Robust Optimal Quantum Gates for Josephson Charge Qubits

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Quantum optimal control theory allows us to design accurate quantum gates. We employ it to design high-fidelity two-bit gates for Josephson charge qubits in the presence of both leakage and noise. Our protocol considerably increases the fidelity of the gate and, more important, it is quite robust in the disruptive presence of $1/f$ noise. The improvement in the gate performances discussed in this work (errors $\sim 10^{-3}$–10^{-4} in realistic cases) allows us to cross the fault tolerance threshold.

One of the fundamental requirements of any proposed implementation of quantum information processing is the ability to perform reliably single- and two-qubit gates. In the last decade there has been intense experimental and theoretical activity to realize suitable schemes for quantum gates in a variety of physical systems such as NMR, ion traps, cold atoms, solid-state devices, just to mention a few [1]. Typically, as compared to single-bit gates, two-qubit gates are much more difficult to realize. The interaction between the qubits is more delicate to control while preserving coherence. Furthermore two-bit gates are more sensitive to imperfections, noise and, whenever present, leakage to noncomputational states. It is therefore of crucial importance to find strategies to alleviate all these problems. A powerful tool to realize accurate gates is quantum optimal control [2], already applied, for example, to quantum computation with cold atoms in an optical lattice [3]. The aim of the present work is to apply optimal control to the realm of solid-state quantum computation, more specifically to qubits realized with superconducting nanocircuits. Josephson-junction qubits [4,5] are considered among the most promising candidates for implementing quantum protocols in solid-state devices. Because of their design flexibility, several different versions of superconducting (charge, flux, phase) qubits have been theoretically proposed and experimentally realized in a series of beautiful experiments [6]. Several schemes for qubit coupling have also been proposed (see the reviews [4,5]). On the experimental side, coupled qubits have been realized in the charge [7,8] and in the phase [9] regimes where a CNOT, $\sqrt{\text{iSWAP}}$ gates have been implemented, respectively. In the experiment of Steffen *et al.* [9] the measured fidelity was of the order of 75% increasing up to 87% after accounting for measurement errors. Further improvements in the accuracy rely on achieving larger decoherence times. In the experiment of Yamamoto *et al.* [8] a direct determination of the fidelity from the data was not possible, but it has been

estimated to be ~80%. Advances in fabrication techniques will play a crucial role in achieving accurate quantum gates; however, as the thresholds for fault tolerant computation [10] are quite demanding, gate optimization is a powerful tool for a considerable improvement of their accuracy. A major open question is the resilience of optimized operations to imperfections affecting a real laboratory implementation, including leakage to states outside the Hilbert subspace employed for logical encoding, inaccurate realization of the desired pulse shape, and classical noise in the system. In this Letter we apply optimal quantum control to superconducting charge qubits (that we choose for illustration purposes). We analyze in detail the effect of noise and leakage, and we show that optimization keeps yielding a considerable improvement in gate fidelities even under such realistic conditions. In the context of superconducting charge qubits, it has been proposed to couple the qubits via a capacitance [7,11], an additional Josephson Junction (JJ) [12] or an inductance [13,14]. The two-bit gate is realized by an appropriate choice of pulses in the gate potentials. For the two cases of capacitive and JJ coupling we construct the optimal pulse shapes, thereby obtaining very high fidelities. For the case of capacitive coupling optimal control has been applied to superconducting qubits for the first time by Spörl *et al.* [15]. Here, we extend their results in two important aspects: First, we compare two different couplings in order to optimize the design. Second, we include the effect of $1/f$ charge noise, believed to be the main source of decoherence in these systems [16,17], and show that the optimal gates are robust against it. We further show that gate accuracy is maintained even under partially distorted pulse shapes.

