Homogeneous binary trees as ground states of quantum critical Hamiltonians

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Many-body states whose wave functions admit a representation in terms of a uniform binary-tree tensor decomposition are shown to obey power-law two-body correlation functions. Any such state can be associated with the ground state of a translationally invariant Hamiltonian which, depending on the dimension of the systems sites, involves at most couplings between third-neighboring sites. Under general conditions it is shown that they describe unfrustrated systems which admit an exponentially large degeneracy of the ground state.

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I. INTRODUCTION

The selection of suitable tailored variational wave functions is a fundamental problem in the study of quantum many-body systems [1]. The variational ansatz must satisfy two basic requirements: it should provide an accurate approximation of the target state (e.g., the ground state), and it should allow for an efficient evaluation of the relevant physical quantities (e.g., local observables and associated correlation functions). Matrix product states (MPSs) are a successful example of this kind [2]. It is possible to quantify their accuracy to approximate the exact wave function [3] and, in some specific cases [4], the ground state itself is in a matrix product form (e.g., see Ref. [5] for a review). MPSs are specifically suited to deal with noncritical, short-range, one-dimensional Hamiltonians. In order to overcome these limitations, several generalizations have been proposed [6–9]. Projected entangled-pair states [6] were introduced to deal with higher dimensions, weighted graph states [7] to treat systems with long-range interactions, and multiscale entanglement renormalization ansatz (MERA) [8] to efficiently address critical systems.

In this work, we focus on one-dimensional quantum critical systems using homogeneous binary-tree states (HBTSs) as variational states. They share some structural properties of scale-invariant MERA states (including the possibility of constructing efficient optimizing algorithms [8,10,11]) but admit a simpler description and provide a prototypical realization of a real-space renormalization process. Although on general grounds it can be argued that these states are suitable candidates to approximate critical systems (e.g., they violate area law [12] with logarithmic corrections [13]), an explicit derivation of their critical properties is still missing. We will prove that HBTSs can describe critical systems by computing the correlation functions and show that they decay in a power-law fashion.

Once ascertained that HBTSs describe the critical ground state it would be important to know if there are cases in which they are actually the exact ground state of a given model Hamiltonian. Despite the large body of work devoted so far to the subject, there is no definite answer for critical systems. (Up to now, only approximated, numerical evidence has been gathered on this issue.) Given their ubiquitous presence in condensed matter and statistical mechanics, this question is of particular relevance both conceptually and for possible numerical implementations. In this work, we show that, in the thermodynamic limit, HBTSs can be associated to (nontrivial) local and translationally invariant parent Hamiltonians [14]. Furthermore, similarly to what was done for MPSs [2,15], we discuss sufficient conditions under which such operators also continue to be parent Hamiltonians for finite sites. By construction, this allows us to identify a class of (nontrivial) unfrustrated Hamiltonians whose characterization has been attracting some recent interest (see, e.g., Ref. [16]). Although we concentrate only on binary trees, the method we present to construct the parent Hamiltonian can be applied to other tensor structures, such as the MERA, which support scale invariance.

The article is organized as follows: Section II is devoted to introduce the basic properties of HTBSs, Sec. II B focuses on how correlation functions can be computed for such systems and shows the critical properties of such quantities, and Sec. III discusses how to construct parent Hamiltonians for HTBSs. Conclusions and remarks are presented in Sec. IV. In the appendix, we discuss how to construct the parent Hamiltonian in the case of a MERA state.

II. HOMOGENEOUS BINARY TREE STATES

Consider a one-dimensional lattice of $N = 2^n$ sites of a given local dimension d and with periodic boundary conditions. A generic pure state of such system can always be expressed as

$$|\psi^{(n)}\rangle = \sum_{\ell_1,\ldots,\ell_N=1}^{d} \mathcal{T}_{\ell_1,\ldots,\ell_N} |\xi_{\ell_1}\ldots\xi_{\ell_N}\rangle, \qquad (1)$$

where $\{|\xi_i\rangle\}_i$ is a canonical basis for the single qudit and $\mathcal{T}$ is a type-$\binom{0}{N}$ tensor. HBTSs of depth n are identified as those $|\psi^{(n)}\rangle$ whose $\mathcal{T}$ can be decomposed in terms of smaller tensors, as in Fig. 1. Following Ref. [8], each node of such a graph represents a tensor (the emerging legs of the node being its indices), while a link connecting any two nodes represents contraction of the corresponding indices. The yellow element

