation results for skewed X with triangular kernel (13), $\alpha = 1.3$ and $\beta = 0.7$ (left), $\beta = -0.5$ (right)

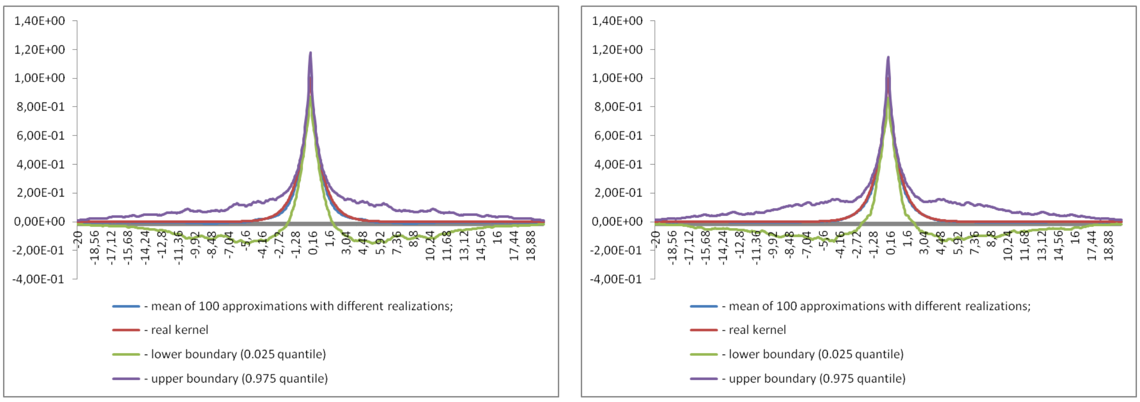

Figure 4: Estimation results for X with infinitely divisible Λ and exponential kernel (15). Parameters of Lévy density (11) are $c_1 = c_2 = 1$, $p_1 = p_2 = 2.5$ (left) and $p_1 = p_2 = 4$ (right)

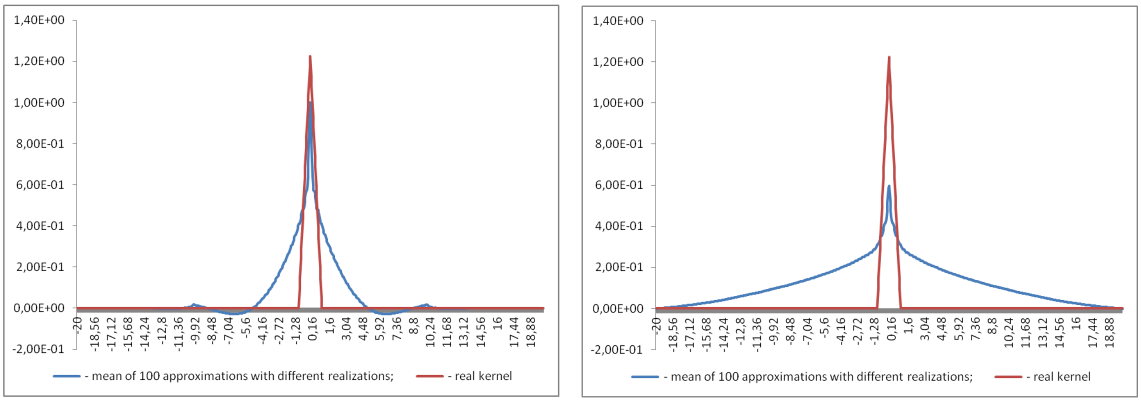

Figure 5: Estimation results for X with triangular kernel (13), Gamma-distributed Λ (left) and skewed infinitely divisible Λ (right). Parameters of Lévy density (11) are $p_1 = 2.1$, $p_2 = 2.7$ and $c_1 = c_2 = 1$

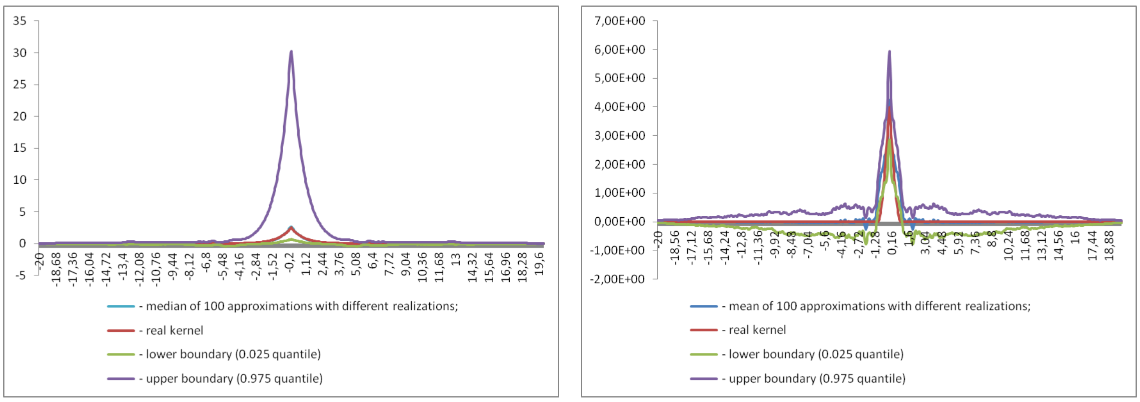

Figure 6: Estimation results for $S\alpha S$ X with unknown norm of f. Here $\alpha = 0.7$ and f is an exponential kernel (15) with $c = 2.5$ (left) and $\alpha = 1.7$ and f is a spherical kernel (14) with $c = 4$ (right)

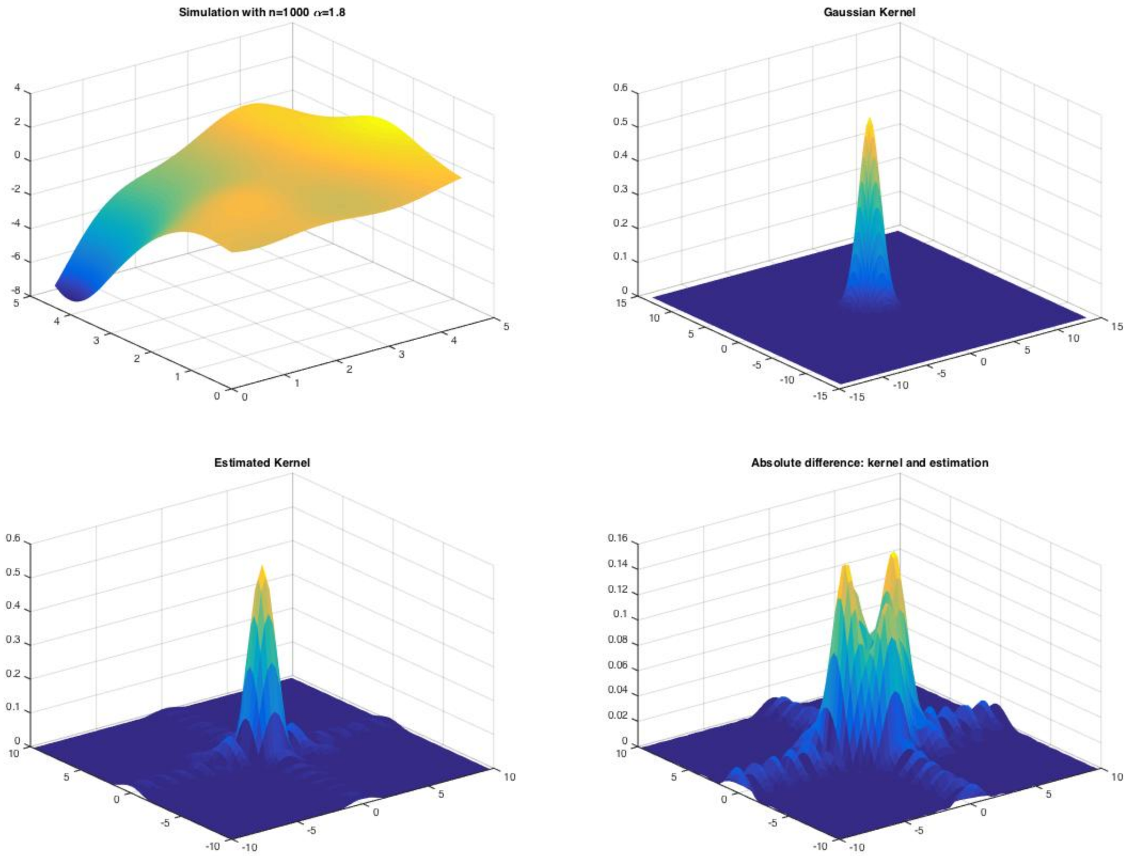

Figure 7: A simulated realization (top left) of $S\alpha S$ random field X ($d = 2$) with Gaussian kernel (16) (top right), a realization of the kernel estimator (bottom left) and its difference to the real kernel (bottom right)

than in the one-dimensional case, we simulated just one realization of the estimator. Figure 7 (bottom row) shows that our estimation method (with the appropriately chosen parameters) performs also well in two dimensions.

5 Summary and open problems

The preceding section showed the good performance of the high frequency estimates of a smooth symmetric bounded rapidly decreasing kernel f of positive type for α–stable moving averages X (both skewed and symmetric) in the case $\alpha \in (0, 2]$. Additionally, we verified empirically the applicability of the method to certain non–stable symmetric infinitely divisible integrators Λ. An open problem is to provide rigorous mathematical proofs for this experimental evidence. Recall that we were able to show the consistency of our estimation methods only in the $S\alpha S$ case. Our working hypothesis is that the results of Theorems 1 and 2 stay true for all stable integrators Λ as well as for symmetric infinitely divisible Λ without a finite second moment (at least lying in the domain of attraction of a stable law).

Another open problem is to prove limit theorems for the estimates of g and f in case of $S\alpha S$ Λ. If f is not symmetric (e.g., it is causal) our estimation ansatz fails to work completely, so new ideas are needed here. This is the subject of future research.

Acknowledgements

We thank I. Liflyand and V.P. Zastavnyi for the discussion on positive definite functions and their Fourier transforms. We are also grateful to our students L. Palianytsia, O. Stelmakh and B. Ströh for doing numerical experiments in Section 4.

Appendix A: Proofs

Kernel f with compact support

Proof of Theorem 1. We first show how (ii) follows from (i). Notice that $|\hat{g}(\lambda)| = \hat{g}(\lambda)$ for all $\lambda \in \mathbb{R}$, since by (F1) f is symmetric and of positive type. In order to prove

$$\int_{-a_n}^{a_n} \left(\sqrt{\Delta_n I_{n,X}^s(\lambda)} - \hat{g}(\lambda)\right)^2 d\lambda \xrightarrow{P} 0, \quad n \to \infty,$$

we use the inequality $|\sqrt{a} - \sqrt{b}| \leq \sqrt{|a-b|}$ for $a, b \geq 0$. We get

$$\begin{aligned}\int_{-a_n}^{a_n} \left(\sqrt{\Delta_n I_{n,X}^s(\lambda)} - \hat{g}(\lambda)\right)^2 d\lambda &\leq \int_{-a_n}^{a_n} \left|\Delta_n I_{n,X}^s(\lambda) - \hat{g}(\lambda)^2\right| d\lambda \\ &\leq \sqrt{2a_n} \cdot \sqrt{\int_{-a_n}^{a_n} \left|\Delta_n I_{n,X}^s(\lambda) - \hat{g}(\lambda)^2\right|^2 d\lambda} \xrightarrow{P} 0\end{aligned}$$

by (i), where the last inequality is due to Cauchy–Schwarz.

Since $\hat{g} \in L^2(\mathbb{R})$ it follows

$$\int_{\{\lambda: |\lambda| > a_n\}} \hat{g}(\lambda)^2 \, d\lambda \to 0, \quad n \to \infty,$$

so we get

$$\int_{\mathbb{R}} \left(\mathbf{1}_{[-a_n, a_n]}(\lambda) \cdot \sqrt{\Delta_n I_{n,X}^s(\lambda)} - \hat{g}(\lambda)\right)^2 d\lambda \xrightarrow{P} 0, \quad n \to \infty.$$

Now Plancherel's equality yields

$$\int_{\mathbb{R}} \left| \frac{1}{2\pi} \int_{[-a_n, a_n]} \sqrt{\Delta_n I_{n,X}^s(\lambda)} e^{it\lambda} \, d\lambda - g(t) \right|^2 dt \xrightarrow{P} 0, \quad n \to \infty,$$

which is equivalent to the statement.

Now let us prove (i). Write

$$I_{n,X}^s(\lambda) = \frac{J_{n,X}^s(\lambda)}{S_{n,X}},$$

where $J_{n,X}^s(\lambda) = \sum_{|m| \leq m_n} W_n(m) \left| \sum_{j=1}^n X(t_{j,n}) e^{i t_{j,n} \nu_n(m,\lambda)} \right|^2, \quad S_{n,X} = \sum_{j=1}^n X(t_{j,n})^2.$

Let f be supported by $[-T, T]$. We will assume that $N = T/\Delta_n$ is integer: this will simplify the exposition while not harming the rigor. The proof is rather long, so we split it into several steps for better readability. Choose $n \geq 2N + 1$.

Step 1. Denominator. We start with investigating the denominator $S_{n,X}$. First we study the behavior of a similar expression with f replaced by its discretized version. Specifically, define

$$X_n(t_{j,n}) = \sum_{k=-N}^{N-1} f(t_{k,n})\Lambda\big(((j-k-1)\Delta_n, (j-k)\Delta_n]\big) = \int_{\mathbb{R}} f_n(t_{j,n}-s)\Lambda(ds),\ j=1,\ldots,n,$$

where $f_n(x) = \sum_{k=-N}^{N-1} f(t_{k,n})\mathbb{1}_{[t_{k,n},t_{k+1,n})}(x)$. Denote $\varepsilon_{l,n} = \Lambda\big(((l-1)\Delta_n, l\Delta_n]\big)$, $l \in \mathbb{Z}$. For fixed n, these variables are independent $S\alpha S$ with scale parameter $\Delta_n^{1/\alpha}$.