Coupled Josephson qubits.—Josephson charge qubits, sketched in Fig. 1, are defined in the regime in which the Josephson coupling is much smaller than the charging energy. The single-qubit Hamiltonian (including also noncomputational states) is defined as [4,5]
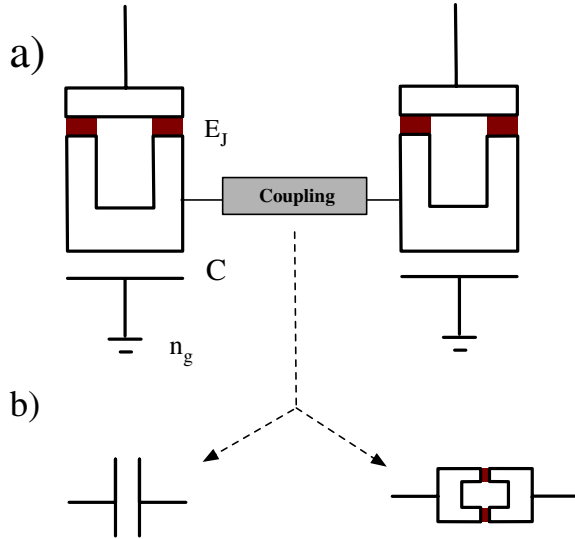
a)

b)

FIG. 1 (color online). (a) A Cooper pair box can implement a charge qubit when tuned in the regime in which only two charge states are relevant. The box between the qubits represents the coupling which is specified below. (b) An extra Josephson junction (right) or of a capacitance (left).

$$\mathcal{H}_i = \sum_{n_i} \left[E_C(n_i - n_g^{(i)})^2 |n_i\rangle\langle n_i| - \frac{E_J^{(i)}}{2}(|n_i\rangle\langle n_i+1| + \text{H.c.}) \right]$$

where n_i is the number of excess Cooper pairs on the *i*th ($i = 1, 2$) qubit, $n_g^{(i)} = C_g V_g^{(i)}/(2e)$ is the offset charge controlled by the gate voltage $V_g^{(i)}$ (C_g is the gate capacitance), E_C is the charging energy, and $E_J^{(i)}$ is the Josephson coupling. By projecting onto the Hilbert space spanned by the states $|0\rangle$, $|1\rangle$ ($D = 2$, D is the dimension of the Hilbert space) one recovers the charge qubit Hamiltonian. We want to include the effect of leakage to the charge states (in this case $D > 2$). Since we have $E_J/E_C \ll 1$, it is sufficient to add few other charge states. We included the charge states from $|-2\rangle$ to $|3\rangle$, i.e., $D = 6$. However, in the range $E_J/E_C \sim 5$–10×10^{-2} [7] we verified that retaining the charge states $|-1\rangle$, $|0\rangle$, $|1\rangle$, $|2\rangle$ is sufficient.

The coupling between the qubits [see Fig. 1(b)] can be either via a capacitor or a Josephson junction. In the case of capacitive coupling, Fig. 1(b) (left), the interaction Hamiltonian reads

$$\mathcal{H}_I^{cc} = E_{cc} \sum_{n_1, n_2} (n_1 - n_{g,1})(n_2 - n_{g,2})|n_1, n_2\rangle\langle n_1, n_2| \quad (1)$$

where E_{cc} is the charging energy associated to the Coulomb interaction between the qubits. If instead the coupling is via a Josephson junction, Fig. 1(b) (right), the coupling Hamiltonian is given by

$$\mathcal{H}_I^{JJ} = \frac{\tilde{E}_{JJ}}{2} \sum_{n_1, n_2} (|n_1\rangle|n_2+1\rangle\langle n_1+1|\langle n_2| + \text{H.c.}) \quad (2)$$

where $\tilde{E}_{JJ}$ is the Josephson energy of the coupling junction [18].