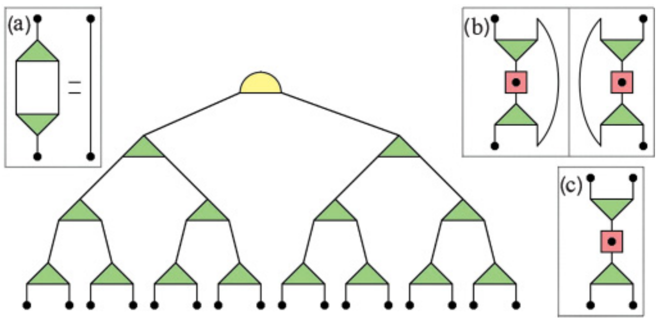

FIG. 1. (Color online) BT network for $16 = 2^n$ sites. Inset (A) shows the isometric property of λ, (B) shows the maps $\mathcal{D}_L$ (left) and $\mathcal{D}_R$, and (C) shows the map $\mathcal{S}$ of Eq. (3).

at the top of Fig. 1 describes a type-$\binom{0}{2}$ tensor $\mathcal{C}$ of elements $\mathcal{C}_{\ell_1,\ell_2}$, while the $2N-1$ triangles represent the same $d \times d^2$ tensor λ of type $\binom{1}{2}$ whose elements $\lambda^u_{\ell_1,\ell_2}$ satisfy the isometric condition

$$\sum_{k_1,k_2} \lambda^u_{k_1,k_2} \bar{\lambda}^{k_1,k_2}_{\ell} = \delta^u_\ell, \tag{2}$$

where δ^u_ℓ is the Kronecker delta and $\bar{\lambda}^{u_1,u_2}_\ell \equiv (\lambda^\ell_{u_1,u_2})^*$ is the adjoint of the λ obtained by exchanging its lower and upper indices and taking the complex conjugate. Together with the condition $\sum_{\ell_1,\ell_2} \mathcal{C}_{\ell_1,\ell_2}\bar{\mathcal{C}}^{\ell_1,\ell_2} = 1$, Eq. (2) automatically guarantees normalization of the HBT state at every level. It has been shown in [13,17] that, under these assumptions, tensor tree states allow for an efficient evaluation of observables and correlation functions. In the generic case each tensor can be chosen to be different from the others. Being interested in scale-invariant systems, it is natural to assume all the tensors λ to be equal. In the rest of the article we will follow the formalism described in [18].

A. Evaluation of local observables

1. *Single-site observables*

In the limit of large n, the physical properties of such states are fully determined by the completely positive trace-preserving (CPT) channel $\mathcal{S}$ of Fig. 1(c). It transforms a single-site density matrix with elements $[\rho]^u_\ell \equiv \langle \xi_\ell|\rho|\xi_u\rangle$ into a two-site state $\mathcal{S}(\rho)$ with elements

$$\langle \xi_{\ell_1},\xi_{\ell_2}|\mathcal{S}(\rho)|\xi_{u_1},\xi_{u_2}\rangle = \sum_{k_1,k_2} \bar{\lambda}^{u_1,u_2}_{k_1}\, [\rho]^{k_1}_{k_2}\, \lambda^{k_2}_{\ell_1,\ell_2}. \tag{3}$$

Consider then a family $\mathcal{F} \equiv \{|\psi^{(n)}\rangle; n = 2,3,\ldots\}$ of HBTSs of increasing size (depth) sharing the same λ and $\mathcal{C}$. The map $\mathcal{S}$ allows us to construct a simple recursive expression for the reduced-density operator:

$$\bar{\rho}^{(n)}_1 \equiv \frac{1}{N}\sum_{\alpha=1}^{N} \rho^{(n)}_\alpha, \tag{4}$$

which describes the averaged single site state of the nth element of $\mathcal{F}$ (here $\rho^{(n)}_\alpha$ is the reduced density matrix of the αth site of the system). Specifically, the isometric property of λ yields

$$\bar{\rho}^{(n+1)}_1 = \mathcal{D}\big(\bar{\rho}^{(n)}_1\big), \tag{5}$$

where $\mathcal{D}$ is the CPT map obtained by taking an equally weighted mixture of the partial traces of the map $\mathcal{S}$ as indicated in Fig. 1(b). This can be expressed as

$$\mathcal{D} \equiv (\mathcal{D}_L + \mathcal{D}_R)/2, \tag{6}$$

where $\mathcal{D}_L(\cdot) \equiv \mathrm{Tr}_2[\mathcal{S}(\cdot)]$ and $\mathcal{D}_R(\cdot) \equiv \mathrm{Tr}_1[\mathcal{S}(\cdot)]$ with $\mathrm{Tr}_{1,2}$ representing partial traces with respect to the first and second site, respectively. Equation (5) allows us to compute the average expectation of a single site observable Θ, for every full depth value n of the tree in terms of a repetitive application of the map $\mathcal{D}$. Indeed, indicating with $\langle\Theta_\alpha\rangle^{(n)}$ the expectation value on the αth site of $|\psi^{(n)}\rangle$, we can write

$$\frac{1}{N}\sum_{\alpha=1}^{2^n}\langle\Theta_\alpha\rangle^{(n)} = \mathrm{Tr}\big[\Theta\bar{\rho}^{(n)}_1\big] = \mathrm{Tr}\big[\Theta\cdot\mathcal{D}^{n-1}(\rho_{\mathrm{hat}})\big], \tag{7}$$

