Exercise 1  Landau levels

The energy spectrum $E$ of an electron of mass $m$ and charge $-e$ in a constant homogeneous magnetic field $\vec{B} \equiv \{0,0,B\}$, directed along the $z$-axis, is determined by solving the stationary Schrödinger equation

$$\frac{1}{2m} \left[ \hat{\vec{p}} - e\vec{A}(\vec{r}) \right]^2 \psi_E(\vec{r}) = E\psi_E(\vec{r})$$

for the wave function $\psi_E(\vec{r})$. Here $\hat{\vec{p}} = -i\hbar \frac{\partial}{\partial \vec{r}}$ is the momentum operator and $\vec{A}(\vec{r})$ is the vector potential, that is $\vec{B} \equiv \nabla \times \vec{A}$.

a) By taking the vector potential in the form

$$\vec{A}_1 = \{-By, 0,0\} ,$$

such as $\nabla \times \vec{A}_1 = \{0,0,B\}$ with div$\vec{A}_1 = 0$ (the Coulomb gauge), find the solutions $\psi_E$ of Eq. (1) and the corresponding energies $E$ in terms of the wave functions and spectrum of the linear harmonic oscillator.

(3 points)

b) Solve the classical equations of motion governed by the corresponding Hamiltonian function

$$\mathcal{H}(\vec{r}, \vec{p}) \equiv \frac{1}{2m} \left[ \vec{p} - e\vec{A}_1(\vec{r}) \right]^2$$

with the initial condition given by $\vec{r}(t = 0) = \vec{r}_0$ and $\vec{p}(t = 0) = \vec{p}_0$.

(3 points)

c) Proof that the vector potential $\vec{A}_2 \equiv \{0,Bx,0\}$ describes the same constant homogeneous magnetic field $\vec{B} \equiv \{0,0,B\}$ with div$\vec{A}_2 = 0$ (the Coulomb gauge) and find the gauge transformation function $\Lambda(\vec{r})$, such as

$$\vec{A}_2 = \vec{A}_1 + \nabla \Lambda .$$

(1 point)

d) What is the relation between the wave functions $\psi_{E}^{(1,2)}(\vec{r})$, which are solutions of Eq. (1) for the same energy $E$ with the vector potentials $\vec{A}_1$ and $\vec{A}_2$, accordingly?

(1 point)