Two-qubit gates.—The goal is to implement the universal two-qubit gates G_{JJ} and G_{cc} for the JJ and capacitive couplings, respectively. They read

$$G_{JJ} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \pm i & 0 & 0 \\ 0 & 0 & \pm i & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad G_{cc} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
(3)

where we used the basis $\{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$ for G_{JJ} and the basis $\{|11\rangle, |10\rangle, |01\rangle, |00\rangle\}$ for G_{cc} ($|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$). Even under ideal operating conditions these gates cannot be implemented exactly [8,12]. As discussed in Ref. [12], G_{JJ} can be approximately realized by tuning both qubits to degeneracy, fixing all the Josephson couplings to be equal in magnitude and turning on the interaction for a time $\tau_{JJ} \simeq 0.972\pi/\tilde{E}_{JJ}$. For G_{cc} we choose the same parameters of the experiment [8] ($E_J/E_C^{(1)} \simeq 0.0777$, $E_J/E_C^{(2)} \simeq 0.0610$, $E_{cc}/E_C^{(1)} \simeq 0.1653$). The time needed for the gate is $\tau_{cc} \simeq 1.18\pi/E_J^{(1)}$. Defining U_τ^α ($\alpha = JJ, cc$) as the time evolution operator associated to the *full* Hamiltonian $\mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_I^\alpha$, a figure of merit to quantify the accuracy of a quantum gate is the error defined as

$$\varepsilon = 1 - \text{Tr}(G_\alpha^\dagger \tilde{U}_\tau^\alpha). \quad (4)$$

The $\tilde{U}$ is the time evolution operator projected onto the computational states (the fidelity of the operation should be tested only on the computational basis $|n_1, n_2\rangle = |00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$). The fidelity is defined as $\mathcal{F} \equiv 1 - \varepsilon$. In the following we determine optimized fidelities, up to a global phase for the gates in Eq. (3) when implemented with charge qubits. In order to be as close as possible to the experimental situation, we search for optimal pulses with the constraint that after the gates the two qubits are in their idle points.

Optimized quantum gates.—Quantum control techniques described in Refs. [2,3] allow us to minimize the error ε defined in Eq. (4). One assumes that the Hamiltonian is controlled by a set of external parameters which can be varied in time. The goal is to find the time dependence of the parameters that minimizes ε. To illustrate it in a little more detail, let us imagine a system governed by the time-dependent Hamiltonian $\mathcal{H}[g(t)]$, where $g(t)$ is the control parameter. The goal of a quantum optimal control algorithm in general is to reach, in a certain time τ, a desired target state $|\psi_T\rangle$ with high fidelity. The algorithm employed here, due to Krotov [2], works as follows: (i) an initial guess $g_0(t)$ is chosen for the control parameter; (ii) the initial state $|\psi_0\rangle$ is evolved in time according to the dynamics dictated by $\mathcal{H}[g(t)]$ until time τ: $|\psi_g(\tau)\rangle = U_\tau(g)|\psi_0\rangle$; (iii) an auxiliary state $|\chi_g(\tau)\rangle \equiv |\psi_T\rangle\langle\psi_T|\psi_E(\tau)\rangle$ is defined, which can be inter-

preted as the part of $|\psi_g(\tau)\rangle$ that has reached the target $|\psi_T\rangle$ and the auxiliary state is evolved backwards in time until $t = 0$; (iv) $|\chi_g(t)\rangle$ and $|\psi_g(t)\rangle$ are propagated again forward in time, while the control parameter is updated with the rule $g(t) \to g(t) + \text{Im}[\langle\chi_g(t)|\partial_g H|\psi_g(t)\rangle]/\lambda(t)$ and the weight function $\lambda(t)$ constrains the initial and final values of the control parameter; (v) steps (iii) and (iv) are repeated until the desired value of the fidelity is obtained. The same procedure can be followed also when the Hamiltonian contains more than one parameter. After a sufficient number of iterations, the algorithm converges and reaches asymptotically a minimum $\varepsilon_{\min}$. In the present case, we consider $E_J^{(i)}$, $n_{g,i}$, E_α as control parameters (Josephson couplings can be tuned by means of an applied magnetic flux), and we look for optimal pulse shapes to improve the fidelity $\mathcal{F}_\alpha$. Although in principle one may consider all the different couplings independently, this is impractical for an experimental point of view. In the case of JJ coupling we keep the gate voltage fixed and consider the same time dependence for all the Josephson couplings. This type of control can be achieved by applying a *uniform* time-dependent magnetic field. In the case of capacitive coupling we allow for time-dependent gates but keep the Josephson couplings fixed. Relaxing these constraints will certainly lead to a further optimization of the fidelity at the cost, however, of a more complex external control. The important point is that already at the level discussed in this work the improvement in the gate performances allows for crossing the fault tolerance threshold [10].