Decompose

$$\sum_{j=1}^{n} X_n(t_{j,n})^2 = \sum_{j=1}^{n}\left(\sum_{l=j-N+1}^{j+N} f(t_{j-l,n})\varepsilon_{l,n}\right)^2$$

$$= \sum_{j=1}^{n}\sum_{l=j-N+1}^{j+N} f(t_{j-l,n})^2\varepsilon_{l,n}^2 + \sum_{j=1}^{n}\sum_{\substack{l_1,l_2=j-N+1\\ l_1\neq l_2}}^{j+N} f(t_{j-l_1,n})f(t_{j-l_2,n})\varepsilon_{l_1,n}\varepsilon_{l_2,n}$$

$$= \left(\sum_{l=N+1}^{n-N}\sum_{j=l-N}^{l+N-1} + \sum_{l=2-N}^{N}\sum_{j=1}^{l+N-1} + \sum_{l=n-N+1}^{n+N}\sum_{j=l-N}^{n}\right) f(t_{j-l,n})^2\varepsilon_{l,n}^2$$

$$+ \sum_{j=1}^{n}\sum_{\substack{l_1,l_2=j-N+1\\ l_1\neq l_2}}^{j+N} f(t_{j-l_1,n})f(t_{j-l_2,n})\varepsilon_{l_1,n}\varepsilon_{l_2,n} =: S_{1,n}+S_{2,n}+S_{3,n}+S_{4,n}.$$

We are going to show that the last three terms are negligible. We use the shorthand $E_n = \sum_{l=N+1}^{n-N}\varepsilon_{l,n}^2$, as this will be our benchmark term. Observe that $S_{1,n} = \sum_{k=-N}^{N-1} f(t_{k,n})^2 E_n$. Thanks to the boundedness and uniform continuity of f we have

$$\left|\Delta_n\sum_{k=-N}^{N-1} f(t_{k,n})^2 - \int_{-T}^{T} f(x)^2\,dx\right| = O(\omega_f(\Delta_n)).$$

Thus

$$\left|S_{1,n} - \frac{1}{\Delta_n}\int_{-T}^{T} f(x)^2dx\cdot E_n\right| = O\big(\Delta_n^{-1}\omega_f(\Delta_n)E_n\big),\quad n\to\infty. \tag{17}$$

On the other hand, by [6, XVII.5, Theorem 3 (i)], we have

$$\frac{E_n}{n^{2/\alpha}\Delta_n^{2/\alpha}} \Rightarrow Z_\alpha,\quad n\to\infty, \tag{18}$$

where Z_α is some positive $\alpha/2$–stable random variable. Therefore, by Slutsky's theorem,

$$\frac{S_{1,n}}{n^{2/\alpha}\Delta_n^{2/\alpha-1}} \Rightarrow Z_\alpha\int_{-T}^{T} f(x)^2dx,\quad n\to\infty. \tag{19}$$

Estimate

$$S_{2,n}+S_{3,n} \le \left(\sum_{l=2-N}^{N} + \sum_{l=n-N+1}^{n+N}\right)\varepsilon_{l,n}^2\sum_{k=-N}^{N-1} f(t_{k,n})^2 =: S_{5,n}.$$

Similarly to (19),

$$\frac{S_{5,n}}{(2N)^{2/\alpha}\Delta_n^{2/\alpha-1}} \Rightarrow Z'_\alpha \int_{-T}^{T} f(x)^2 dx, \quad n \to \infty. \tag{20}$$

Since $N\Delta_n = T$, we have

$$S_{2,n} + S_{3,n} = O_P(\Delta_n^{-1}) = O_P\big((n\Delta_n)^{-2/\alpha} n^{2/\alpha}\Delta_n^{2/\alpha-1}\big) = O_P\big(S_{1,n}(n\Delta_n)^{-2/\alpha}\big),$$

$n \to \infty$. Now write $S_{4,n}$ as

$$S_{4,n} = \sum_{\substack{l_1,l_2=2-N \\ l_1\neq l_2}}^{n+N} b_{l_1,l_2,n}\varepsilon_{l_1,n}\varepsilon_{l_2,n},$$

where $|b_{l_1,l_2,n}| \le 2N\|f\|_\infty^2$ and $b_{l_1,l_2,n} = 0$ whenever $|l_1 - l_2| \ge 2N$. Hence,

$$\sum_{\substack{l_1,l_2=2-N \\ l_1\neq l_2}}^{n+N} |b_{l_1,l_2,n}|^2 \le 16N^3 n\|f\|_\infty^4,$$

so Lemma 3 implies $S_{4,n} = O_P(N^{3/2}n^{2/\alpha-1/2}\Delta_n^{2/\alpha}) = O_P((n\Delta_n)^{-1/2}S_{1,n})$.

Summing up, we have $\sum_{j=1}^n X_n(t_{j,n})^2 = S_{1,n}\big(1 + O_P((n\Delta_n)^{-1/2})\big)$, $n \to \infty$, and $S_{1,n}$ is of order $n^{2/\alpha}\Delta_n^{2/\alpha-1}$, in the sense of (19).

Now we get back to the denominator of $I_{n,X}(\lambda)$. For any positive vanishing sequence $\{\delta_n, n \ge 1\}$ write the following simple estimate:

$$\left|a^2 - b^2\right| \le 2\left|a(a-b)\right| + \left|a-b\right|^2 \le \delta_n a^2 + (1+\delta_n^{-1})\left|a-b\right|^2. \tag{21}$$

Then we obtain

$$\left|\sum_{j=1}^n X_n(t_{j,n})^2 - S_{n,X}\right| \le \delta_n \sum_{j=1}^n X_n(t_{j,n})^2 + (1+\delta_n^{-1})\sum_{j=1}^n \left(X_n(t_{j,n}) - X(t_{j,n})\right)^2.$$

From Lemma 4 it follows that

$$\sum_{j=1}^n \left(X_n(t_{j,n}) - X(t_{j,n})\right)^2 = O_P(\|f_n - f\|_\infty^2 n^{2/\alpha}\Delta_n^{2/\alpha-1}) = O_P(\omega_f(\Delta_n)^2 S_{1,n}), \quad n \to \infty.$$

By assumption (F2), it holds $\omega_f(\Delta_n) \to 0$, $n \to \infty$. Putting $\delta_n = \omega_f(\Delta_n)$, we get that

$$\left|\sum_{j=1}^n X_n(t_{j,n})^2 - S_{n,X}\right| = O_P(\omega_f(\Delta_n)S_{1,n}) = o_P(S_{1,n}), \quad n \to \infty.$$

We conclude that

$$S_{n,X} = S_{1,n}\big(1 + O_P((n\Delta_n)^{-1/2} + \omega_f(\Delta_n))\big) = S_{1,n}(1 + o_P(1)), \quad n \to \infty,$$

whence, using (17),

$$\Delta_n S_{n,X} = \big(\|f\|_2^2 + O_P((n\Delta_n)^{-1/2} + \omega_f(\Delta_n))\big)\, E_n \overset{P}{\sim} \|f\|_2^2 E_n, \ n \to \infty. \tag{22}$$

Step 2. Whole expression
We turn to the expression in the left-hand side of (6). Recall that

$$J_{n,X}^{s}(\lambda)=\sum_{|m|\le m_n} W_n(m)\left|\sum_{j=1}^{n} X(t_{j,n})e^{it_{j,n}\nu_n(m,\lambda)}\right|^2$$

is the numerator of $I_{n,X}^{s}(\lambda)$ and write

$$\begin{aligned}
a_n\int_{-a_n}^{a_n}\left|\Delta_n I_{n,X}^{s}(\lambda)-|\hat{g}(\lambda)|^2\right|^2 d\lambda &= a_n\int_{-a_n}^{a_n}\left|\frac{\Delta_n^2 J_{n,X}^{s}(\lambda)}{\Delta_n S_{n,X}}-\frac{\left|\hat{f}(\lambda)\right|^2}{\|f\|_2^2}\right|^2 d\lambda\\
&\le 2a_n\int_{-a_n}^{a_n}\left|\frac{\Delta_n^2 J_{n,X}^{s}(\lambda)}{\Delta_n S_{n,X}}-\frac{\left|\hat{f}(\lambda)\right|^2 E_n}{\Delta_n S_{n,X}}\right|^2 d\lambda\\
&\quad+2a_n\int_{-a_n}^{a_n}\left|\frac{\left|\hat{f}(\lambda)\right|^2 E_n}{\Delta_n S_{n,X}}-\frac{\left|\hat{f}(\lambda)\right|^2}{\|f\|_2^2}\right|^2 d\lambda\\
&= 2a_n\left[\int_{-a_n}^{a_n}\left|\frac{\Delta_n^2 J_{n,X}^{s}(\lambda)-\left|\hat{f}(\lambda)\right|^2 E_n}{\Delta_n S_{n,X}}\right|^2 d\lambda+\left|\frac{\|f\|_2^2E_n-\Delta_n S_{n,X}}{\Delta_n\|f\|_2^2 S_{n,X}}\right|^2\int_{-a_n}^{a_n}\left|\hat{f}(\lambda)\right|^4 d\lambda\right].
\end{aligned}$$

Thanks to (22),

$$a_n\int_{-a_n}^{a_n}\left|\frac{\Delta_n^2 J_{n,X}^{s}(\lambda)-\left|\hat{f}(\lambda)\right|^2 E_n}{\Delta_n S_{n,X}}\right|^2 d\lambda \overset{P}{\sim} \frac{a_n}{\|f\|_2^4 E_n^2}\int_{-a_n}^{a_n}\left|\Delta_n^2 J_{n,X}^{s}(\lambda)-\left|\hat{f}(\lambda)\right|^2 E_n\right|^2 d\lambda$$

and

$$2a_n\left|\frac{\|f\|_2^2E_n-\Delta_n S_{n,X}}{\Delta_n\|f\|_2^2 S_{n,X}}\right|^2\int_{-a_n}^{a_n}\left|\hat{f}(\lambda)\right|^4 d\lambda = O_P\big(a_n((n\Delta_n)^{-1}+\omega_f(\Delta_n)^2)\big)=o_P(1)$$

as $n\to\infty$. Thus, it remains to prove that

$$a_n\int_{-a_n}^{a_n}\left|\Delta_n^2 J_{n,X}^{s}(\lambda)-\left|\hat{f}(\lambda)\right|^2 E_n\right|^2 d\lambda = o_P(E_n^2),\quad n\to\infty. \tag{23}$$

Step 3. Numerator. As with the denominator, we start with examining the discretized version of $J_{n,X}^{s}(\lambda)$:

$$\begin{aligned}
R_n(\lambda) &= \sum_{|m|\le m_n} W_n(m)\left|\sum_{j=1}^{n} X_n(t_{j,n})e^{it_{j,n}\nu_n(m,\lambda)}\right|^2\\
&= \sum_{|m|\le m_n} W_n(m)\left|\sum_{j=1}^{n}\sum_{l=j-N+1}^{j+N} f(t_{j-l,n})\varepsilon_{l,n}e^{it_{j,n}\nu_n(m,\lambda)}\right|^2.
\end{aligned}$$

We proceed in three substeps, first considering the following expression

$$R_{1,n}(\lambda) = \sum_{|m|\le m_n} W_n(m) \left| \sum_{l=N+1}^{n-N} \sum_{j=l-N}^{l+N-1} f(t_{j-l,n})\varepsilon_{l,n} e^{it_{j,n}\nu_n(m,\lambda)} \right|^2 .$$

<u>Step 3a):</u> We shall show

$$a_n \int_{-a_n}^{a_n} \left| \left| \hat{f}(\lambda) \right|^2 E_n - \Delta_n^2 R_{1,n}(\lambda) \right|^2 d\lambda = o_P(E_n^2), \quad n \to \infty. \tag{24}$$

We have for $\lambda \in [-a_n, a_n]$ that

$$\begin{aligned} R_{1,n}(\lambda) &= \sum_{|m|\le m_n} W_n(m) \left| \sum_{l=N+1}^{n-N} \varepsilon_{l,n} e^{it_{l,n}\nu_n(m,\lambda)} \sum_{j=l-N}^{l+N-1} f(t_{j-l,n}) e^{it_{j-l,n}\nu_n(m,\lambda)} \right|^2 \\ &= \sum_{|m|\le m_n} W_n(m) \left| \sum_{k=-N}^{N-1} f(t_{k,n}) e^{it_{k,n}\nu_n(m,\lambda)} \right|^2 \left| \sum_{l=N+1}^{n-N} \varepsilon_{l,n} e^{it_{l,n}\nu_n(m,\lambda)} \right|^2 \\ &= F_n(\lambda) \sum_{l=N+1}^{n-N} \varepsilon_{l,n}^2 + \sum_{N+1\le l_1\ne l_2\le n-N} a_{l_1,l_2,n}(\lambda)\varepsilon_{l_1,n}\varepsilon_{l_2,n}, \end{aligned}$$

where

$$F_n(\lambda) = \sum_{|m|\le m_n} W_n(m) \left| \sum_{k=-N}^{N-1} f(t_{k,n}) e^{it_{k,n}\nu_n(m,\lambda)} \right|^2 ,$$

$$a_{l_1,l_2,n}(\lambda) = \sum_{|m|\le m_n} W_n(m) \left| \sum_{k=-N}^{N-1} f(t_{k,n}) e^{it_{k,n}\nu_n(m,\lambda)} \right|^2 e^{i(l_1-l_2)\Delta_n\nu_n(m,\lambda)} \mathbb{1}_{[-a_n,a_n]}(\lambda).$$

With the help of Lemma 5, we obtain

$$\begin{aligned} \left| \left| \hat{f}(\lambda) \right|^2 E_n - \Delta_n^2 R_{1,n}(\lambda) \right| &\le O\Big(\big(W_n^{(2)}\big)^{1/2} (n\Delta_n)^{-1} + \omega_f(\Delta_n) + (1+|\lambda|)\Delta_n \Big) E_n \\ &\quad + \Delta_n^2 \left| \sum_{N+1\le l_1\ne l_2\le n-N} a_{l_1,l_2,n}(\lambda)\varepsilon_{l_1,n}\varepsilon_{l_2,n} \right|, \quad n\to\infty. \end{aligned}$$

