Alternatives to conventional Monte Carlo

Recursive Numerical Integration
&
Symmetrized Configurations

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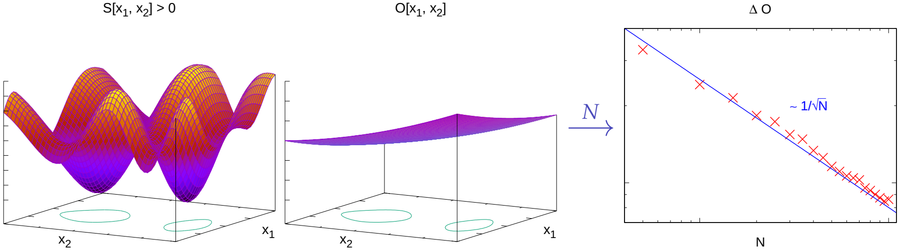
Monte Carlo importance sampling

$$\langle O \rangle = \frac{\int_{D^d} \mathrm{d}x\, O[x]\, \mathrm{e}^{-S[x]}}{\int_{D^d} \mathrm{d}x\ \mathrm{e}^{-S[x]}}, \qquad \text{prob. density} \quad p(x) = \frac{\mathrm{e}^{-S[x]}}{\int_{D^d} \mathrm{d}x\ \mathrm{e}^{-S[x]}}$$

$$N \cdot [x] \qquad \langle O \rangle = \frac{1}{N} \sum_{i=1}^{N} O_p[x]$$

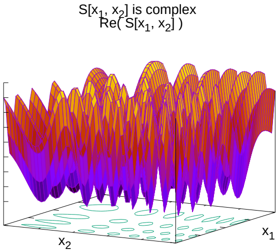


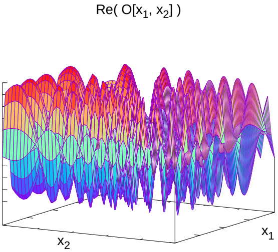
Monte Carlo importance sampling

$$\langle O \rangle = \frac{\int_{D^d} \mathrm{d}x\, O[x]\, \mathrm{e}^{-S[x]}}{\int_{D^d} \mathrm{d}x\ \mathrm{e}^{-S[x]}},$$

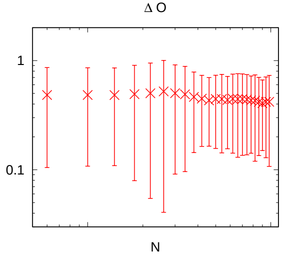
prob. density $\quad p(x) = \dfrac{\mathrm{e}^{-S[x]}}{\int_{D^d} \mathrm{d}x\ \mathrm{e}^{-S[x]}}$

$N \cdot [x] \quad \langle O \rangle = \dfrac{1}{N} \sum_{i=1}^{N} O_p[x]$





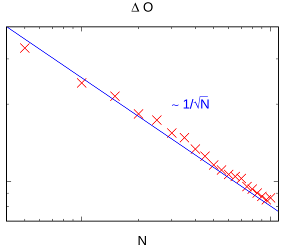
$\xrightarrow{N}$



Solutions

Take specific integration points

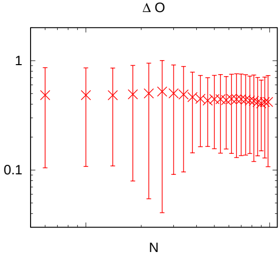
$S > 0$



$\longrightarrow$ Recursive Numerical Integration

Gauss quadrature points

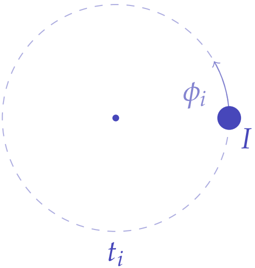
$S \in \mathbb{C}$



$\longrightarrow$ Symmetrized Configurations

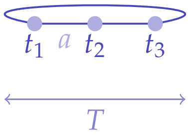
Spherical t-designs

Topological Oscillator



$$S(\phi) = \int_0^T \mathrm{d}t \, \frac{I}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 \qquad \phi \in [0, 2\pi)$$

$$\to \frac{I}{a} \sum_{i=1}^{3} \left(1 - \cos(\phi_{i+1} - \phi_i)\right)$$



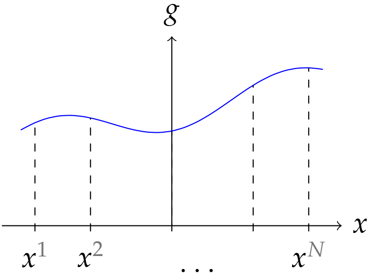
Partition function

$$Z = \int_{[0,2\pi)^3} \mathrm{d}\phi_1 \, \mathrm{d}\phi_2 \, \mathrm{d}\phi_3 \, \mathrm{e}^{-S[\phi_1, \phi_2, \phi_3]}$$