The presence of leakage may be disruptive for two-bit gates in Josephson charge qubits [19]. Optimization, however, fully compensates for leakage in both of the schemes depicted in Fig. 1. In the case of JJ coupling, Fig. 2 (left panel), we have only one control parameter, the Josephson coupling energy [$E_J^{(1)}(t) = E_J^{(2)}(t) = \tilde{E}_{JJ}(t)$]. The nonoptimized gate (open circles) is realized as described in Ref. [12] while the optimized curve, for the qubits of Ref. [8] ($E_J/E_C \sim 3 \times 10^{-2}$), gives an error of the order
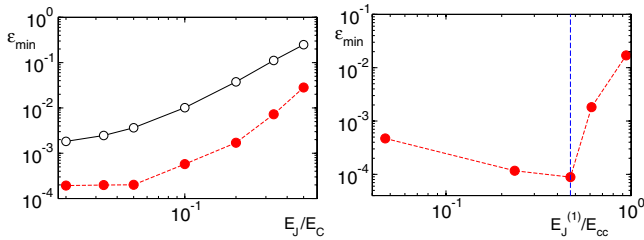
of 10^{-4}. This error is not appreciably influenced by the choice of the initial pulse, but rather it is physically determined by the constraints imposed on the pulse itself—for instance, requiring it to start and end at an optimal working point away from degeneracy, as we do here. In both cases we include leakage and the small effect of a finite charging energy E_{cc}. In the case of capacitive coupling, we build on the results obtained in Ref. [15] and use their pulse sequence as the initial guess. Thus we present here only the optimized gate. Our results are shown in Fig. 2 (right panel). In this setup, which coincides with that of the experiment of Ref. [8] the coupling E_{cc} cannot be changed. The values of the parameters $E_J^{(i)}/E_C^{(i)}$, $E_J^{(i)}/E_{cc}$ ($i = 1, 2$), and τ_{cc} should be chosen properly in order to realize the gate G_{cc}. Consistently, if $E_J^{(i)}/E_{cc}$ is changed by a given factor, τ_{cc} should be divided by the same factor. For the experimental value of $E_J^{(i)}/E_{cc}$, the error is of the order $\varepsilon_{\min} \simeq 10^{-4}$. Note that increasing $E_J^{(i)}/E_{cc}$ results in a faster gate, thus reducing the effect of decoherence. Here, in the best case, we can reduce the gate time to $\tau \sim 30$ ps, while keeping the fidelity constant.

An important question to be addressed is to what extent our optimized gates are robust against noise. For this reason we check how stable the fidelity (optimized in the absence of noise) is when the environment is taken into account [20]. The most important source of decoherence in charge qubits is $1/f$ charge noise [21]. Although its understanding is far from complete, $1/f$ noise is believed to originate from two-level fluctuators present in the substrate and/or in the insulating barrier. Several theoretical works have recently studied the relation between $1/f$ noise and decoherence in charge qubits (see Refs. [16,17] and references therein). Here we follow the approach of Ref. [17] and model the environment as a superposition of bistable classical fluctuators resulting in an additional random contribution $\delta n_g^{(i)}(t)$ to the gate charge. A distribution of switch rates γ behaving as $P(\gamma) \propto 1/\gamma$ in a range $[\gamma_{\min}:\gamma_{\max}]$ results in a noise power spectrum $S_{n_g}(\omega) = \langle\delta n_g^{(i)}(t)\delta n_g^{(i)}(0)\rangle_\omega \sim \omega^{-1}$. Following [17] we chose the switching rates such that the typical frequency of the gates
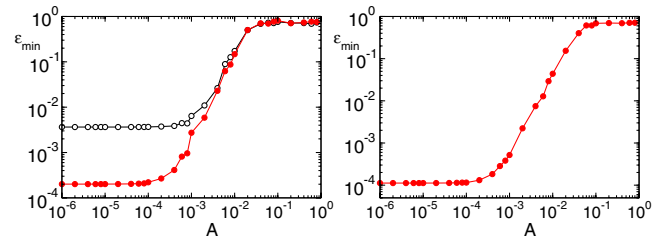