where $\rho_{\mathrm{hat}} \equiv \bar{\rho}^{(1)}_1$ is the single-site density matrix of elements $\langle\xi_\ell|\rho_{\mathrm{hat}}|\xi_u\rangle \equiv \sum_k[\mathcal{C}^*_{u,k}\mathcal{C}_{\ell,k} + \mathcal{C}^*_{k,u}\mathcal{C}_{k,\ell}]/2$, and where $\mathcal{D}^n \equiv \mathcal{D}\circ\cdots\circ\mathcal{D}$, with "$\circ$" representing the composition of CPT maps.

2. *Two-site observables*

An analogous procedure can be used to express averages of operators defined on $\nu = 2$ neighboring sites. All we have to do is to consider the density matrix

$$\bar{\rho}^{(n)}_2 \equiv \frac{1}{N}\sum_{\alpha=1}^{N}\rho^{(n)}_{\alpha,\alpha+1} \tag{8}$$

and build for this quantity a level-recursive mapping which is the two-nearest-neighboring-sites version of Eq. (5) (here, $\rho^{(n)}_{\alpha,\alpha+1}$ represents the reduced density matrix of the sites α and $\alpha+1$ associate with a HBTS of depth n). The calculation is straightforward, so we just write the result:

$$\bar{\rho}^{(n+1)}_2 = \tfrac{1}{2}(\mathcal{D}_R\otimes\mathcal{D}_L)\big(\bar{\rho}^{(n)}_2\big) + \tfrac{1}{2}\mathcal{S}\big(\bar{\rho}^{(n)}_1\big). \tag{9}$$

The above expression allows us also to deal with the case of observables operating on ν neighboring sites. Indeed, for $\nu \geqslant 3$, it can be shown that any average density matrix $\bar{\rho}^{(n)}_\nu$ can be expressed in terms of $\{\bar{\rho}^{(m)}_2\}_{m<n}$ via the application of a proper CPT map derived from $\mathcal{S}$. As an example, we write explicitly the case for $\nu = 3$ and 4:

$$\bar{\rho}^{(n)}_3 = \tfrac{1}{2}[\mathcal{D}_R\otimes\mathcal{S} + \mathcal{S}\otimes\mathcal{D}_L]\big(\bar{\rho}^{(n-1)}_2\big), \tag{10}$$

$$\begin{aligned}\bar{\rho}^{(n)}_4 = {}& \tfrac{1}{2}[\mathcal{S}\otimes\mathcal{S}]\big(\bar{\rho}^{(n-1)}_2\big) + \tfrac{1}{4}[\mathcal{D}_R\otimes\mathcal{S}\otimes\mathcal{D}_L]\,\circ \\ &\circ[\mathcal{D}_R\otimes\mathcal{S} + \mathcal{S}\otimes\mathcal{D}_L]\big(\bar{\rho}^{(n-2)}_2\big).\end{aligned} \tag{11}$$

3. *Thermodynamic limit*

In the limit of infinitely many sites, it follows from Eq. (5) that if the average single-site state associated with an HBTS of infinite depth characterized by a given isometry λ is defined, then it must be a fixed point of the map $\mathcal{D}$. Since CPT maps have a unique fixed point except for a subset of zero probability [19], the fixed point is almost always defined. Similarly, we can also provide an explicit formula for the thermodynamic limit of the two-site state (9); namely,

$$\bar{\rho}_2^{(\infty)} \equiv \lim_{n\to\infty} \bar{\rho}_2^{(n)}. \tag{12}$$

This can be written either as a self-consistent equation or like a series in terms of $\bar{\rho}_1^{(\infty)}$ by exploiting the identity (9)

$$\bar{\rho}_2^{(\infty)} = \frac{1}{2}\sum_{m=0}^{(\infty)}\left[\frac{1}{2^m}(\mathcal{D}_R\otimes\mathcal{D}_L)^m\right]\circ\mathcal{S}\big(\bar{\rho}_1^{(\infty)}\big). \tag{13}$$

The series is convergent in any norm, thanks to the geometric factor and the fact that CPTs are nonexpansive. Such an argument becomes even simpler when dealing with three or more sites density matrices. Indeed, one can show that, for any integer ν, there exists a CPT map $\mathcal{D}_{2\to\nu}$ such that the ν-nearest-neighbor-site density matrix $\bar{\rho}_\nu^{(\infty)}$ (averaged over translations) in the thermodynamic limit is given by

$$\bar{\rho}_\nu^{(\infty)} = \mathcal{D}_{2\to\nu}\big(\bar{\rho}_2^{(\infty)}\big). \tag{14}$$