Therefore,

$$\begin{aligned} a_n \int_{-a_n}^{a_n} \left| \left| \hat{f}(\lambda) \right|^2 E_n - \Delta_n^2 R_{1,n}(\lambda) \right|^2 d\lambda &= O\Big(a_n^2 W_n^{(2)} (n\Delta_n)^{-2} + a_n^2 \omega_f(\Delta_n)^2 + a_n^4\Delta_n^2 \Big) E_n^2 \\ &\quad + a_n \Delta_n^4 \int_{\mathbb{R}} \left| \sum_{N+1\le l_1\ne l_2\le n-N} a_{l_1,l_2,n}(\lambda)\varepsilon_{l_1,n}\varepsilon_{l_2,n} \right|^2 d\lambda, \quad n\to\infty. \end{aligned}$$

By Lemma 2,

$$\int_{\mathbb{R}} \left| \sum_{N+1\le l_1\ne l_2\le n-N} a_{l_1,l_2,n}(\lambda)\varepsilon_{l_1,n}\varepsilon_{l_2,n} \right|^2 d\lambda = O_P(A_n n^{4/\alpha-2}\Delta_n^{4/\alpha}),$$

where $A_n = \int_{-a_n}^{a_n} \sum_{N+1\le l_1\ne l_2\le n-N} |a_{l_1,l_2,n}(\lambda)|^2\, d\lambda$. By Lemma 6,

$$\sum_{N+1\le l_1\ne l_2\le n-N} |a_{l_1,l_2,n}(\lambda)|^2 = O(W_n^*(K_n^* n)^2), \quad n\to\infty,$$

where

$$K_n^* = \sup_{|m|\le m_n} \left|\sum_{k=-N}^{N-1} f(t_{k,n}) e^{it_{k,n}\nu_n(m,\lambda)}\right|^2 \le \left(\sum_{k=-N}^{N-1} |f(t_{k,n})|\right)^2 \sim \Delta_n^{-2}\|f\|_1^2,\ n\to\infty.$$

Hence,

$$\begin{aligned} a_n\Delta_n^4 \int_{\mathbb{R}} \left|\sum_{N+1\le l_1\ne l_2\le n-N} a_{l_1,l_2,n}(\lambda)\varepsilon_{l_1,n}\varepsilon_{l_2,n}\right|^2 d\lambda &= O_P(a_n\Delta_n^4 a_n W_n^*(K_n^* n)^2 n^{4/\alpha-2}\Delta_n^{4/\alpha}) \\ &= O_P(a_n^2 W_n^*(n\Delta_n)^{4/\alpha}) = O_P\left(a_n^2 W_n^* E_n^2\right),\ n\to\infty. \end{aligned}$$

Combining the estimates, we get (24).

Step 3b): We get

$$a_n \int_{-a_n}^{a_n} \left|\Delta_n^2 R_n(\lambda) - \left|\hat f(\lambda)\right|^2 E_n\right|^2 d\lambda = o_P(E_n^2), \quad n\to\infty. \tag{25}$$

Indeed, write

$$\begin{aligned} &a_n \int_{-a_n}^{a_n} \left|\Delta_n^2 R_n(\lambda) - \left|\hat f(\lambda)\right|^2 E_n\right|^2 d\lambda \\ &\le 2a_n \int_{-a_n}^{a_n} \Delta_n^4 |R_n(\lambda) - R_{1,n}(\lambda)|^2 d\lambda + 2a_n \int_{-a_n}^{a_n} \left|\Delta_n^2 R_{1,n}(\lambda) - \left|\hat f(\lambda)\right|^2 E_n\right|^2 d\lambda \\ &= 2a_n\Delta_n^4 \int_{-a_n}^{a_n} |R_n(\lambda) - R_{1,n}(\lambda)|^2 d\lambda + o_P(E_n^2), \quad n\to\infty. \end{aligned}$$

Let us estimate the first expression. Take some positive vanishing sequence $\{\theta_n, n\ge 1\}$, which will be specified later. Using (21), we have

$$\begin{aligned} &|R_{1,n}(\lambda) - R_n(\lambda)| \le \theta_n R_{1,n}(\lambda) + (1+\theta_n^{-1}) \\ &\times \sum_{|m|\le m_n} W_n(m) \left|\left(\sum_{l=2-N}^{N}\sum_{j=1}^{l+N-1} + \sum_{l=n-N+1}^{n+N}\sum_{j=l-N}^{n}\right) \varepsilon_{l,n} e^{it_{l,n}\nu_n(m,\lambda)} f(t_{j-l,n}) e^{it_{j-l,n}\nu_n(m,\lambda)}\right|^2 \\ &\le \theta_n R_{1,n}(\lambda) + (1+\theta_n^{-1})\left(R_{2,n}(\lambda) + R_{3,n}(\lambda)\right), \end{aligned}$$

where

$$R_{2,n}(\lambda) = \sum_{|m|\le m_n} W_n(m) \left|\sum_{l=2-N}^{N} \varepsilon_{l,n} e^{it_{l,n}\nu_n(m,\lambda)} \sum_{k=1-l}^{N-1} f(t_{k,n}) e^{it_{k,n}\nu_n(m,\lambda)}\right|^2,$$

$$R_{3,n}(\lambda) = \sum_{|m|\le m_n} W_n(m) \left|\sum_{l=n-N+1}^{n+N} \varepsilon_{l,n} e^{it_{l,n}\nu_n(m,\lambda)} \sum_{k=-N}^{n-l} f(t_{k,n}) e^{it_{k,n}\nu_n(m,\lambda)}\right|^2.$$

Hence,

$$
\begin{aligned}
&a_n \int_{-a_n}^{a_n} |R_n(\lambda) - R_{1,n}(\lambda)|^2 \, d\lambda \\
&\le 2a_n\theta_n^2 \int_{-a_n}^{a_n} R_{1,n}(\lambda)^2 d\lambda + 4a_n(1+\theta_n^{-1})^2 \int_{-a_n}^{a_n} \big(R_{2,n}(\lambda)^2 + R_{3,n}(\lambda)^2\big) d\lambda.
\end{aligned}
$$

Now

$$
\begin{aligned}
R_{2,n}(\lambda) &= \sum_{l=2-N}^{N} \varepsilon_{l,n}^2 \sum_{|m|\le m_n} W_n(m) \left| \sum_{k=1-l}^{N-1} f(t_{k,n}) e^{it_{k,n}\nu_n(m,\lambda)} \right|^2 \\
&\quad + \sum_{\substack{l_1,l_2=2-N \\ l_1\neq l_2}}^{N-1} b_{l_1,l_2,n}(\lambda)\varepsilon_{l_1,n}\varepsilon_{l_2,n} \\
&\le \sum_{l=2-N}^{N} \varepsilon_{l,n}^2 \left(\sum_{k=-N}^{N-1} |f(t_{k,n})| \right)^2 + \sum_{\substack{l_1,l_2=2-N \\ l_1\neq l_2}}^{N} b_{l_1,l_2,n}(\lambda)\varepsilon_{l_1,n}\varepsilon_{l_2,n} =: R_{4,n} + R_{5,n}(\lambda),
\end{aligned}
$$

where

$$
b_{l_1,l_2,n}(\lambda) = \sum_{|m|\le m_n} W_n(m) e^{i(l_1-l_2)\Delta_n\nu_n(m,\lambda)} \sum_{k_1=1-l_1}^{N-1} \sum_{k_2=1-l_2}^{N-1} f(t_{k_1,n}) f(t_{k_2,n}) e^{i(k_1-k_2)\Delta_n\nu_n(m,\lambda)}.
$$

The functions $b_{l_1,l_2,n}$ satisfy

$$
|b_{l_1,l_2,n}(\lambda)| \le \left(\sum_{k=-N}^{N-1} |f(t_{k,n})| \right)^2 \sim \Delta_n^{-2} \|f\|_1^2, \ n\to\infty.
$$

Thus

$$
\int_{\mathbb{R}} \sum_{1\le l_1<l_2\le N} |b_{l_1,l_2,n}(\lambda)|^2 \, \mathbb{1}_{[-a_n,a_n]}(\lambda) \, d\lambda = O(a_n N^2 \Delta_n^{-4}), \ n\to\infty,
$$

and therefore Lemma 2 implies

$$
\int_{-a_n}^{a_n} R_{5,n}(\lambda)^2 d\lambda = O_P(a_n N^2 \Delta_n^{4/\alpha-4} N^{4/\alpha-2}) = O_P(a_n\Delta_n^{-4}), \quad n\to\infty.
$$

Further,

$$
R_{4,n} \sim \Delta_n^{-2}\|f\|_1^2 \sum_{l=2-N}^{N} \varepsilon_{l,n}^2, \ n\to\infty,
$$

so thanks to (20), $R_{4,n} = O_P(N^{2/\alpha}\Delta_n^{2/\alpha-2}) = O_P(\Delta_n^{-2})$, $n\to\infty$. Thus, we get

$$
\int_{-a_n}^{a_n} R_{2,n}(\lambda)^2 \, d\lambda = O_P(a_n\Delta_n^{-4}), n\to\infty.
$$

Similarly, $\int_{-a_n}^{a_n} R_{3,n}(\lambda)^2 \, d\lambda = O_P(a_n\Delta_n^{-4})$, $n\to\infty$.

Setting $\theta_n = (n\Delta_n)^{-2/(3\alpha)}$, we get by (A3) that

$$a_n(1+\theta_n^{-1})^2 \int_{-a_n}^{a_n} \big(R_{2,n}(\lambda)^2 + R_{3,n}(\lambda)^2\big) d\lambda = O_P(a_n^2 \Delta_n^{-4} (n\Delta_n)^{4/(3\alpha)}) = o_P(n^{4/\alpha}\Delta_n^{4/\alpha - 4})$$

as $n \to \infty$. Therefore, we arrive at

$$\begin{aligned} &a_n \int_{-a_n}^{a_n} \left|\Delta_n^2 R_n(\lambda) - \Delta_n^2 R_{1,n}(\lambda)\right|^2 d\lambda \\ &\le 2a_n (n\Delta_n)^{-4/(3\alpha)} \Delta_n^4 \int_{-a_n}^{a_n} R_{1,n}(\lambda)^2 d\lambda + o_P(n^{4/\alpha}\Delta_n^{4/\alpha}), \quad n \to \infty. \end{aligned}$$

Noting that $o_P(n^{4/\alpha}\Delta_n^{4/\alpha}) = o_P(E_n^2)$, $n \to \infty$ and by Step 3a)

$$\begin{aligned} \Delta_n^4 \int_{-a_n}^{a_n} R_{1,n}(\lambda)^2 d\lambda &\le 2\left(\int_{-a_n}^{a_n} \left|\hat{f}(\lambda)\right|^4 d\lambda \cdot E_n^2 + \int_{-a_n}^{a_n} \left|\left|\hat{f}(\lambda)\right|^2 E_n - \Delta_n^2 R_{1,n}(\lambda)\right|^2 d\lambda\right) \\ &= \left(2\int_{-a_n}^{a_n} \left|\hat{f}(\lambda)\right|^4 d\lambda + o_P(1)\right) E_n^2, \quad n \to \infty, \end{aligned}$$

we get by (A3) $a_n \int_{-a_n}^{a_n} \left|\Delta_n^2 R_n(\lambda) - \Delta_n^2 R_{1,n}(\lambda)\right|^2 d\lambda = o_P(E_n^2)$, $\quad n \to \infty$, whence (25) follows from (24).

Step 3c): Finally we have

$$a_n \Delta_n^4 \int_{-a_n}^{a_n} \left|J_{n,X}^s(\lambda) - R_n(\lambda)\right|^2 d\lambda = o_P(E_n^2), \quad n \to \infty. \tag{26}$$

Using (21) again, write

$$\begin{aligned} \left|J_{n,X}^s(\lambda) - R_n(\lambda)\right| &= \left| R_n(\lambda) - \sum_{|m|\le m_n} W_n(m) \left|\sum_{j=1}^n X(t_{j,n}) e^{it_{j,n}\nu_n(m,\lambda)}\right|^2 \right| \\ &\le \delta_n R_n(\lambda) + (1+\delta_n^{-1}) \sum_{|m|\le m_n} W_n(m) \left|\sum_{j=1}^n (X_n(t_{j,n}) - X(t_{j,n})) e^{it_{j,n}\nu_n(m,\lambda)}\right|^2 \\ &= \delta_n R_n(\lambda) + (1+\delta_n^{-1}) \sum_{|m|\le m_n} W_n(m) \left| \int_{\mathbb{R}} \sum_{j=1}^n \big(f_n(t_{j,n}-s) - f(t_{j,n}-s)\big) e^{it_{j,n}\nu_n(m,\lambda)} \Lambda(ds)\right|^2 \\ &=: \delta_n R_n(\lambda) + (1+\delta_n^{-1}) R_{6,n}(\lambda) \end{aligned} \tag{27}$$

for $\delta_n = \omega_f(\Delta_n)^{2/3}$. Hence,

$$a_n \int_{-a_n}^{a_n} \left|J_{n,X}^s(\lambda) - R_n(\lambda)\right|^2 d\lambda \le 2a_n\delta_n^2 \int_{-a_n}^{a_n} R_n(\lambda)^2 d\lambda + 2a_n(1+\delta_n^{-1})^2 \int_{-a_n}^{a_n} R_{6,n}(\lambda)^2 d\lambda.$$

Define $h_{n,m}(s,\lambda) = \sum_{j=1}^n \big(f_n(t_{j,n}-s) - f(t_{j,n}-s)\big) e^{it_{j,n}\nu_n(m,\lambda)} \mathbb{1}_{[-a_n,a_n]}(\lambda)$. Note that the summands do not exceed $\omega_f(\Delta_n)$, and at most $2N$ of them are not zero. Hence, $\|h_{n,m}(\cdot,\lambda)\|_\infty \le 2N\omega_f(\Delta_n)$. Applying Lemma 7, we get

$$a_n \int_{-a_n}^{a_n} R_{6,n}(\lambda)^2 d\lambda = O_P(a_n^2 N^4 \omega_f(\Delta_n)^4 (n\Delta_n)^{4/\alpha}) = O_P(a_n^2 \omega_f(\Delta_n)^4 n^{4/\alpha} \Delta_n^{4/\alpha - 4}),\ n \to \infty.$$

Recalling that $\delta_n \to 0$ and $a_n\omega_f(\Delta_n)^{4/3} \to 0$ as $n \to \infty$, we ultimately obtain

$$a_n\Delta_n^4 \int_{-a_n}^{a_n} \left|J^s_{n,X}(\lambda) - R_n(\lambda)\right|^2 d\lambda$$

$$= O_P\Big(a_n(\omega_f(\Delta_n))^{4/3}\Big(\int_{\mathbb{R}} \left|\hat{f}(\lambda)\right|^2 d\lambda + o_P(1)\Big)E_n^2 + \Delta_n^4(\omega_f(\Delta_n))^{-4/3}a_n^2\omega_f(\Delta_n)^4 n^{4/\alpha}\Delta_n^{4/\alpha-4}\Big)$$

$$= o_p(E_n^2 + n^{4/\alpha}\Delta_n^{4/\alpha}) = o_p(E_n^2), \quad n \to \infty.$$

Combining (25) and (26), we come to (23). □

Kernel f with unbounded support

Proof of Theorem 2. Part (ii) is derived from part (i) exactly the same way as in Theorem 1.