$$= \int_{[0,2\pi)^3} \mathrm{d}\phi_1 \, \mathrm{d}\phi_2 \, \mathrm{d}\phi_3 \prod_{i=1}^{3} \underbrace{\mathrm{e}^{-\frac{I}{a}(1 - \cos(\phi_{i+1} - \phi_i))}}_{f(\phi_i, \phi_{i+1})}$$

$$= \int_0^{2\pi} \mathrm{d}\phi_1 \int_0^{2\pi} \mathrm{d}\phi_2 \, f(\phi_1, \phi_2) \int_0^{2\pi} \mathrm{d}\phi_3 \, f(\phi_2, \phi_3) f(\phi_3, \phi_1)$$

Gauss quadrature points



Gauss

$$g(x) \approx P_{2N-1}$$

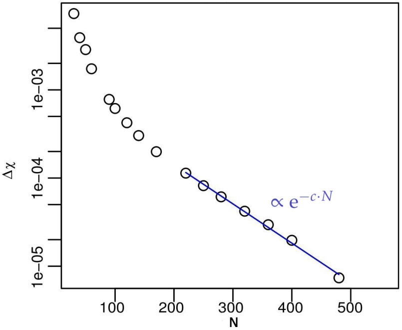
$$\int_{-1}^{1} \mathrm{d}x\, g(x) = \sum_{r=1}^{N} w^r g(x^r) + \mathcal{O}\left(\frac{1}{(2N)!}\right)$$

x^r, w^r : $L_N(x)$ Legendre polynoms

$$Z = \int_0^{2\pi} \mathrm{d}\phi_1 \int_0^{2\pi} \mathrm{d}\phi_2\, f(\phi_1, \phi_2) \int_0^{2\pi} \mathrm{d}\phi_3\, f(\phi_2, \phi_3) f(\phi_3, \phi_1)$$

$$\approx \sum_{t=1}^{N} w^t \cdot \sum_{s=1}^{N} w^s \cdot f(\phi^s, \phi^t) \sum_{r=1}^{N} w^r \cdot f(\phi^s, \phi^r) \cdot f(\phi^r, \phi^t)$$

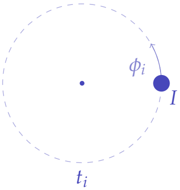
Truncation Error Scaling



Error $\Delta\chi_i = |\chi_i - \chi(N = 560)|$

Constants $I = 0.25$

$a = 0.4, T = 20$



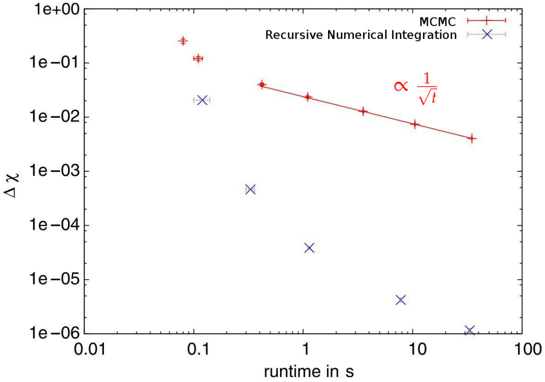
Topological Charge

$$Q(\phi) = \frac{1}{2\pi}\int_0^T \mathrm{d}t \left(\frac{\partial\phi}{\partial t}\right)$$

Topological Susceptibility

$$\chi = \frac{\langle Q^2(\phi)\rangle}{T}$$

RNI - Comparison with MCMC



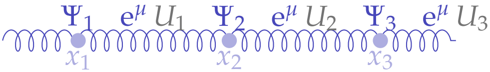
Error $\Delta\chi_{i,Gauss} = |\chi_i - \chi(N = 400)|$

$\Delta\chi_{i,Cluster}$: 10 runs

Constants $I = 0.25$

$a = 0.1, T = 20$

1D-QCD



Ψ : mass m

$$S[U, \overline{\Psi}, \Psi] = \sum_i m\overline{\Psi}_i \Psi_i + e^{\mu}\, \overline{\Psi}_i U_i \Psi_{i+1} + e^{-\mu}\, \overline{\Psi}_{i-1} U_i^* \Psi_i$$

$$= \overline{\Psi}\, \mathfrak{D}[U]\, \Psi, \qquad U_i \in \mathcal{G}, \text{ e.g. } \mathcal{U}(N), \mathcal{SU}(N)$$