This provides a complete characterization of the local properties of our infinitely deep HBTS. For future reference, we report the expression for the cases $\nu = 3$ and 4:

$$\mathcal{D}_{2\to3} = (\mathcal{D}_R\otimes\mathcal{S} + \mathcal{S}\otimes\mathcal{D}_L)/2, \tag{15}$$

$$\mathcal{D}_{2\to4} = [\mathcal{S}\otimes\mathcal{S} + (\mathcal{D}_R\otimes\mathcal{S}\otimes\mathcal{D}_L)\circ\mathcal{D}_{2\to3}]/2, \tag{16}$$

[notice that $\mathcal{D}_{2\to3}$ is exactly the channel which enters in Eq. (10)]. Finally, we notice that all these quantities are independent from the element $\mathcal{C}$ of the HBTS, implying that, in the thermodynamical limit, the local structure of the state loses all its dependence from this element. As the physics of the system is determined by the algebra of the local observables, this implies that all HBTS of infinite depth, associated with a given λ but with different $\mathcal{C}$, describe the same state.

B. Correlation functions

Similarly to what has been done for the local observables in the previous section, correlation functions can also be expressed in terms of iterative applications of certain maps. Most important for our work is to show that this procedure leads naturally, in the case of homogeneous trees, to power-law decays for the correlator—the exponents being related to the eigenvalues of the map [18].

In the following, we focus on two-point correlation functions. As discussed before, since HBTSs are not manifestly translationally invariant, an average over translations has to be made. We thus introduce the quantities

$$\mathfrak{C}_{\Delta\alpha}^{(n)} \equiv \frac{1}{2^n}\sum_{\beta=1}^{2^n}\big[\langle\Theta_\beta\Theta'_{\beta+\Delta\alpha}\rangle^{(n)} - \langle\Theta_\beta\rangle^{(n)}\langle\Theta'_{\beta+\Delta\alpha}\rangle^{(n)}\big],$$

where Θ and Θ' are two single-site observables. A remarkable simplification is achieved for any distance $\Delta\alpha$ equal to a power of 2. Under this condition, we find that

$$\mathfrak{C}_{\Delta\alpha=2^m}^{(n)} = \mathrm{Tr}\big[(\Theta\otimes\Theta')\mathscr{D}^m\big(\bar{\rho}_2^{(n-m)} - \bar{\eta}_{1,1}^{(n-m)}\big)\big], \tag{17}$$

where $\mathscr{D}$ is the map $\mathscr{D} \equiv \frac{1}{2}(\mathcal{D}_L\otimes\mathcal{D}_L + \mathcal{D}_R\otimes\mathcal{D}_R)$. The quantity $\bar{\eta}_{1,1}^{(n)}$ is the averaged two-site-nearest-neighbor density matrix after we traced away every quantum correlation, while keeping eventual classical correlations intact;that is,

$$\bar{\eta}_{1,1}^{(n)} = \frac{1}{2^n}\sum_{\alpha=1}^{2^n}\rho_\alpha^{(n)}\otimes\rho_{\alpha+1}^{(n)}. \tag{18}$$

Take then $n\to\infty$ while keeping $m = \log_2\Delta\alpha$ fixed. In this context, it is important to notice that, like $\bar{\rho}_2^{(n)}$, $\bar{\eta}_{1,1}^{(n)}$ also has a well-defined limit. It coincides with the two-site state

$$\bar{\eta}_{1,1}^{(\infty)} = \frac{1}{2}\sum_{m=0}^{(\infty)}\left[\frac{1}{2^m}(\mathcal{D}_R\otimes\mathcal{D}_L)^m\right]\circ(\mathcal{D}_L\otimes\mathcal{D}_R)(\not{\sigma}), \tag{19}$$

with $\not{\sigma}$ being the fixed point of $\mathscr{D}$. Exploiting this fact, we can thus write the thermodynamic limit of the correlation as

$$\begin{aligned}\mathfrak{C}_{\Delta\alpha=2^m}^{(\infty)} &= \mathrm{Tr}\big[(\Theta\otimes\Theta')\mathscr{D}^m\big(\bar{\rho}_2^{(\infty)} - \bar{\eta}_{1,1}^{(\infty)}\big)\big]\\ &= \mathrm{Tr}\big[\big(\bar{\rho}_2^{(\infty)} - \bar{\eta}_{1,1}^{(\infty)}\big)\mathscr{A}^m(\Theta\otimes\Theta')\big],\end{aligned} \tag{20}$$

where $\mathscr{A}$ is the adjoint superoperator of $\mathscr{D}$ (with respect to the operator scalar product $\langle A,B\rangle = \mathrm{Tr}[A^\dagger B]$). Recall that we are keeping track of the quantum correlations only and, moreover, since $\bar{\rho}_2^{(\infty)} - \bar{\eta}_{1,1}^{(\infty)}$ is a traceless matrix, we are guaranteed that $\mathfrak{C}_{\Delta\alpha}^{(\infty)} \to 0$ for $\Delta\alpha\to\infty$ because $\mathscr{D}^\infty(X) = \not{\sigma}\,\mathrm{Tr}[X]$. This shows that the residual influence on m is only kept through the number of times we have to apply the appropriate map to its matrix argument. By decomposing $\mathscr{A}$ in Jordan blocks, one finds its set of eigenoperators. If we assume that $\Theta\otimes\Theta'$ is one of these operators related to the eigenvalue κ, then the correlation function is expressed as follows:

$$\mathfrak{C}_{2^m}^{(\infty)} = g\,\Delta\alpha^{\log_2\kappa}, \tag{21}$$

where we separated the prefactor

$$g = \mathfrak{C}_1^{(\infty)} = \mathrm{Tr}\big[\big(\bar{\rho}_2^{(\infty)} - \bar{\eta}_{1,1}^{(\infty)}\big)\Theta\otimes\Theta'\big]. \tag{22}$$

The critical exponents are defined by the spectrum of $\mathscr{A}$, and the related primary fields are given by the respective eigenoperators. Notice that such exponents have always negative real parts, since all $|\kappa| \leqslant 1$ because $\mathscr{A}$ is a completely positive unital operator (and with the mixing condition only $|\kappa| < 1$ and $\kappa = 1$ are allowed; e.g., see Ref. [20]). This guarantees that such correlations are actually decaying power-law functions.

If the observables $\Theta\otimes\Theta'$ are not eigenoperators of $\mathscr{A}$, their correlator is typically a sum of power-laws and may manifest logarithmic corrections (arising from the fact that $\mathscr{A}$ in general cannot be diagonalized):

$$\big|\mathfrak{C}_{\Delta\alpha}^{(\infty)}\big| \simeq \sum_\kappa \Delta\alpha^{\log_2|\kappa|}g_\kappa(\log_2\Delta\alpha), \tag{23}$$

where the sum spans over the spectrum of $\mathscr{A}$, and $g_k(\cdot)$ are polynomials in their main argument. The present considerations prove the criticality character of HBTSs.

III. PARENT HAMILTONIANS

In the previous section we showed that HBTSs support a power-law decay of correlators, and we related the associated critical exponents to the tensors which the define the state. Is there any case where an HBTS is the exact ground state of a short-range critical Hamiltonian? In this section we show how to construct local translationally invariant Hamiltonians for which a given homogeneous HBTS is the exact ground state. First, we focuses on the case of infinite-dimensional systems (thermodynamical limit). Next, we show how the analysis can be extended to the case of finite-dimensional HBTSs.

A. Thermodynamic limit

Consider a Hamiltonian which involves at most $(\nu-1)$ neighboring couplings of the form

$$\mathcal{H}=\frac{1}{N}\sum_{\alpha=1}^{N}H_\nu(\alpha). \tag{24}$$

The factor $1/N$ is introduced to keep a finite spectrum in the thermodynamical limit, and $H_\nu(\alpha)$ is an interaction term that couples ν consecutive sites starting from the αth (i.e., the sites $\alpha,\ldots,\nu-1+\alpha$). The expectation value over the infinite HBTS of this Hamiltonian is

$$\langle\mathcal{H}\rangle^{(\infty)}=\mathrm{Tr}\left[H_\nu\bar{\rho}_\nu^{(\infty)}\right], \tag{25}$$

where $\bar{\rho}_\nu^{(\infty)}$ (the averaged ν-neighboring-sites density matrix) is a quantity we can calculate as discussed in the previous sections.

Let us for a moment assume that the rank of $\bar{\rho}_\nu^{(\infty)}$ is less than its maximum d^ν. This means that such density matrix has a nontrivial kernel, which can be decomposed in a basis of vectors $\{|\phi_\nu(k)\rangle\}_k$. Therefore, we take

$$H_\nu=\sum_k E_k|\phi_\nu(k)\rangle\langle\phi_\nu(k)|, \tag{26}$$

with E_k being arbitrary positive constants. This is positive by construction and so is the associated $\mathcal{H}$. Then, since the image of H_ν belongs to the kernel of $\bar{\rho}_\nu^{(\infty)}$, it is clear that $H_\nu\bar{\rho}_\nu^{(\infty)}=0$, and so $\langle\mathcal{H}\rangle^{(\infty)}=0$ as well. In the end, we built a positive, translation-invariant Hamiltonian, with $(\nu-1)$ neighboring coupling terms, whose expectation value over our HBTS is zero. This means that the state is a ground state for $\mathcal{H}$. The only caveat to make it work is to demonstrate that, for some ν, we have

$$\mathrm{rank}\big(\bar{\rho}_\nu^{(\infty)}\big)<d^\nu, \tag{27}$$

otherwise H_ν would be the trivial null operator. The fundamental ingredient to verify this is to notice that the channel $\mathcal{S}$ of Eq. (3) preserves rank while increasing dimensions (i.e., it is an isometric mapping). Let thus investigate the case $\nu=3$. We know that the state $\bar{\rho}_3^{(\infty)}$ is obtained by exploiting the first of the mapping of Eq. (16). Specifically, we have