FIG. 2 (color online). Error $\varepsilon_{\min}$ as a function of leakage for two-qubit gates: for Josephson-junction coupling (including a residual capacitive coupling with $E_{cc}/\tilde{E}_{JJ} = 0.05$), with and without optimization (lower and upper curve, respectively) (left panel); for capacitive coupling, with optimization, as a function of the ratio $E_J^{(1)}/E_{cc}$ (which we use here since the two charging energies in the experiment $E_C^{(i)}$ are different) (right panel). The experimental value $E_J^{(1)}/E_C = 0.47$ from Ref. [8] is marked.

FIG. 3 (color online). Optimized gate error $\varepsilon_{\min}$ under noise having power spectrum $S_{n_g}(\omega) = A/\omega$ as a function of noise strength A: for Josephson coupling with $E_J/E_C = 0.05$, with and without optimization (lower and upper curves, respectively) (left panel); for capacitive coupling with $E_J^{(1)}/E_{cc} = 0.47$ (right panel). Typical experimental value are around $A \sim 10^{-5}$.
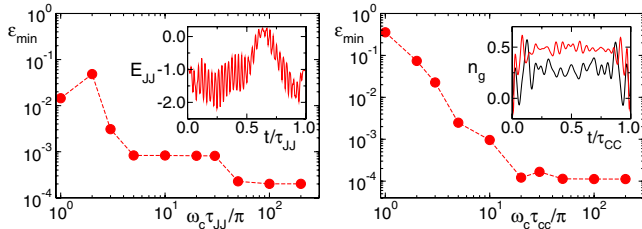
P H Y S I C A L   R E V I E W   L E T T E R S

FIG. 4 (color online).   Gate error $\varepsilon_{\min}$ as a function of the pulse spectral cutoff ω_c for Josephson coupling with $E_J/E_C = 1/20$ (left) and for capacitive coupling with $E_J^{(1)}/E_{cc} = 0.47$ (right). Insets: Corresponding optimal pulses.

is centered in between the 2 orders of magnitude over which the $1/f$ noise extends (We checked the stability of our results with the choice of $\gamma_{\min}$ and $\gamma_{\max}$; data not shown). We considered up to 1000 independent fluctuators coupled weakly to the qubits and we assumed that the charge noise on the two separate qubits is uncorrelated. The results of our analysis are reported in Fig. 3: regardless of the coupling scheme, the fidelity turns out to be quite robust against noise. Moreover, the error rates remain orders of magnitude better than without application of the quantum control algorithm, even under significant noise strengths, up to $A \sim 10^{-4}$–10^{-3}. We checked these results also with different kinds of noise (white noise, homogeneous frequency broadening in the control pulses) and we found similar conclusions (see also Ref. [15]). We finally investigated the dependence of the gate error on the experimentally unavoidable inaccuracies of the pulse shapes. To this end we applied a filter to suppress the contribution of harmonics above a cutoff ω_c in the shape of the optimal pulses. In Fig. 4 we show the dependence of the error on the number of frequencies that compose the optimized pulses. In both cases the most important corrections are those at lower frequencies, as already pointed out in Ref. [15]. This explains the robustness of both optimized gates against noise processes: the fidelity is just marginally influenced by new frequencies introduced by the noise. Although the realization of (nearly) optimal pulses is demanding, it definitely leads to accurate gate operation. One can then imagine realizing the two-bit gates in a longer time, in which case the shape of the pulse should be easier to realize. On the other hand, if the gate is too slow decoherence becomes relevant. It is then important to find an optimal gate time for which these two competing effects are minimized. We believe that this may be an avenue to realizing high-fidelity computations with Josephson nanocircuits.

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