In order to prove part (i), we need to show that

$$a_n \int_{-a_n}^{a_n} \left|\Delta_n I^s_{n,X}(\lambda) - |\hat{g}(\lambda)|^2\right|^2 d\lambda \xrightarrow{P} 0, n \to \infty.$$

We start by setting $N_n = \lfloor \omega_f(\Delta_n)^{-1/a}\Delta_n^{-1}\rfloor$, $n \geq 1$, so that $T_n := N_n\Delta_n \sim \omega_f(\Delta_n)^{-1/a}$, $n \to \infty$. Recall that $\varepsilon_{l,n} = \Lambda([(l-1)\Delta_n, l\Delta_n])$, $l \in \mathbb{Z}$, and set $E_n = \sum_{l=N_n+1}^{n-N_n} \varepsilon^2_{l,n}$.

The rest of the proof follows the same scheme as that of Theorem 1. Specifically, examining Step 2 of the latter, it is enough to show that

(i) $S_{n,X} = \frac{1}{\Delta_n}\int_{\mathbb{R}} f(x)^2 dx \cdot E_n\Big(1 + O_P\big(\omega_f(\Delta_n)^{1-1/(2a)} + N_n^{2/\alpha}n^{-2/\alpha} + (n\Delta_n)^{-1/2}\big)\Big)$, $n \to \infty$;

(ii) $a_n\int_{-a_n}^{a_n}\left|\Delta_n^2 J^s_{n,X}(\lambda) - \left|\hat{f}(\lambda)\right|^2 E_n\right|^2 d\lambda = o_P(E_n^2)$, $n \to \infty$.

We thus split the proof into two steps, establishing these relations.

Step 1.

Define $f_n(x) = \sum_{k=-N_n}^{N_n-1} f(t_{k,n})\mathbb{1}_{[t_{k,n},t_{k+1,n})}(x)$, $X_n(t) = \int_{\mathbb{R}} f_n(t-s)\Lambda(ds)$. Write

$$S_{n,X_n} = \sum_{j=1}^{n} X_n(t_{j,n})^2 = \sum_{j=1}^{n}\left(\sum_{l=j-N_n+1}^{j+N_n} f(t_{j-l,n})\varepsilon_{l,n}\right)^2$$

$$= \sum_{j=1}^{n}\sum_{l=j-N_n+1}^{j+N_n} f(t_{j-l,n})^2\varepsilon^2_{l,n} + \sum_{j=1}^{n}\sum_{\substack{l_1,l_2=j-N_n+1\\ l_1\neq l_2}}^{j+N_n} f(t_{j-l_1,n})f(t_{j-l_2,n})\varepsilon_{l_1,n}\varepsilon_{l_2,n}$$

$$= \left(\sum_{l=N_n+1}^{n-N_n}\sum_{j=l-N_n}^{l+N_n-1} + \sum_{l=2-N_n}^{N_n}\sum_{j=1}^{l+N_n-1} + \sum_{l=n-N_n+1}^{n+N_n}\sum_{j=l-N_n}^{n}\right) f(t_{j-l,n})^2\varepsilon^2_{l,n}$$

$$+ \sum_{j=1}^{n}\sum_{\substack{l_1,l_2=j-N_n+1\\ l_1\neq l_2}}^{j+N_n} f(t_{j-l_1,n})f(t_{j-l_2,n})\varepsilon_{l_1,n}\varepsilon_{l_2,n} =: S_{1,n} + S_{2,n} + S_{3,n} + S_{4,n}.$$

First note that

$$\left|S_{1,n} - \Delta_n^{-1}\int_{\mathbb{R}} f(x)^2 dx \cdot E_n\right| = \left|\sum_{k=-N_n}^{N_n-1} f(t_{k,n})^2 - \Delta_n^{-1}\int_{\mathbb{R}} f(x)^2 dx\right| E_n$$

$$
\begin{aligned}
&\le \left|\sum_{k=-N_n}^{N_n-1} f(t_{k,n})^2 - \Delta_n^{-1}\int_{-T_n}^{T_n} f(x)^2dx\right| E_n + \Delta_n^{-1}\left|\int_{\{x:|x|>T_n\}} f(x)^2dx\right| E_n \\
&= O_P\big((T_n\omega_f(\Delta_n) + T_n^{1-2a})\Delta_n^{-1}E_n\big) = O_P\big(\omega_f(\Delta_n)^{1-1/a}n^{2/\alpha}\Delta_n^{2/\alpha-1}\big), \quad n\to\infty,
\end{aligned}
$$

where we have used that $T_n\omega_f(\Delta_n)^{1/a}$ is bounded away both from zero and from infinity. Similarly to the finite support case,

$$
S_{2,n}+S_{3,n} = O_P(N_n^{2/\alpha}\Delta_n^{2/\alpha-1}) = O_P(N_n^{2/\alpha}n^{-2/\alpha}n^{2/\alpha}\Delta_n^{2/\alpha-1}),\ n\to\infty,
$$

and

$$
S_{4,n} = 2\sum_{2-N_n\le l_1<l_2\le n+N_n} a_{l_1,l_2,n}\varepsilon_{l_1,n}\varepsilon_{l_2,n},
$$

where

$$
a_{l_1,l_2,n} = \sum_{j=(l_2-N_n)\vee 1}^{(l_1+N_n)\wedge n} f(t_{j-l_1,n})f(t_{j-l_2,n}).
$$

Estimate

$$
\begin{aligned}
\sum_{\substack{l_1,l_2=1-N_n\\ l_1<l_2}}^{n+N_n} |a_{l_1,l_2,n}|^2 &\le \sum_{\substack{l_1,l_2=1-N_n\\ l_1<l_2}}^{n+N_n}\left(\sum_{j=(l_2-N_n)\vee 1}^{(l_1+N_n)\wedge n}|f(t_{j-l_1,n})f(t_{j-l_2,n})|\right)^2 \\
&\sim \Delta_n^{-4}\int_{-T_n}^{n\Delta_n+T_n}\int_x^{n\Delta_n+T_n}\left(\int_{(y-T_n)\vee 0}^{(x+T_n)\wedge(n\Delta_n)}|f(z-x)f(z-y)|dz\right)^2 dy\,dx \\
&\le \Delta_n^{-4}\int_{-T_n}^{n\Delta_n+T_n}\int_0^{n\Delta_n+T_n-x}\left(\int_{y'-T_n}^{T_n}|f(z')f(z'-y')|dz'\right)^2 dy'\,dx = O(n\Delta_n^{-3}),\ n\to\infty,
\end{aligned}
$$

since

$$
\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|f(z)f(z-y)|dz\right)^2 dy = \||f|\star|f|\|_2^2 = \|\widehat{|f|}^2\|_2^2 < \infty,
$$

where $\star$ is the convolution operation. Hence by Lemma 3,

$$
S_{4,n} = O_P(n^{1/2}\Delta_n^{-3/2}n^{2/\alpha-1}\Delta_n^{2/\alpha}) = O_P((n\Delta_n)^{-1/2}n^{2/\alpha}\Delta_n^{2/\alpha-1}),\quad n\to\infty.
$$

Collecting the estimates, we get

$$
\begin{aligned}
&\left|S_{n,X_n} - \Delta_n^{-1}\int_{\mathbb{R}} f(x)^2dx\cdot E_n\right| \\
&= O_P\big((\omega_f(\Delta_n))^{1-1/a} + N_n^{2/\alpha}n^{-2/\alpha} + (n\Delta_n)^{-1/2}\big)n^{2/\alpha}\Delta_n^{2/\alpha-1}\big)
\end{aligned}
\tag{28}
$$

as $n\to\infty$.

Further, by (21), for some vanishing sequence $\{\delta_n, n\ge 1\}$

$$
|S_{n,X_n} - S_{n,X}| \le \delta_n S_{n,X_n} + (1+\delta_n^{-1})S_{5,n}, \tag{29}
$$

where

$$
S_{5,n} = \sum_{j=1}^n (X_n(t_{j,n}) - X(t_{j,n}))^2 = \sum_{j=1}^n\left(\int_{\mathbb{R}}\big(f(t_{j,n}-s) - f_n(t_{j,n}-s)\big)\Lambda(ds)\right)^2
$$

$$\leq 3\left(\sum_{j=1}^{n}\left(\int_{t_{j,n}-T_n}^{t_{j,n}+T_n}\big(f(t_{j,n}-s)-f_n(t_{j,n}-s)\big)\Lambda(ds)\right)^2\right.$$
$$+\sum_{j=1}^{n}\left(\int_{\{s:|s-t_{j,n}|\in[T_n,n\Delta_n]\}} f(t_{j,n}-s)\Lambda(ds)\right)^2$$
$$\left.+\sum_{j=1}^{n}\left(\int_{\{s:|s-t_{j,n}|>n\Delta_n\}} f(t_{j,n}-s)\Lambda(ds)\right)^2\right) =: S_{6,n}+S_{7,n}+S_{8,n},$$

where $T_n = N_n\Delta_n$. Since $|f_n(y)-f(y)|\leq \omega_f(\Delta_n)$ for $y\in[-N_n\Delta_n, N_n\Delta_n]$, by Lemma 4,

$$S_{6,n} = O_P\big(\omega_f(\Delta_n)^2 T_n n^{2/\alpha}\Delta_n^{2/\alpha-1}\big) = o_P\big(\omega_f(\Delta_n)^{2-1/a} n^{2/\alpha}\Delta_n^{2/\alpha-1}\big),\ n\to\infty.$$

To estimate $S_{7,n}$, we use Lemma 1. For each $n\geq 1$, the process

$$Y_{t,n} = \int_{\{s:|s-t_{j,n}|\in[T_n,n\Delta_n]\}} f(t-s)\Lambda(ds),\quad t\in[0,n\Delta_n],$$

has the same distribution as

$$\tilde{Y}_{t,n} = C_\alpha^{1/\alpha}(3n\Delta_n)^{1/\alpha}\sum_{k=1}^{\infty}\Gamma_k^{-1/\alpha} f(t-\xi_k)\mathbb{1}_{|\xi_k-t|\in[T_n,n\Delta_n]}\zeta_k,\quad t\in[0,n\Delta_n],$$

where $\{\Gamma_k, k\geq 1\}$ and $\{\zeta_k, k\geq 1\}$ are as in Lemma 1, $\{\xi_k, k\geq 1\}$ are iid uniformly distributed over $[-n\Delta_n, 2n\Delta_n]$. Since we are concerned with convergence in probability, we can freely assume that $Y_{t,n} = \tilde{Y}_{t,n}$. Then, taking into account (F3′),

$$\mathsf{E}\left[\sum_{j=1}^{n} Y_{t_{j,n},n}^2 \,\Big|\, \Gamma\right] \leq C_\alpha^{2/\alpha}(3n\Delta_n)^{2/\alpha}\sum_{j=1}^{n}\sum_{k=1}^{\infty}\Gamma_k^{-2/\alpha}\mathsf{E}\left[f(t_{j,n}-\xi_k)^2\mathbb{1}_{|\xi_k-t_{j,n}|\in[T_n,n\Delta_n]}\right]$$
$$\leq C(n\Delta_n)^{2/\alpha-1}\sum_{k=1}^{\infty}\Gamma_k^{-2/\alpha}\sum_{j=1}^{n}\int_{\{x:|x-t_{j,n}|\in[T_n,n\Delta_n]\}} f(t_{j,n}-x)^2 dx$$
$$\leq C(n\Delta_n)^{2/\alpha-1}\sum_{k=1}^{\infty}\Gamma_k^{-2/\alpha}\sum_{j=1}^{n}\int_{\{x:|x-t_{j,n}|\geq T_n\}} |t_{j,n}-x|^{-2a}dx$$
$$\leq Cn^{2/\alpha}\Delta_n^{2/\alpha-1}T_n^{1-2a}\sum_{k=1}^{\infty}\Gamma_k^{-2/\alpha}.$$

Since by the strong law of large numbers, $\Gamma_k\sim k$, $k\to\infty$, a.s., the last series converges almost surely, therefore, given Γ, $\sum_{j=1}^{n} Y_{t_{j,n},n}^2 = O_P(T_n^{1-2a}n^{2/\alpha}\Delta_n^{2/\alpha-1})$, $n\to\infty$. Consequently, $S_{7,n} = O_P(T_n^{1-2a}n^{2/\alpha}\Delta_n^{2/\alpha-1}) = O_P\big(\omega_f(\Delta_n)^{2-1/a}n^{2/\alpha}\Delta_n^{2/\alpha-1}\big)$, $n\to\infty$.