$$\bar{\rho}_3^{(\infty)}=\mathcal{D}_{2\to3}\big(\bar{\rho}_2^{(\infty)}\big)=(\mathcal{I}\otimes\mathcal{S})(A)+(\mathcal{S}\otimes\mathcal{I})(B), \tag{28}$$

with $\mathcal{I}$ being the single-site identity mapping and with A and B being some $d^2\times d^2$ positive matrices. The maps $\mathcal{I}\otimes\mathcal{S}$ and $\mathcal{S}\otimes\mathcal{I}$ preserve the rank, and the rank of the sum is less than or equal to the sum of ranks, thus leading to the inequality

$$\mathrm{rank}\big(\bar{\rho}_3^{(\infty)}\big)\leqslant 2d^2, \tag{29}$$

over a maximum of d^3. Therefore, if the local dimension d is 3 (spin 1) or greater, then we already achieved our goal of finding a $\bar{\rho}_\nu^{(\infty)}$ matrix with nonmaximal rank.

For $d=2$ (spin 1/2), we have to move to $\nu=4$, instead. In this case, the state to consider is

$$\begin{aligned}\bar{\rho}_4^{(\infty)}&=\mathcal{D}_{2\to4}\big(\bar{\rho}_2^{(\infty)}\big)\\&=(\mathcal{S}\otimes\mathcal{S})(A')+(\mathcal{I}\otimes\mathcal{S}\otimes\mathcal{I})(B').\end{aligned} \tag{30}$$

Since its rank obeys the inequality

$$\mathrm{rank}\big(\bar{\rho}_4^{(\infty)}\big)\leqslant d^2+d^3, \tag{31}$$

we have found a state that already for $d=2$ possesses a nontrivial kernel [indeed, in this case $\mathrm{rank}(\bar{\rho}_4^{(\infty)})\leqslant 12$, which is strictly less than the total dimension $d^4=16$].

In summary, this shows that any given infinite HBTS always admits a local translationally invariant nontrivial parent Hamiltonian $\mathcal{H}$, which can be constructed explicitly as in Eq. (26). For $d\geqslant3$, such an $\mathcal{H}$ can be chosen to have interactions which involve up to second-neighboring couplings. For $d=2$, we instead can always choose $\mathcal{H}$ with up to third-neighboring couplings. Our analysis deals with the worst-case scenario, if it occurs that $\bar{\rho}_2^{(\infty)}$ is nonfull rank by accident, one can construct a shorter-ranged (i.e., nearest neighbor) parent Hamiltonian using (26).

B. Finite-size case

The above approach formally applies to the case of infinitely many sites, and in general there is no guarantee that the selected $\mathcal{H}$ will be a parent Hamiltonian of the HBTS $|\psi^{(n)}\rangle$ when n is finite. Nonetheless, the proof can be extend to also cover this case in most situations. This will allow us to prove that, for even N, $\mathcal{H}$ is unfrustrated and its ground space must have dimension D_{gr} larger than $d^{N/2}$.

To show this, we focus on the case $d\geqslant3$ and assume that our HBTS has $\bar{\rho}_2^{(\infty)}$ of full rank (generalization to $d=2$ shall be dealt with later); this guarantees that Eq. (26) provides a parent Hamiltonian $\mathcal{H}$ for the thermodynamical state with a three-body interaction H_3. Consider then a generic state $|\psi\rangle$ of $N/2$ sites and "grow" a BT level from it, using the same λ isometry we used to build $\mathcal{H}$. This way we obtain a N-sited state

$$|\phi\rangle=\lambda^{\otimes N/2}|\psi\rangle, \tag{32}$$

which, by varying $|\psi\rangle$, spans a subspace $\mathfrak{S}$ of dimension $d^{N/2}$ (when N is a power of 2 one element of such a subspace is, for instance, the HBTS we started with). The expectation value $\langle\phi|\mathcal{H}|\phi\rangle$ can then be expressed as

$$\mathrm{Tr}\,[\bar{q}_3H_3]=\mathrm{Tr}[\mathcal{D}_{2\to3}(\bar{r}_2)H_3]=\mathrm{Tr}[\bar{r}_2\mathcal{A}_{3\to2}(H_3)],$$

where $\bar{q}_3$ is the averaged reduced-density matrices of three-neighboring sites of $|\phi\rangle$, $\bar{r}_2$ is the averaged reduced density matrices of two-neighboring sites of $|\psi\rangle$, while $\mathcal{A}_{3\to2}$ is the Heisenberg conjugate map of $\mathcal{D}_{2\to3}$. At this point, we observe that $\mathcal{A}_{3\to2}(H_3)$ is the null operator. This follows from Eq. (16), which allows us to write