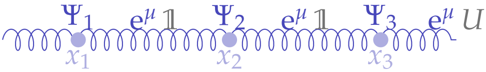
PARTITION FUNCTION

$$Z[U] = \int_{\mathcal{G}} dh_G(U_1) \int_{\mathcal{G}} dh_G(U_2) \int_{\mathcal{G}} dh_G(U_3) \int d\overline{\Psi} \int d\Psi\; e^{-S[U, \overline{\Psi}, \Psi]},$$

$$= \int_{\mathcal{G}^3} dh^3_{\mathcal{G}}(U)\, \det \mathfrak{D}[U]$$

$$= \int_{\mathcal{G}^3} dh^3_{\mathcal{G}}(U) \left(c(m) + \quad 2^{-3}\, e^{-3\mu} \left(\prod_{j=1}^{3} U_j \right)^* + (-1)^3\, 2^{-3}\, e^{3\mu} \left(\prod_{j=1}^{3} U_j \right) \right)$$

1D-QCD



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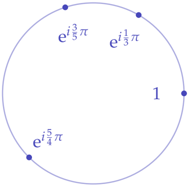
$$= \int_{\mathcal{G}} dh_{\mathcal{G}}(U)\left(c(m) + \quad 2^{-3}\,e^{-3\mu}\,U^* \right.$$

$$\left. +(-1)^3\,2^{-3}\,e^{3\mu}\,U\right)$$

Why difficult for MC?

$$Z[U] = \int_{\mathcal{G}} \mathrm{d}h_{\mathcal{G}}(U)\ \det\left(c(m) + 2^{-3}\,\mathrm{e}^{-3\mu}\,U^* + (-1)^3 2^{-3}\,\mathrm{e}^{3\mu}\,U\right) Z[U] =$$

$$\int_{\mathcal{U}(1)} \mathrm{d}U\,U = \int_0^{2\pi} \mathrm{d}\theta\ \mathrm{e}^{i\theta} = 0$$



$$\overset{\mathrm{MC}}{\approx} \frac{1}{4}\left(1 + \mathrm{e}^{i\frac{1}{3}\pi} + \mathrm{e}^{i\frac{3}{5}\pi} + \mathrm{e}^{i\frac{5}{4}\pi}\right) \approx 0.48 + 1.11i$$

$$\Delta_{MC} = \mathcal{O}(1) \overset{N\to\infty}{\longrightarrow} \mathcal{O}(0)$$

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$$Z[U] = \int_{\mathcal{G}} \mathrm{d}h_{\mathcal{G}}(U)\ \det\left(c(m) + 2^{-3}\,\mathrm{e}^{-3\mu}\, U^{*} + (-1)^{3} 2^{-3}\,\mathrm{e}^{3\mu}\, U\right) Z[U] =$$

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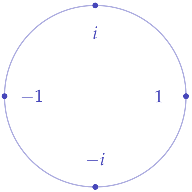

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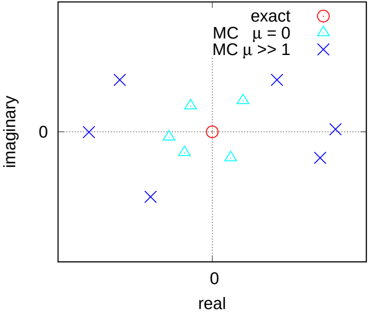
$$\overset{\mathrm{sym}}{\approx}\ \frac{1}{4}\left(1 + i + (-i) + (-1)\right) = 0$$

Problem of the chemical potential

$$Z[U] = \int_{\mathcal{U}(1)} \mathrm{d}U \ \left(c(m) + 2^{-3}\,\mathrm{e}^{-3\mu}\,U^* + (-1)^3 2^{-3}\,\mathrm{e}^{3\mu}\,U\right)$$



$$\int_{\mathcal{U}(1)} \mathrm{d}U\, U = 0$$

$$\downarrow \mu > 0$$

$$\int_{\mathcal{U}(1)} \mathrm{d}U \ \mathrm{e}^{3\mu}\, U = 0$$

Spherical t-designs

$$\mathcal{G} = S^n, \qquad f(U) \approx P_t$$

$$\int_{\mathcal{G}} \mathrm{d}h_{\mathcal{G}}(U)\, f(U) \approx \frac{1}{t+1} \sum_{k=1}^{t+1} f(U^k)$$