To estimate $S_{8,n}$, let $Z_{t,n} = \int_{\{s:|s-t|>n\Delta_n\}} f(t-s)\Lambda(ds)$ and define for some $b>0$ the positive density over $\mathbb{R}$:

$$\varphi(x) = K_b|x|^{-1}\big(|\log|x||+1\big)^{-1-b}, \tag{30}$$

where $K_b = \left(\int_{\mathbb{R}}|x|^{-1}\big(|\log|x||+1\big)^{-1-b}dx\right)^{-1}$ is the normalizing constant. As before, we can assume that

$$Z_{t,n} = C_\alpha^{1/\alpha}\sum_{k=1}^{\infty}\Gamma_k^{-1/\alpha} f(t-\eta_k)\varphi(\eta_k)^{-1/\alpha}\mathbb{1}_{\{|\eta_k-t|>n\Delta_n\}}\zeta_k,\quad t\geq 0,$$

where $\{\Gamma_k, k \geq 1\}$ and $\{\zeta_k, k \geq 1\}$ are as in Lemma 1, $\{\eta_k, k \geq 1\}$ are iid with density φ. Then

$$\mathsf{E}\left[\sum_{j=1}^{n} Z_{t_{j,n},n}^2 \,\middle|\, \Gamma\right] = C_\alpha^{2/\alpha} \sum_{j=1}^{n} \sum_{k=1}^{\infty} \Gamma_k^{-2/\alpha} \mathsf{E}\left[\, f(t_{j,n} - \eta_k)^2 \varphi(\eta_k)^{-2/\alpha} \mathbb{1}_{|\eta_k - t_{j,n}| > n\Delta_n} \,\right]$$

$$= C_\alpha^{2/\alpha} \sum_{j=1}^{n} \int_{\{x: |x - t_{j,n}| > n\Delta_n\}} f(t_{j,n} - x)^2 \varphi(x)^{1-2/\alpha} dx \sum_{k=1}^{\infty} \Gamma_k^{-2/\alpha} =: C_\alpha^{2/\alpha} L_n \sum_{k=1}^{\infty} \Gamma_k^{-2/\alpha}.$$

It is enough to study the term L_n:

$$\begin{aligned} L_n &\leq C \sum_{j=1}^{n} \int_{\{x: |x - t_{j,n}| > n\Delta_n\}} |t_{j,n} - x|^{-2a} \varphi(x)^{1-2/\alpha} dx \\ &\sim C \Delta_n^{-1} \int_0^{n\Delta_n} \int_{\{x: |x-t| > n\Delta_n\}} |t - x|^{-2a} \varphi(x)^{1-2/\alpha} dx\, dt \\ &= C \Delta_n^{-1} \int_{\{y: |y| > n\Delta_n\}} |y|^{-2a} \int_0^{n\Delta_n} \varphi(t - y)^{1-2/\alpha} dt\, dy. \end{aligned} \tag{31}$$

Since φ is monotonically decreasing on $[1, \infty)$, we have

$$\begin{aligned} L_n &\leq C \Delta_n^{-1} \int_{\{y: |y| > n\Delta_n\}} |y|^{-2a} n\Delta_n (n\Delta_n + |y|)^{2/\alpha - 1} \log^r(n\Delta_n + |y|) dy \\ &\leq C \Delta_n^{-1} \big(n\Delta_n\big)^{2/\alpha + 1 - 2a} \log^r(n\Delta_n), \end{aligned} \tag{32}$$

due to L'Hospital's rule, where $r = (2/\alpha - 1)(1 + b)$. Since $a > 2$, we get

$$S_{8,n} = O_P\big((n\Delta_n)^{-1} n^{2/\alpha} \Delta_n^{2/\alpha - 1}\big), n \to \infty.$$

Summing everything up,

$$S_{5,n} = n^{2/\alpha} \Delta_n^{2/\alpha - 1} \cdot O_P\big(\omega_f(\Delta_n)^{2-1/a} + (n\Delta_n)^{-1}\big),\ n \to \infty.$$

Now set $\delta_n = \big(\omega_f(\Delta_n)^{2-1/a} + (n\Delta_n)^{-1}\big)^{1/2}$ in (29). Clearly, $\delta_n \to 0$, $n \to \infty$. Thus, using (28), we arrive at

$$S_{n,X} = \Delta_n^{-1} \left(\int_{\mathbb{R}} f(x)^2 dx + O_P\big(\omega_f(\Delta_n)^{1-1/(2a)} + N_n^{2/\alpha} n^{-2/\alpha} + (n\Delta_n)^{-1/2}\big) \right) \cdot E_n,\ n \to \infty,$$

as claimed.

Step 2. This goes similar to Step 3 of Theorem 1, so we omit some details. First, similarly to Step 3a, write

$$\begin{aligned} R_{1,n}(\lambda) &:= \sum_{|m| \leq m_n} W_n(m) \left| \sum_{l=N_n+1}^{n-N_n} \sum_{j=l-N_n}^{l+N_n-1} f(t_{j-l,n}) \varepsilon_{l,n} e^{it_{j,n}\nu_n(m,\lambda)} \right|^2 \\ &= F_n(\lambda) \cdot E_n + \sum_{N_n+1 \leq l_1 \neq l_2 \leq n-N_n} a_{l_1,l_2,n}(\lambda) \varepsilon_{l_1,n} \varepsilon_{l_2,n}, \end{aligned}$$

where

$$F_n(\lambda) = \sum_{|m| \leq m_n} W_n(m) \left| \sum_{k=-N_n}^{N_n-1} f(t_{k,n}) e^{it_{k,n}\nu_n(m,\lambda)} \right|^2,$$

$$a_{l_1,l_2,n}(\lambda)=\sum_{|m|\le m_n}W_n(m)\left|\sum_{k=-N_n}^{N_n-1}f(t_{k,n})e^{it_{k,n}\nu_n(m,\lambda)}\right|^2 e^{i(l_1-l_2)\Delta_n\nu_n(m,\lambda)}\mathbb{1}_{[-a_n,a_n]}(\lambda).$$

Using Lemma 5, we get

$$\left|\left|\hat f(\lambda)\right|^2 E_n-\Delta_n^2R_{1,n}(\lambda)\right|\le O\Big(\big(W_n^{(2)}\big)^{1/2}(n\Delta_n)^{-1}+T_n\omega_f(\Delta_n)+T_n^{1-a}+|\lambda|\,\Delta_n\Big)E_n$$
$$+\Delta_n^2\left|\sum_{N_n+1\le l_1\ne l_2\le n-N_n}a_{l_1,l_2,n}(\lambda)\varepsilon_{l_1,n}\varepsilon_{l_2,n}\right|.$$

Then, using Lemmas 6 and 2, we get

$$\begin{aligned}&a_n\int_{-a_n}^{a_n}\left|\left|\hat f(\lambda)\right|^2 E_n-\Delta_n^2R_{1,n}(\lambda)\right|^2d\lambda\\ &=O\Big(a_n^2W_n^{(2)}(n\Delta_n)^{-2}+a_n^2T_n^2\omega_f(\Delta_n)^2+a_n^2T_n^{2-2a}+a_n^4\Delta_n^2\Big)E_n^2+O_P\big(a_n^2W_n^*(n\Delta_n)^{4/\alpha}\big)\\ &=o_P(E_n^2),\quad n\to\infty.\end{aligned}\tag{33}$$

Secondly, we define

$$R_n(\lambda)=\sum_{|m|\le m_n}W_n(m)\left|\sum_{j=1}^{n}\sum_{l=j-N_n+1}^{j+N_n}f(t_{j-l,n})\varepsilon_{l,n}e^{it_{j,n}\nu_n(m,\lambda)}\right|^2$$

and write, using the above,

$$\begin{aligned}&a_n\int_{-a_n}^{a_n}\left|\Delta_n^2R_n(\lambda)-\left|\hat f(\lambda)\right|^2E_n\right|^2d\lambda\\ &\le 2a_n\Delta_n^4\int_{-a_n}^{a_n}|R_n(\lambda)-R_{1,n}(\lambda)|^2d\lambda+o_P(E_n^2),\quad n\to\infty.\end{aligned}$$

In turn, for some positive sequence $\{\theta_n,n\ge1\}$,

$$\begin{aligned}&a_n\int_{-a_n}^{a_n}|R_n(\lambda)-R_{1,n}(\lambda)|^2d\lambda\\ &\le 2a_n\theta_n^2\int_{-a_n}^{a_n}R_{1,n}(\lambda)^2d\lambda+4a_n(1+\theta_n^{-1})^2\int_{-a_n}^{a_n}\big(R_{2,n}(\lambda)^2+R_{3,n}(\lambda)^2\big)d\lambda\end{aligned}$$

with

$$R_{2,n}(\lambda)=\sum_{|m|\le m_n}W_n(m)\left|\sum_{l=2-N_n}^{N_n}\varepsilon_{l,n}e^{it_{l,n}\nu_n(m,\lambda)}\sum_{k=1-l}^{N_n-1}f(t_{k,n})e^{it_{k,n}\nu_n(m,\lambda)}\right|^2,$$
$$R_{3,n}(\lambda)=\sum_{|m|\le m_n}W_n(m)\left|\sum_{l=n-N_n+1}^{n+N_n}\varepsilon_{l,n}e^{it_{l,n}\nu_n(m,\lambda)}\sum_{k=-N_n}^{n-l}f(t_{k,n})e^{it_{k,n}\nu_n(m,\lambda)}\right|^2.$$

As in Step 3b, these terms are estimated in a similar fashion, e.g.

$$R_{2,n}(\lambda)\le\sum_{l=2-N_n}^{N_n}\varepsilon_{l,n}^2\left(\sum_{k=-N_n}^{N_n-1}|f(t_{k,n})|\right)^2+\sum_{\substack{l_1,l_2=2-N_n\\ l_1\ne l_2}}^{N_n}b_{l_1,l_2,n}(\lambda)\varepsilon_{l_1,n}\varepsilon_{l_2,n}=:R_{4,n}+R_{5,n}(\lambda),$$

with

$$b_{l_1,l_2,n}(\lambda) = \sum_{|m|\le m_n} W_n(m) e^{i(l_1-l_2)\Delta_n \nu_n(m,\lambda)}$$

$$\times \sum_{k_1=1-l_1}^{N_n-1} \sum_{k_2=1-l_2}^{N_n-1} f(t_{k_1,n}) f(t_{k_2,n}) e^{i(k_1-k_2)\Delta_n\nu_n(m,\lambda)} \mathbb{1}_{[-a_n,a_n]}(\lambda).$$

Then from Lemma 2

$$\int_{-a_n}^{a_n} R_{5,n}(\lambda)^2 d\lambda = O_P(a_n N_n^2 \Delta_n^{4/\alpha-4} N_n^{4/\alpha-2}) = O_P(a_n T_n^{4/\alpha}\Delta_n^{-4}), \quad n\to\infty,$$

and from (20), $R_{4,n} = O_P(N_n^{2/\alpha}\Delta_n^{2/\alpha-2}) = O_P(T_n^{2/\alpha}\Delta_n^{-2})$, $n\to\infty$. Consequently,

$$a_n \int_{-a_n}^{a_n} R_{2,n}(\lambda)^2 d\lambda = O_P(a_n^2 T_n^{4/\alpha}\Delta_n^{-4}), \quad n\to\infty,$$

and similarly for $R_{3,n}$. Observing

$$\Delta_n^4 \int_{-a_n}^{a_n} R_{1,n}(\lambda)^2 d\lambda = O_P(E_n^2)$$

by (33) and putting $\theta_n = a_n^{1/4} N_n^{1/\alpha} n^{-1/\alpha}$ yields

$$a_n \Delta_n^4 \int_{-a_n}^{a_n} \left|R_n(\lambda) - R_{1,n}(\lambda)\right|^2 d\lambda = O_P(a_n^{3/2} N_n^{2/\alpha} n^{-2/\alpha} E_n^2), \quad n\to\infty.$$

Since $a_n^{3/4} N_n^{1/\alpha} n^{-1/\alpha} = o(1)$, $n\to\infty$, by (F4′), we get

$$a_n \Delta_n^4 \int_{-a_n}^{a_n} \left|R_n(\lambda) - R_{1,n}(\lambda)\right|^2 d\lambda = o_P(E_n^2), \quad n\to\infty.$$

Finally, by upper bound (27), write for any $\lambda\in[-a_n,a_n]$ and for some positive vanishing sequence $\{\delta_n, n\ge 1\}$

$$\begin{aligned}
\left|J^s_{n,X}(\lambda) - R_n(\lambda)\right| &\le \delta_n R_n(\lambda) + 3(1+\delta_n^{-1})\Bigg(\sum_{|m|\le m_n} W_n(m)\left|\int_{\mathbb{R}} h_{n,m}(s,\lambda)\Lambda(ds)\right|^2 \\
&+ \sum_{|m|\le m_n} W_n(m)\left|\sum_{j=1}^n \int_{\{s:|s-t_{j,n}|\in[T_n,n\Delta_n]\}} f(t_{j,n}-s)e^{it_{j,n}\nu_n(m,\lambda)}\Lambda(ds)\right|^2 \\
&+ \sum_{|m|\le m_n} W_n(m)\left|\sum_{j=1}^n \int_{\{s:|s-t_{j,n}|>n\Delta_n\}} f(t_{j,n}-s)e^{it_{j,n}\nu_n(m,\lambda)}\Lambda(ds)\right|^2\Bigg) \\
&=: \delta_n R_n(\lambda) + 3(1+\delta_n^{-1})\big(R_{6,n}(\lambda) + R_{7,n}(\lambda) + R_{8,n}(\lambda)\big),
\end{aligned}$$

where

$$h_{n,m}(s,\lambda) = \sum_{j=1}^n \big(f_n(t_{j,n}-s) - f(t_{j,n}-s)\big)e^{it_{j,n}\nu_n(m,\lambda)}\mathbb{1}_{|s-t_{j,n}|\le T_n}\mathbb{1}_{[-a_n,a_n]}(\lambda).$$