$$\begin{aligned}0&=\mathrm{Tr}\big[\bar{\rho}_3^{(\infty)}H_3\big]=\mathrm{Tr}\big[\mathcal{D}_{2\to3}\big(\bar{\rho}_2^{(\infty)}\big)H_3\big]\\&=\mathrm{Tr}\big[\bar{\rho}_2^{(\infty)}\mathcal{A}_{3\to2}(H_3)\big],\end{aligned} \tag{33}$$

where the first identity simply states that $\mathcal{H}$ is the parent Hamiltonian of the HBTS at thermodynamical limit. Since $\bar{\rho}_2^{(\infty)}$ has maximal support by hypothesis and $\mathcal{A}_{3\to 2}(H_3)$ is positive semidefinite by construction, Eq. (33) implies $\mathcal{A}_{3\to 2}(H_3) = 0$. Equation (33) then leads to $\langle\phi|\mathcal{H}|\phi\rangle = 0$ which, together with the fact that $\mathcal{H}$ is positive, tells us that each one of the vectors $|\phi\rangle$ of the subspace $\mathfrak{S}$ is a ground state of the parent Hamiltonian $\mathcal{H}$.

Let us now deal briefly with the case $d = 2$. If $\bar{\rho}_3^{(\infty)}$ is nonfull rank then we can build a three-body interacting parent Hamiltonian just like the $d \geqslant 3$ case, and the generalization to the finite setting is identical. Otherwise, via $\bar{\rho}_4^{(\infty)}$, we can build a positive parent Hamiltonian $\mathcal{H}$ of the thermodynamical state with four-body interactions H_4. Evaluating its expectation value on the N-site state (32) it is then easy to verify that it becomes null (the proof is similar to the previous case, and it exploits the fact that $\bar{\rho}_3^{(\infty)}$ has full rank).

The above discussion proves that, for all even N, the Hamiltonian $\mathcal{H}$ (respectively $\mathcal{H}'$) has a ground eigenspace which is at least $d^{N/2}$ dimensional [21]. The presence of a wide ground state degeneracy is in accordance with symmetry predictions: since a finite HBT state $|\psi\rangle$ breaks the translational symmetry at every length scale, the whole space generated by $\{|\psi\rangle, T|\psi\rangle, T^2|\psi\rangle, \ldots, T^N|\psi\rangle\}$ must be embedded within the ground space. The present argument also implies that $\mathcal{H}$ represents an unfrustrated system. Indeed if $\langle\phi|\mathcal{H}|\phi\rangle = 0$ then each local component of $\mathcal{H}$ needs to become null on $|\phi\rangle$; that is, $\langle\phi|H_v(\alpha)|\phi\rangle = 0\,\forall\alpha$. As an example in Fig. 2, we report the eigenvalue degeneracies for a parent Hamiltonian $\mathcal{H}$ generated from an isometry λ defined by the mapping

$$|0\rangle \to |01\rangle, \tag{34}$$

$$|1\rangle \to \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{35}$$

[here $d = 2$ while $\mathcal{H}$ was generated by taking the free-parameters E_k of Eq. (26) to be uniform]. For $N = 4,6,8$

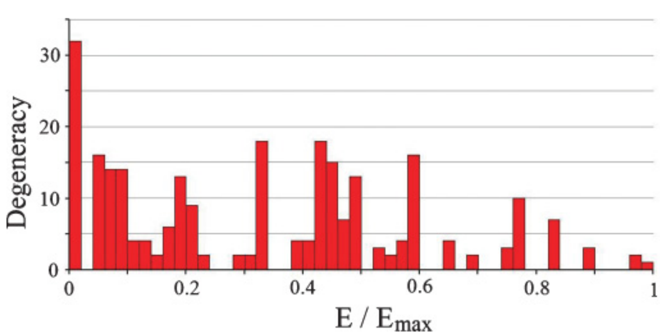

FIG. 2. (Color online) Unnormalized density of states of the parent Hamiltonian generated from a sample HBTS for $N = 8$ sites (energy levels have been rescaled to the maximum energy eigenvalue). For this example, it occurs that the ground-state degeneracy is twice the lower bound we discuss in the article: 32-fold over 256 states. More precisely, we have shown that the ground space of the Hamiltonian in this case coincides with the direct sum $\mathfrak{S} \oplus T(\mathfrak{S})$ of the space $\mathfrak{S}$ formed by the vectors defined in Eq. (32), and by its translated version $T(\mathfrak{S})$; T being the one-site translation (we verified that, in this case, the two sectors form a linearly independent subspace).

the ground state degeneracy turns out to be exactly $2\,2^{N/2}$, showing that in this case $\mathfrak{S}$ and $T(\mathfrak{S})$ [21] saturate completely the corresponding eigenspace (the figure only reports the case $N = 8$). We also checked numerically the case of odd N, for which the previous theoretical analysis does not hold. In this case, the ground state energy is not null, which shows that $\mathcal{H}$ is frustrated and that its degeneracy is much smaller than $d^{N/2}$ and controllable by choosing the E_k parameters appropriately.