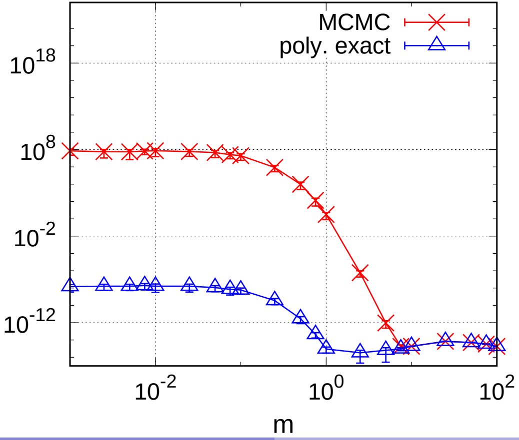
- rules for S^1, S^3 and S^5 [Genz 2003]
- use isomorphisms to get rules for $\mathcal{U}(1)$, $\mathcal{U}(2)$, $\mathcal{U}(3)$, $\mathcal{SU}(2)$, $\mathcal{SU}(3)$ [Ammon 2016]

Result Partition Function

$$Z[U] = \int_{\mathcal{U}(1)} \mathrm{d}U \ \left(c(m) + 2^{-3}\, \mathrm{e}^{-3\mu}\, U^* + (-1)^3 2^{-3}\, \mathrm{e}^{3\mu}\, U\right)$$

$$\frac{\langle Z\rangle_{\text{quadrature}} - \langle Z\rangle_{\text{analytic}}}{\langle Z\rangle_{\text{analytic}}}$$

- $\mathcal{G} = \mathcal{U}(1)$
- double prec
- $\mu = 1$
- 20 lattice points

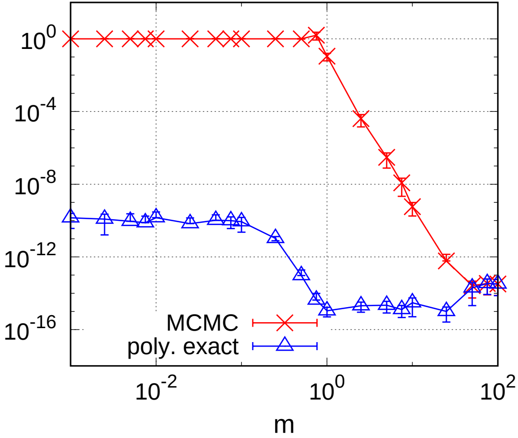


Result Chiral Condensate

$$\langle \overline{\Psi}\Psi \rangle = \partial_m \ln Z = \frac{\int_{\mathcal{G}} \mathrm{d}h_{\mathcal{G}}\, \partial_m \det \mathfrak{D}}{\int_{\mathcal{G}} \mathrm{d}h_{\mathcal{G}}\, \det \mathfrak{D}}$$

$$\frac{\langle \overline{\Psi}\Psi \rangle_{\text{quadrature}} - \langle \overline{\Psi}\Psi \rangle_{\text{analytic}}}{\langle \overline{\Psi}\Psi \rangle_{\text{analytic}}}$$

- $\mathcal{G} = \mathcal{U}(2)$
- double prec
- $\mu = 1$
- 8 lattice points



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- 1024 bit prec
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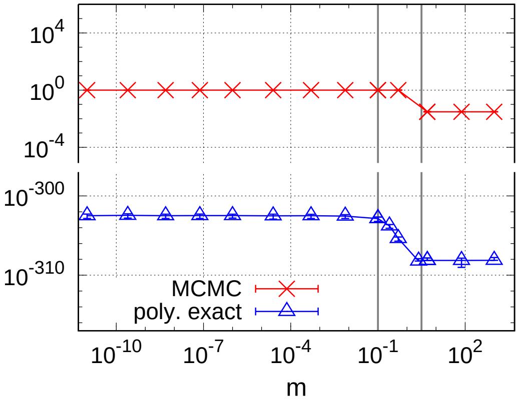
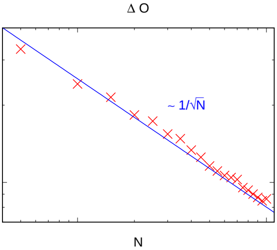
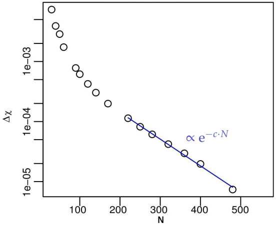


Conclusions

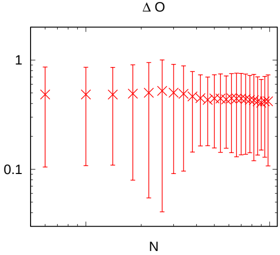
$S > 0$



[1503.05088] $\longrightarrow$ topol. osci.



$S \in \mathbb{C}$



[1607.05027] $\longrightarrow$ 1d-QCD