Therefore,

$$
\begin{aligned}
&a_n \int_{-a_n}^{a_n} \left| J^s_{n,X}(\lambda) - R_n(\lambda) \right|^2 d\lambda \\
&\le 2a_n\delta_n^2 \int_{-a_n}^{a_n} R_n(\lambda)^2 d\lambda + 54a_n(1+\delta_n^{-1})^2 \int_{-a_n}^{a_n} \big(R_{6,n}(\lambda)^2 + R_{7,n}(\lambda)^2 + R_{8,n}(\lambda)^2\big) d\lambda.
\end{aligned}
$$

As in Step 3c, noting that $\|h_{n,m}(\cdot,\lambda)\|_\infty \le 2N_n\omega_f(\Delta_n)$ and applying Lemma 7, we get

$$
\begin{aligned}
\int_{-a_n}^{a_n} R_{6,n}(\lambda)^2 d\lambda &= O_P(a_n N_n^4 \omega_f(\Delta_n)^4 (n\Delta_n)^{4/\alpha}) \\
&= O_P(a_n\omega_f(\Delta_n)^4 T_n^4 n^{4/\alpha}\Delta_n^{4/\alpha-4}),\ n\to\infty.
\end{aligned}
$$

To estimate $R_{7,n}(\lambda)$, define

$$
g_{n,m}(s,\lambda) = \sum_{j=1}^n f(t_{j,n}-s) e^{it_{j,n}\nu_n(m,\lambda)} \mathbb{1}_{|s-t_{j,n}|\in[T_n,n\Delta_n]} \mathbb{1}_{[-a_n,a_n]}(\lambda),
$$

$$
G_{n,m}(\lambda) = \int_{\mathbb{R}} g_{n,m}(s,\lambda)\Lambda(ds)
$$

so that $R_{7,n}(\lambda) = \sum_{|m|\le m_n} W_n(m)\,|G_{n,m}(\lambda)|^2$. As before, we can assume that

$$
G_{n,m}(\lambda) = C_\alpha^{1/\alpha}(3n\Delta_n)^{1/\alpha} \sum_{k=1}^\infty \Gamma_k^{-1/\alpha} g_{n,m}(\xi_k,\lambda)\zeta_k, \quad t\in[0,n\Delta_n],
$$

where $\{\Gamma_k, k\ge1\}$ and $\{\zeta_k, k\ge 1\}$ are as in Lemma 1, $\{\xi_k, k\ge1\}$ are iid uniformly distributed over $[-n\Delta_n, 2n\Delta_n]$. Then, similarly to the proof of Lemma 7,

$$
\begin{aligned}
&\mathsf{E}\left[\int_{\mathbb{R}} R_{7,n}(\lambda)^2 d\lambda \,\middle|\, \Gamma,\xi \right] \\
&\le C_\alpha^{4/\alpha}(3n\Delta_n)^{4/\alpha} \int_{\mathbb{R}} \sum_{|m|\le m_n} W_n(m)\mathsf{E}\left[\left| \sum_{k=1}^\infty \Gamma_k^{-1/\alpha} g_{n,m}(\xi_k,\lambda)\zeta_k \right|^4 \,\middle|\, \Gamma,\xi \right] d\lambda \\
&\le C(n\Delta_n)^{4/\alpha} \int_{\mathbb{R}} \sum_{|m|\le m_n} W_n(m)\mathsf{E}\left[\left| \sum_{k=1}^\infty \Gamma_k^{-1/\alpha} g_{n,m}(\xi_k,\lambda)\zeta_k \right|^2 \,\middle|\, \Gamma,\xi \right]^2 d\lambda \\
&= C(n\Delta_n)^{4/\alpha} \int_{\mathbb{R}} \sum_{|m|\le m_n} W_n(m)\Big(\sum_{k=1}^\infty \Gamma_k^{-2/\alpha} |g_{n,m}(\xi_k,\lambda)|^2 \Big)^2 d\lambda.
\end{aligned}
$$

Now estimate

$$
\begin{aligned}
|g_{n,m}(s,\lambda)| &\le \sum_{j=1}^n |f(t_{j,n}-s)|\,\mathbb{1}_{|s-t_{j,n}|\ge T_n} \le C \sum_{j:|s-t_{j,n}|\ge T_n} |s-t_{j,n}|^{-a} \\
&\le C\Delta_n^{-a} \sum_{k:|k|\ge N_n} |k|^{-a} \le C\Delta_n^{-a} N_n^{1-a} = C\Delta_n^{-1} T_n^{1-a}.
\end{aligned}
$$

Since $g_{n,m}(s,\cdot)$ vanishes outside $[-a_n, a_n]$, we get

$$\begin{aligned} &\mathsf{E}\Big[\int_{\mathbb{R}} R_{7,n}(\lambda)^2 d\lambda \,\Big|\, \Gamma\Big] \\ &\le C(n\Delta_n)^{4/\alpha} \int_{\mathbb{R}} \sum_{|m|\le m_n} W_n(m)\Big(\sum_{k=1}^{\infty} \Gamma_k^{-2/\alpha}(\Delta_n^{-1}T_n^{1-a})^2 \mathbb{1}_{[-a_n,a_n]}(\lambda)\Big)^2 d\lambda \\ &= C a_n T_n^{4-4a} n^{4/\alpha} \Delta_n^{4/\alpha-4}\Big(\sum_{k=1}^{\infty}\Gamma_k^{-2/\alpha}\Big)^2, \end{aligned}$$

whence, as usual, $\int_{\mathbb{R}} R_{7,n}(\lambda)^2 d\lambda = O_P(a_n T_n^{4-4a} n^{4/\alpha}\Delta_n^{4/\alpha-4})$, $n\to\infty$.

To estimate $R_{8,n}(\lambda)$, define

$$z_{n,m}(s,\lambda) = \sum_{j=1}^{n} f(t_{j,n}-s) e^{it_{j,n}\nu_n(m,\lambda)} \mathbb{1}_{|s-t_{j,n}|>n\Delta_n} \mathbb{1}_{[-a_n,a_n]}(\lambda),$$
$$Z_{n,m}(\lambda) = \int_{\mathbb{R}} z_{n,m}(s,\lambda)\Lambda(ds).$$

and let φ be as in (30). As before, we can assume that

$$Z_{n,m}(\lambda) = C_\alpha^{1/\alpha}\sum_{k=1}^{\infty}\Gamma_k^{-1/\alpha} z_{n,m}(\xi_k,\lambda)\varphi(\xi_k)^{-1/\alpha}\zeta_k, \quad t\ge 0,$$

where $\{\Gamma_k, k\ge 1\}$ and $\{\zeta_k, k\ge 1\}$ are as in Lemma 1, $\{\xi_k, k\ge 1\}$ are iid with density φ.

As in the proof of Lemma 7,

$$\begin{aligned} &\mathsf{E}\Big[\int_{\mathbb{R}} R_{8,n}(\lambda)^2 d\lambda \,\Big|\, \Gamma,\xi\Big] \\ &\le C_\alpha^{4/\alpha}\int_{\mathbb{R}}\sum_{|m|\le m_n} W_n(m)\mathsf{E}\left[\left|\sum_{k=1}^{\infty}\Gamma_k^{-1/\alpha} z_{n,m}(\xi_k,\lambda)\varphi(\xi_k)^{-1/\alpha}\zeta_k\right|^4 \,\Big|\, \Gamma,\xi\right] d\lambda \\ &\le C\int_{\mathbb{R}}\sum_{|m|\le m_n} W_n(m)\mathsf{E}\left[\left|\sum_{k=1}^{\infty}\Gamma_k^{-1/\alpha} z_{n,m}(\xi_k,\lambda)\varphi(\xi_k)^{-1/\alpha}\zeta_k\right|^2 \,\Big|\, \Gamma,\xi\right]^2 d\lambda \\ &= C\int_{\mathbb{R}}\sum_{|m|\le m_n} W_n(m)\Big(\sum_{k=1}^{\infty}\Gamma_k^{-2/\alpha}\left|z_{n,m}(\xi_k,\lambda)\right|^2\varphi(\xi_k)^{-2/\alpha}\Big)^2 d\lambda \\ &= C\int_{\mathbb{R}}\sum_{|m|\le m_n} W_n(m)\Big(\sum_{k=1}^{\infty}\Gamma_k^{-4/\alpha}\left|z_{n,m}(\xi_k,\lambda)\right|^4\varphi(\xi_k)^{-4/\alpha} \\ &+ \sum_{\substack{k_1,k_2=1\\ k_1\ne k_2}}^{\infty}\Gamma_{k_1}^{-2/\alpha}\Gamma_{k_2}^{-2/\alpha}\left|z_{n,m}(\xi_{k_1},\lambda)\right|^2\varphi(\xi_{k_1})^{-2/\alpha}\left|z_{n,m}(\xi_{k_2},\lambda)\right|^2\varphi(\xi_{k_2})^{-2/\alpha}\Big) d\lambda. \end{aligned}$$

It follows that

$$\mathsf{E}\Big[\int_{\mathbb{R}} R_{8,n}(\lambda)^2 d\lambda \,\Big|\, \Gamma\Big] \le C\int_{\mathbb{R}}\sum_{|m|\le m_n} W_n(m)\Big(\sum_{k=1}^{\infty}\Gamma_k^{-4/\alpha}\mathsf{E}\left[\left|z_{n,m}(\xi_1,\lambda)\right|^4\varphi(\xi_1)^{-4/\alpha}\right]$$

$$+ \sum_{\substack{k_1,k_2=1\\k_1\neq k_2}}^{\infty} \Gamma_{k_1}^{-2/\alpha}\Gamma_{k_2}^{-2/\alpha}\mathsf{E}\left[\, |z_{n,m}(\xi_{k_1},\lambda)|^2\,\varphi(\xi_{k_1})^{-2/\alpha}\,|z_{n,m}(\xi_{k_2},\lambda)|^2\,\varphi(\xi_{k_2})^{-2/\alpha}\,\right]\Bigg)\,d\lambda$$

$$\leq C\int_{\mathbb{R}}\sum_{|m|\leq m_n} W_n(m)\Bigg(\sum_{k=1}^{\infty}\Gamma_k^{-4/\alpha}\mathsf{E}\left[\,|z_{n,m}(\xi_1,\lambda)|^4\,\varphi(\xi_1)^{-4/\alpha}\,\right]$$

$$+\Bigg(\sum_{k=1}^{\infty}\Gamma_k^{-2/\alpha}\Bigg)^2\mathsf{E}\left[\,|z_{n,m}(\xi_1,\lambda)|^2\,\varphi(\xi_1)^{-2/\alpha}\,\right]^2\Bigg)\,d\lambda.$$

Similarly to (31) and (32),

$$\mathsf{E}\left[\,|z_{n,m}(\xi_1,\lambda)|^2\,\varphi(\xi_1)^{-2/\alpha}\,\right]\leq C\int_{\mathbb{R}}\Bigg(\sum_{j=1}^{n}|x-t_{j,n}|^{-a}\mathbb{1}_{|x-t_{j,n}|>n\Delta_n}\Bigg)^2\varphi(x)^{1-2/\alpha}dx$$

$$\sim C\Delta_n^{-2}\int_{\mathbb{R}}\Bigg(\int_0^{n\Delta_n}|t-x|^{-a}\,\mathbb{1}_{|t-x|>n\Delta_n}dt\Bigg)^2\varphi(x)^{1-2/\alpha}dx$$

$$\leq C\Delta_n^{-2}(n\Delta_n)\int_{\mathbb{R}}\int_0^{n\Delta_n}|t-x|^{-2a}\,\mathbb{1}_{|t-x|>n\Delta_n}dt\,\varphi(x)^{1-2/\alpha}dx$$

$$\leq C\Delta_n^{-2}\big(n\Delta_n\big)^{2/\alpha+2-2a}\log^r(n\Delta_n),$$

$$\mathsf{E}\left[\,|z_{n,m}(\xi_1,\lambda)|^4\,\varphi(\xi_1)^{-4/\alpha}\,\right]\leq C\int_{\mathbb{R}}\Bigg(\sum_{j=1}^{n}|x-t_{j,n}|^{-a}\mathbb{1}_{|x-t_{j,n}|>n\Delta_n}\Bigg)^4\varphi(x)^{1-4/\alpha}dx$$

$$\sim C\Delta_n^{-4}\int_{\mathbb{R}}\Bigg(\int_0^{n\Delta_n}|t-x|^{-a}\,\mathbb{1}_{|t-x|>n\Delta_n}dt\Bigg)^4\varphi(x)^{1-4/\alpha}dx$$

$$\leq C\Delta_n^{-4}(n\Delta_n)^3\int_{\mathbb{R}}\int_0^{n\Delta_n}|t-x|^{-4a}\,\mathbb{1}_{|t-x|>n\Delta_n}dt\,\varphi(x)^{1-4/\alpha}dx$$

$$\leq C\Delta_n^{-4}\big(n\Delta_n\big)^{4/\alpha+4-4a}\log^s(n\Delta_n)$$

where $r=(2/\alpha-1)(1+b)$, $s=(4/\alpha-1)(1+b)$. Therefore,

$$\mathsf{E}\left[\int_{\mathbb{R}}R_{8,n}(\lambda)^2d\lambda\,\middle|\,\Gamma\right]\leq Ca_n\Delta_n^{-4}\big(n\Delta_n\big)^{4/\alpha+4-4a}\log^s(n\Delta_n),$$

whence $\int_{\mathbb{R}}R_{8,n}(\lambda)^2d\lambda=O_P(a_n\Delta_n^{-4}\big(n\Delta_n\big)^{4/\alpha+4-4a}\log^s(n\Delta_n))$, $n\to\infty$.