This procedure to construct a parent Hamiltonian for a binary tree can also be applied to other tensor structures. In the appendix we discuss the case of a MERA.

IV. CONCLUSIONS

In this article we analyzed the potential of binary trees to simulate efficiently quantum critical systems. Previous works [13] presented evidence that such states could lead to violation of the area law with logarithmic corrections. In this article, we focused on homogenous configurations which allow for an explicit analytic treatment of the thermodynamical limit. Their hierarchical, scale invariant structure suggest that they should be capable of exhibiting critical behavior, at least once a proper averaging over translations has been performed to compensate for their explicit lack of translational invariance [22]. For instance, by looking at their tensorial decomposition it is clear that HBTSs do not violate the area law for all possible partitions of the sites (e.g., since the left side of the graph is connected with the right side by only a single link, the resulting block entropy will be independent of the number of sites). However, it is reasonable to believe that such"anomalies" will wash away when averaging over all possible translations (a legitimate operation when simulating translationally invariant systems). In the case of the block entropy this can be heuristically verified by noticing that, indeed, the average number of tensor links that needs to be cut in order to disconnect the causal cone of a block of consecutive sites from the rest scales almost logarithmically with the block size. To test the validity of these arguments, we focused in our article on the behavior of two-point correlation functions in the thermodynamical limit of infinitely many sites. Once averaged over all possible translations, we proved that these quantities can be explicitly computed and showed that they decay as a power law, in agreement with the criticality character of HBTS.

In the second part of the article, we show that HBTSs are the exact ground states of short-range interacting Hamiltonians. In particular, we gave a procedure to build such a parent Hamiltonian, which appears to be frustration free, and which carries a ground space degeneracy scaling with the square root of the Hilbert space dimension. Similar results can also be obtained (see the appendix) for MERA states.

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APPENDIX: PARENT HAMILTONIAN FOR MERA STATES

In this appendix, we discuss how to generalize the analysis of Sec. III to the case of scale-invariant (i.e., homogeneous) MERA states [8]. Indeed, it is also possible for a MERA to establish upper bounds for the rank of the states $\bar{\rho}_\nu^{(\infty)}$ (the translationally averaged ν-neighboring-sites density matrix in the thermodynamic limit) by exploiting growth superoperator properties for various block sizes ν. Therefore, by finding the suitable (smallest) ν for which such a rank is not maximal, the construction of a parent Hamiltonian interaction term according to Eq. (26) is straightforward.

Precisely, such minimal parent interaction length ν depends on the topology of the original MERA [23] and its local dimension d. For a binary MERA structure, we find that $\mathrm{rank}(\bar{\rho}_5^{(\infty)}) \leqslant 2d^4$ whose value is not maximal for $d \geqslant 3$. For completeness $\mathrm{rank}(\bar{\rho}_6^{(\infty)}) \leqslant d^4 + d^5$ takes care of the case $d = 2$, so it is always possible to build a parent Hamiltonian with a 5-or 6-body interactions. When considering a ternary MERA structure, we have to involve a block of seven nearest neighbors to achieve the meaningful bounding $\mathrm{rank}(\bar{\rho}_7^{(\infty)}) \leqslant 3d^5$, which is always nonmaximal regardless of d. Furthermore, one can produce analogous conditions under which $\mathcal{H}$ will still be a parent (unfrustrated) Hamiltonian for the finite-site case and verify that it posses a ground-state degeneracy which is exponentially large (order $d^{N/2}$ or $d^{N/3}$). A main difference between this case and the previous one is that a parent Hamiltonian for a finite HBT state also always admits a dimerized ground state [just pick up a vector of Eq. (32) that is build by taking $|\psi\rangle$ as a product state], while in the MERA context there is no such proof of triviality.

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[21] Clearly, our argument does not necessarily provide a complete characterization of the ground space, as D_{gr} can be even larger than $d^{N/2}$. In particular it is possible that the space $\mathfrak{S}$ will not be invariant under translation by one site (notice, however, that if T is the translation by one site, the subspace $\mathfrak{S}$ is explicitly invariant under T^2).
[22] Notice that such averaging needs not be performed in a classical fashion (i.e., by replacing the state $|\psi^{(n)}\rangle$ with a mixture of all possible translations of the same vector). Instead, the same average can be performed in a coherent fashion by replacing $|\psi^{(n)}\rangle$ with a pure state $|\bar{\psi}^{(n)}\rangle$ obtained by coherently superimposing all its translated versions. On the base of "ultraviolet catastrophe" arguments, we can indeed argue that, in the thermodynamical limit, the statistics associated with any local observable will be identical in the two cases.
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