Collecting the estimates and setting $\delta_n=\omega_f(\Delta_n)T_n+T_n^{1-a}+(n\Delta_n)^{1-a}\log^{s/4}(n\Delta_n)$, we arrive at

$$a_n\Delta_n^4\int_{-a_n}^{a_n}\left|J^s_{n,X}(\lambda)-R_n(\lambda)\right|^2d\lambda$$

$$=O_P\Big(a_n^2\big(\omega_f(\Delta_n)^2T_n^2+T_n^{2-2a}+(n\Delta_n)^{2-2a}\log^{s/2}(n\Delta_n)\big)(n\Delta_n)^{4/\alpha}\Big)$$

$$=O_P\Big(a_n^2(\omega_f(\Delta_n)^{2-2/a}+(n\Delta_n)^{2-2a}\log^{s/2}(n\Delta_n))E_n^2\Big)=o_P(E_n^2),\quad n\to\infty,$$

since combining (F2') and (F4') we obtain

$$a_n^{1+3(a-1)/(4\alpha)}(n\Delta_n)^{1-a}\to 0,\quad n\to\infty.$$

Hence we conclude exactly as in the proof of Theorem 1. □

Appendix B: Auxiliary statements

Lemma 1. *Let $(E, \mathcal{E}, \nu)$ be a σ-finite measure space, Λ be an independently scattered SαS random measure on E with the control measure ν, and $\{f_t, t \in \mathbf{T}\} \subset L^\alpha(E, \mathcal{E}, \nu)$ be a family of functions indexed by some parameter set $\mathbf{T}$, φ be a positive probability density on E. Then*

$$X_t = \int_E f_t(x)\Lambda(dx), \quad t \in \mathbf{T},$$

has the same finite-dimensional distributions as the almost-surely convergent series

$$X'_t = C_\alpha^{1/\alpha} \sum_{k=1}^{\infty} \Gamma_k^{-1/\alpha} \varphi(\xi_k)^{-1/\alpha} f_t(\xi_k)\zeta_k, \quad t \in \mathbf{T},$$

where $\{\zeta_k, k \geq 1\}$ are iid standard Gaussian random variables, $\{\xi_k, k \geq 1\}$ are iid random elements of E with density φ, $\Gamma_k = \eta_1 + \cdots + \eta_k$, $\{\eta_k, k \geq 1\}$ are iid $\mathrm{Exp}(1)$–distributed random variables, and these three sequences are independent;

$$C_\alpha = \left(\mathsf{E}\left[\,|g_1|^\alpha\,\right] \int_0^\infty x^{-\alpha} \sin x \, dx\right)^{-1} = \begin{cases} \frac{(1-\alpha)\sqrt{\pi}}{2^{\alpha/2}\Gamma((\alpha+1)/2)\Gamma(2-\alpha)\cos(\pi\alpha/2)}, & \alpha \neq 1, \\ \sqrt{2/\pi}, & \alpha = 1. \end{cases}$$

Proof. The statement follows from [15, Section 3.11] by noting that

$$X_t = \int_E f_t(x)\varphi(x)^{-1/\alpha} M(dx),$$

where M is an independently scattered SαS random measure on E defined by

$$M(A) = \int_A \varphi^{1/\alpha}(x)\Lambda(x), \quad A \in \mathcal{E},$$

so that the control measure of M has ν-density φ. □

Lemma 2. *Let, for each $n \geq 1$, $\{\varepsilon_{m,n}, m = 1, \ldots, n\}$ be iid SαS random variables with scale parameter σ_n. Let also $\{a_{j,l,n},\ 1 \leq j < l \leq n\}$ be a collection of measurable functions $a_{j,l,n} \colon \mathbb{R} \to \mathbb{C}$ such that*

$$A_n = \int_{\mathbb{R}} \sum_{1 \leq j < l \leq n} |a_{j,l,n}(\lambda)|^2 \, d\lambda < \infty.$$

Then

$$\int_{\mathbb{R}} \left| \sum_{1 \leq j < l \leq n} a_{j,l,n}(\lambda)\varepsilon_{j,n}\varepsilon_{l,n} \right|^2 d\lambda = O_P(A_n \sigma_n^4 n^{4/\alpha - 2}), \quad n \to \infty.$$

Proof. W.l.o.g. we can assume that $\sigma_n = 1$. We shall use the LePage series representation. Clearly, for each $n \geq 1$, the variables $\{\varepsilon_{m,n}, m = 1, \ldots, n\}$ have the same joint distribution as $\{\Lambda([m-1, m]), m = 1, \ldots, n\}$, where Λ is an independently scattered SαS random measure on $[0, n]$ with the Lebesgues control measure. By Lemma 1, this distribution coincides with that of

$$\tilde{\varepsilon}_{m,n} = n^{1/\alpha} C_\alpha^{1/\alpha} \sum_{k=1}^{\infty} \Gamma_k^{-1/\alpha} \mathbb{1}_{[m-1,m]}(\xi_k)\zeta_k, \quad m = 1, \ldots, n,$$

where $\{\Gamma_k, k \geq 1\}$ and $\{\zeta_k, k \geq 1\}$ are as in Lemma 1, $\{\xi_k, k \geq 1\}$ are iid $U([0,n])$. Since the boundedness in probability relies only on marginal distributions (for fixed n), we can assume that $\varepsilon_{m,n} = \tilde{\varepsilon}_{m,n}$. Let $\Xi_n(\lambda) = \sum_{1 \leq j < l \leq n} a_{j,l,n}(\lambda)\varepsilon_{j,n}\varepsilon_{l,n}$. A generic term in the expansion of $|\Xi_n(\lambda)|^2$ has, up to a non-random constant, the form

$$\Gamma_{k_1}^{-1/\alpha}\Gamma_{k_1'}^{-1/\alpha}\Gamma_{k_2}^{-1/\alpha}\Gamma_{k_2'}^{-1/\alpha}\mathbb{1}_{[j_1-1,j_1]}(\xi_{k_1})\mathbb{1}_{[l_1-1,l_1]}(\xi_{k_1'})\mathbb{1}_{[j_2-1,j_2]}(\xi_{k_2})\mathbb{1}_{[l_2-1,l_2]}(\xi_{k_2'})\zeta_{k_1}\zeta_{k_1'}\zeta_{k_2}\zeta_{k_2'}.$$

Recall that $\{\zeta_k, k \geq 1\}$ are independent and centered. Then, given Γ's and ξ's, such term has a non-zero expectation only if $k_1 = k_2$, $k_1' = k_2'$ or $k_1 = k_2'$, $k_2 = k_1'$ (for $k_1 = k_1'$ it is zero since $j_1 \neq l_1$), so we must also have $j_1 = j_2$, $l_1 = l_2$ or $j_1 = l_2$, $j_2 = l_1$ respectively so that the product of indicators is not zero. The latter, however, is impossible, since $j_1 < l_1$ and $j_2 < l_2$. Consequently, the Lemma of Fatou implies

$$\begin{aligned}
\mathsf{E}\left[\,|\Xi_n(\lambda)|^2 \mid \Gamma\,\right] &\leq C_\alpha^{4/\alpha} n^{4/\alpha} \sum_{k \neq k'}^{\infty} \Gamma_k^{-2/\alpha}\Gamma_{k'}^{-2/\alpha} \sum_{1 \leq j < l \leq n}^{n} |a_{j,l,n}(\lambda)|^2\, \mathsf{E}\left[\,\mathbb{1}_{[j-1,j]}(\xi_k)\mathbb{1}_{[l-1,l]}(\xi_{k'})\,\right] \\
&= C_\alpha^{4/\alpha} n^{4/\alpha} \sum_{k \neq k'}^{\infty} \Gamma_k^{-2/\alpha}\Gamma_{k'}^{-2/\alpha} \sum_{1 \leq j < l \leq n}^{n} |a_{j,l,n}(\lambda)|^2\, P\left(\xi_k \in [j-1,j]\right) P\left(\xi_{k'} \in [l-1,l]\right) \\
&\leq C_\alpha^{4/\alpha} n^{4/\alpha - 2} \sum_{1 \leq j < l \leq n} |a_{j,l,n}(\lambda)|^2 \left(\sum_{k=1}^{\infty} \Gamma_k^{-2/\alpha}\right)^2.
\end{aligned}$$

Integrating over λ, we get

$$\mathsf{E}\left[\int_{\mathbb{R}} |\Xi_n(\lambda)|^2 d\lambda \,\Big|\, \Gamma\right] \leq C_\alpha^{4/\alpha} n^{4/\alpha-2} A_n \left(\sum_{k=1}^{\infty} \Gamma_k^{-2/\alpha}\right)^2.$$

By the strong law of large numbers, $\Gamma_k \sim k$, $k \to \infty$, a.s. Therefore, given Γ's, $\int_{\mathbb{R}} |\Xi_n(\lambda)|^2 d\lambda = O_P(A_n n^{4/\alpha-2})$, $n \to \infty$, whence the required statement follows. □

The following lemma is an immediate corollary of the proof of Lemma 2.

Lemma 3. *Let, for each $n \geq 1$, $\{\varepsilon_{m,n}, m = 1, \ldots, n\}$ be iid $S\alpha S$ random variables with scale parameter σ_n. Let also $\{b_{j,l,n},\ 1 \leq j < l \leq n\}$ be a set of complex numbers with*

$$B_n = \sum_{1 \leq j < l \leq n} |b_{j,l,n}|^2.$$

Then

$$\sum_{1 \leq j < l \leq n} b_{j,l,n}\varepsilon_{j,n}\varepsilon_{l,n} = O_P(B_n^{1/2}\sigma_n^2 n^{2/\alpha-1}), \quad n \to \infty.$$

In the following lemmas $\{\Delta_n, n \geq 1\}$ is some vanishing sequence, $\{N_n, n \geq 1\}$ is a sequence of positive integers such that $N_n \to \infty$, $n \to \infty$, and $N_n = o(n)$, $n \to \infty$. We denote $t_{k,n} = k\Delta_n$, $k \in \mathbb{Z}$, $T_n = N_n\Delta_n$, $n \geq 1$.

Lemma 4. *Let $\{h_n, n \geq 1\}$ be a sequence of bounded functions supported by $[-T_n, T_n]$ and $Y_{t,n} = \int_{\mathbb{R}} h_n(t-s)\Lambda(ds)$, $t \in \mathbb{R}$. Then*

$$\sum_{j=1}^{n} Y_{t_{j,n},n}^2 = O_P\left(\|h_n\|_\infty^2 T_n n^{2/\alpha}\Delta_n^{2/\alpha-1}\right), \quad n \to \infty.$$

Proof. We can assume that $\|h_n\|_\infty = 1$. As in Lemma 2, we also use the LePage representation, so small details will be omitted. Namely, for each $n \geq 1$, the process $\{Y_{t,n}, t \in [0, n\Delta_n]\}$ has the same distribution as

$$\tilde{Y}_{t,n} = C_\alpha^{1/\alpha}(n\Delta_n + 2T_n)^{1/\alpha} \sum_{k=1}^{\infty} \Gamma_k^{-1/\alpha} h_n(t - \xi_k)\zeta_k, \quad t \in [0, n\Delta_n],$$

where $\{\Gamma_k, k \geq 1\}$ and $\{\zeta_k, k \geq 1\}$ are as in Lemma 1, $\{\xi_k, k \geq 1\}$ are iid $U([-T_n, n\Delta_n + T_n])$. We can assume that $Y_{t,n} = \tilde{Y}_{t,n}$. Then

$$\begin{aligned} \mathsf{E}\left[\sum_{j=1}^{n} Y_{t_{j,n},n}^2 \,\middle|\, \Gamma\right] &\leq C_\alpha^{2/\alpha}(n\Delta_n + 2T_n)^{2/\alpha} \sum_{j=1}^{n}\sum_{k=1}^{\infty} \Gamma_k^{-2/\alpha} P(\xi_k \in [t_{j,n} - T_n, t_{j,n} + T_n]) \\ &\leq 2C_\alpha^{2/\alpha} nT_n(n\Delta_n + 2T_n)^{2/\alpha - 1} \sum_{k=1}^{\infty} \Gamma_k^{-2/\alpha}. \end{aligned}$$

It follows that, given Γ, $\sum_{j=1}^{n} Y_{t_{j,n},n}^2 = O_P(T_n n^{2/\alpha}\Delta_n^{2/\alpha - 1})$, $n \to \infty$, which yields the statement. $\square$

Lemma 5. *Let a bounded uniformly continuous function $f\colon \mathbb{R} \to \mathbb{R}$ satisfy* (F3′) *and let m_n, $W_n(m)$ and $\nu_n(m, \lambda)$ be as defined in Section 2 fulfilling* (W1), (W2) *and* (W4). *If the support of f is bounded, let it be contained in $[-T, T]$ and put $T_n := T$. If it is unbounded, then choose a sequence $(T_n)_{n\in\mathbb{N}}$ with $T_n \to \infty$ and $T_n\omega_f(\Delta_n) \to 0$ as $n \to \infty$. W.l.o.g. $N_n := T_n/\Delta_n$ is a sequence of integers. Put*

$$F_n(\lambda) = \sum_{|m| \leq m_n} W_n(m) \left| \sum_{k=-N_n}^{N_n - 1} f(t_{k,n}) e^{it_{k,n}\nu_n(m,\lambda)} \right|^2.$$

Then

$$\left| \left|\hat{f}(\lambda)\right|^2 - \Delta_n^2 F_n(\lambda) \right| = O\Big(\big(W_n^{(2)}\big)^{1/2} (n\Delta_n)^{-1} + T_n\omega_f(\Delta_n) + T_n^{1-a} + |\lambda|\, \Delta_n \Big), \quad n \to \infty.$$

If f is supported by $[-T_n, T_n]$, then

$$\left| \left|\hat{f}(\lambda)\right|^2 - \Delta_n^2 F_n(\lambda) \right| = O\Big(\big(W_n^{(2)}\big)^{1/2} (n\Delta_n)^{-1} + \omega_f(\Delta_n) + |\lambda|\, \Delta_n \Big), \quad n \to \infty.$$

Proof. Start by studying the expression

$$F_{1,n}(\lambda) = \sum_{|m| \leq m_n} W_n(m) \left| \hat{f}(\nu_n(m, \lambda)) \right|^2.$$

Using (21), it can be shown that for any $\delta > 0$

$$\left| \left|\hat{f}(\lambda)\right|^2 - F_{1,n}(\lambda) \right| \leq \delta \left|\hat{f}(\lambda)\right|^2 + (1 + \delta^{-1}) \sum_{|m| \leq m_n} W_n(m) \Big(\left|\hat{f}(\lambda)\right| - \left|\hat{f}(\nu_n(m, \lambda))\right| \Big)^2.$$

By (F3$'$) $\hat{f}'$ is bounded since obviously $\widehat{f(t)}' = -i\widehat{tf(t)}$ and $tf(t)$ is integrable if $a > 2$. So

$$\sum_{|m|\le m_n} W_n(m)\Big(\left|\hat{f}(\lambda)\right| - \left|\hat{f}(\nu_n(m,\lambda))\right|\Big)^2 \le \|\hat{f}'\|_\infty^2 \sum_{|m|\le m_n} W_n(m)\big(\lambda - \nu_n(m,\lambda))\big)^2$$

$$\le \frac{\|\hat{f}'\|_\infty^2}{(n\Delta_n)^2} \sum_{|m|\le m_n} m^2 W_n(m) = \frac{\|\hat{f}'\|_\infty^2 W_n^{(2)}}{(n\Delta_n)^2}.$$

Setting $\delta = \big(W_n^{(2)}\big)^{1/2}(n\Delta_n)^{-1}$, we get

$$\left|\left|\hat{f}(\lambda)\right|^2 - F_{1,n}(\lambda)\right| = O\big(\big(W_n^{(2)}\big)^{1/2}(n\Delta_n)^{-1}\big), \quad n\to\infty.$$

Further, denote $F_{2,n}(\lambda) = \sum_{|m|\le m_n} W_n(m)\left|\widehat{f_n}(\nu_n(m,\lambda))\right|^2$ and write for some $\theta > 0$, using (21),

$$|F_{1,n}(\lambda) - F_{2,n}(\lambda)| \le \sum_{|m|\le m_n} W_n(m)\big|\hat{f}(\nu_n(m,\lambda))^2 - \hat{f}_n(\nu_n(m,\lambda))^2\big|$$

$$\le \theta F_{1,n}(\lambda) + (1+\theta^{-1})2\sum_{|m|\le m_n} W_n(m)\left|\int_{-T_n}^{T_n} \big(f(s) - f_n(s)\big)e^{is\nu_n(m,\lambda)}ds\right|^2$$

$$+(1+\theta^{-1})2\sum_{|m|\le m_n} W_n(m)\left|\int_{\{s:|s|>T_n\}} f(s)e^{is\nu_n(m,\lambda)}ds\right|^2$$

$$\le \theta F_{1,n}(\lambda) + 8T_n^2(1+\theta^{-1})\omega_f(\Delta_n)^2 + C(1+\theta^{-1})\left(\int_{\{s:|s|>T_n\}} |s|^{-a}ds\right)^2$$

$$\le \theta F_{1,n}(\lambda) + C(1+\theta^{-1})\big(T_n^2\omega_f(\Delta_n)^2 + T_n^{2-2a}\big).$$

Setting $\theta = T_n\omega_f(\Delta_n) + T_n^{1-a}$, we get

$$|F_{1,n}(\lambda) - F_{2,n}(\lambda)| = O(T_n\omega_f(\Delta_n) + T_n^{1-a}), \quad n\to\infty.$$

Finally, for $\kappa > 0$

$$\big|F_{2,n}(\lambda) - \Delta_n^2 F_n(\lambda)\big|$$

$$\le \kappa F_{2,n}(\lambda) + (1+\kappa^{-1})\sum_{|m|\le m_n} W_n(m)\left|\sum_{k=-N_n}^{N_n-1}\int_{t_{k,n}}^{t_{k+1,n}} f(t_{k,n})\big(e^{is\nu_n(m,\lambda)} - e^{it_{k,n}\nu_n(m,\lambda)}\big)ds\right|^2$$

$$\le \kappa F_{2,n}(\lambda) + (1+\kappa^{-1})\sum_{|m|\le m_n} W_n(m)\left(\sum_{k=-N_n}^{N_n-1}\int_{t_{k,n}}^{t_{k+1,n}} |f(t_{k,n})|\,\Delta_n\nu_n(m,\lambda)ds\right)^2$$

$$\le \kappa F_{2,n}(\lambda) + C(1+\kappa^{-1})\sum_{|m|\le m_n} W_n(m)\|f\|_1^2\big(\Delta_n\nu_n(m,\lambda)\big)^2$$

$$\le \kappa F_{2,n}(\lambda) + C(1+\kappa^{-1})\|f\|_1^2\Delta_n^2\sum_{|m|\le m_n} W_n(m)\left(\lambda^2 + \frac{m^2}{(n\Delta_n)^2}\right)$$

$$= \kappa F_{2,n}(\lambda) + C(1+\kappa^{-1})\|f\|_1^2\Delta_n^2\left(\lambda^2 + \frac{W_n^{(2)}}{(n\Delta_n)^2}\right).$$

Taking $\kappa = \Delta_n\big(|\lambda| + \big(W_n^{(2)}\big)^{1/2}(n\Delta_n)^{-1}\big)$, we obtain

$$\big|F_{2,n}(\lambda) - \Delta_n^2 F_n(\lambda)\big| = O\left(\Delta_n\big(|\lambda| + \big(W_n^{(2)}\big)^{1/2}(n\Delta_n)^{-1}\big)\right), \quad n \to \infty.$$

Combining the estimates, we arrive at

$$\left|\left|\hat{f}(\lambda)\right|^2 - \Delta_n^2 F_n(\lambda)\right| = O\Big(\big(W_n^{(2)}\big)^{1/2}(n\Delta_n)^{-1} + T_n\omega_f(\Delta_n) + T_n^{1-a} + |\lambda|\,\Delta_n\Big), \quad n \to \infty,$$

as required. The second statement follows easily, since in this case

$$\int_{\{s:|s|>T_n\}} f(s)e^{is\nu_n(m,\lambda)}ds = 0. \qquad \square$$

Lemma 6. *Let $\{m_n, n \ge 1\}$ be a sequence of positive integers such that $m_n \to \infty$, $m_n = o(n)$, $n \to \infty$, $\{W_n(m), n \ge 1, m = -m_n, \dots, m_n\}$ be a sequence of filters satisfying* (W1)–(W2)*, and $\{K_n(m), n \ge 1, m = -m_n, \dots, m_n\}$ be a sequence of real numbers. Then*

$$S_n = \sum_{j_1,j_2=1}^{n} \left|\sum_{|m|\le m_n} W_n(m)K_n(m)e^{i(j_1-j_2)m/n}\right|^2 = O(W_n^*(K_n^* n)^2), \ n \to \infty$$

with $W_n^* = \max_{|m|\le m_n} W_n(m)$, $K_n^* = \max_{|m|\le m_n}|K_n(m)|$.

Proof. Write

$$\begin{aligned}
S_n &= \sum_{j_1,j_2=1}^{n}\sum_{m,m'=-m_n}^{m_n} W_n(m)K_n(m)W_n(m')K_n(m')e^{i(j_1-j_2)(m-m')/n} \\
&= \sum_{m,m'=-m_n}^{m_n} W_n(m)K_n(m)W_n(m')K_n(m')\left|\sum_{j=1}^{n} e^{i\,j(m-m')/n}\right|^2 \\
&= n^2\sum_{|m|\le m_n} W_n(m)^2K_n(m)^2 + 2\sum_{m<m'} W_n(m)K_n(m)W_n(m')K_n(m')\left|\frac{e^{i(m-m')}-1}{e^{i(m-m')/n}-1}\right|^2 \\
&\le W_n^*(K_n^*)^2\left(n^2\sum_{|m|\le m_n} W_n(m) + 2\sum_{-m_n\le m<m'\le m_n} W_n(m)\left|\frac{e^{i(m-m')}-1}{e^{i(m-m')/n}-1}\right|^2\right) \\
&\le W_n^*(K_n^*)^2\left(n^2 + 8\sum_{|m|\le m_n} W_n(m)\sum_{k=1}^{2m_n}\frac{1}{|e^{ik/n}-1|^2}\right).
\end{aligned}$$

Note that for $x \in [0,1]$, $|e^{ix}-1| \ge L\,|x|$ with some positive constant L. Since $m_n = o(n)$, $n \to \infty$, for all n large enough it holds $m_n \le n/2$. Therefore, $\left|e^{ik/n}-1\right| \ge L\,|k/n|$ for $k = 1, \dots, 2m_n$, whence

$$S_n \le W_n^*(K_n^*)^2\left(n^2 + 8L^{-2}n^2\sum_{|m|\le m_n} W_n(m)\sum_{k=1}^{2m_n}\frac{1}{k^2}\right) \le W_n^*(K_n^*)^2 n^2\left(1 + 8L^{-2}\sum_{k=1}^{\infty}\frac{1}{k^2}\right),$$

as required. $\square$

Lemma 7. *Let $\{m_n, n \geq 1\}$ be a sequence of positive integers such that $m_n \to \infty$ as $n \to \infty$. For a number sequence $\{W_n(m), n \geq 1, m = -m_n, \dots, m_n\}$ satisfying* (W1)–(W2) *and continuous functions $h_{n,m}\colon [-T_n, n\Delta_n+T_n]\times\mathbb{R} \to \mathbb{C}$, $n \geq 1, m = -m_n, \dots, m_n$, define $R_n(\lambda) = \sum_{|m|\leq m_n} W_n(m) \left|\int_{-T_n}^{n\Delta_n+T_n} h_{n,m}(t,\lambda)\Lambda(dt)\right|^2$. Then*

$$\int_{\mathbb{R}} R_n(\lambda)^2 d\lambda = O_P\big(H_n^*(n\Delta_n)^{4/\alpha}\big), \quad n \to \infty,$$

where $H_n^* = \int_{\mathbb{R}} H(\lambda)\, d\lambda$ for $H(\lambda) = \sum_{|m|\leq m_n} W_n(m)\|h_{n,m}(\cdot,\lambda)\|_\infty^4$.

Proof. By Lemma 1, for each $n \geq 1$ the family

$$H_{n,m}(\lambda) = \int_{-T_n}^{n\Delta_n+T_n} h_{n,m}(t,\lambda)\Lambda(dt), \quad |m| \leq m_n, \lambda \in \mathbb{R},$$

has the same distribution as

$$\tilde{H}_{n,m}(\lambda) = C_\alpha^{1/\alpha}(n\Delta_n + 2T_n)^{1/\alpha}\sum_{k=1}^{\infty}\Gamma_k^{-1/\alpha} h_{n,m}(\xi_k,\lambda)\zeta_k, \quad |m| \leq m_n, \lambda \in \mathbb{R},$$

where the variables $\Gamma_k, \xi_k, \zeta_k, k \geq 1$, are the same as in the proof of Lemma 4. Again, we can assume that $\tilde{H}_{n,m}(\lambda) = \tilde{H}_{n,m}(\lambda)$. Jensen's inequality implies $(\sum_{|m|\leq m_n} W_n(m)a_m)^2 \leq \sum_{|m|\leq m_n} W_n(m)a_m^2$. Thus we estimate

$$\begin{aligned}
&\mathsf{E}\left[\int_{\mathbb{R}} R_n(\lambda)^2 d\lambda \,\middle|\, \Gamma, \xi\right] \\
&\leq C_\alpha^{4/\alpha}(n\Delta_n + 2T_n)^{4/\alpha}\int_{\mathbb{R}}\sum_{|m|\leq m_n} W_n(m)\mathsf{E}\left[\left|\sum_{k=1}^{\infty}\Gamma_k^{-1/\alpha}h_{n,m}(\xi_k,\lambda)\zeta_k\right|^4 \,\middle|\, \Gamma,\xi\right] d\lambda \\
&\leq C(n\Delta_n)^{4/\alpha}\int_{\mathbb{R}}\sum_{|m|\leq m_n} W_n(m)\mathsf{E}\left[\left|\sum_{k=1}^{\infty}\Gamma_k^{-1/\alpha}h_{n,m}(\xi_k,\lambda)\zeta_k\right|^2 \,\middle|\, \Gamma,\xi\right]^2 d\lambda \\
&= C(n\Delta_n)^{4/\alpha}\int_{\mathbb{R}}\sum_{|m|\leq m_n} W_n(m)\left(\sum_{k=1}^{\infty}\Gamma_k^{-2/\alpha}\,|h_{n,m}(\xi_k,\lambda)|^2\right)^2 d\lambda \\
&\leq C(n\Delta_n)^{4/\alpha}\int_{\mathbb{R}}\sum_{|m|\leq m_n} W_n(m)\left(\sum_{k=1}^{\infty}\Gamma_k^{-2/\alpha}\right)^2\|h_{n,m}(\cdot,\lambda)\|_\infty^4 d\lambda,
\end{aligned}$$

for some generic constant $C > 0$ where we have used that, given Γ and ξ, the series $\sum_{k=1}^{\infty}\Gamma_k^{-1/\alpha}h_{n,m}(\xi_k,\lambda)\zeta_k$ has a centered Gaussian distribution. Therefore,

$$\begin{aligned}
&\mathsf{E}\left[\int_{\mathbb{R}} R_n(\lambda)^2 d\lambda \,\middle|\, \Gamma\right] \\
&\leq C(n\Delta_n)^{4/\alpha}\left(\sum_{k=1}^{\infty}\Gamma_k^{-2/\alpha}\right)^2\int_{\mathbb{R}}\sum_{|m|\leq m_n} W_n(m)\|h_{n,m}(\cdot,\lambda)\|_\infty^4\, d\lambda \\
&= C(n\Delta_n)^{4/\alpha}\left(\sum_{k=1}^{\infty}\Gamma_k^{-2/\alpha}\right)^2 H_n^*.
\end{aligned}$$

As a result, given Γ, $\int_{\mathbb{R}} R_n(\lambda)^2 d\lambda = O_P\big(H_n^*(n\Delta_n)^{4/\alpha}\big)$, $n \to \infty$, which implies the statement. ☐

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