Theoretical Quantum Optics

Sheet 1 **SS 2017**
Date of issue: 23-04-2017 Discussion: 28-04-2017

Exercise 1 *Landau levels*

The energy spectrum E of an electron of mass m and charge $-e$ in a constant homogeneous magnetic field $\vec{B} \equiv \{0, 0, B\}$, directed along the z-axis, is determined by solving the stationary Schrödinger equation

$$\frac{1}{2m}\left[\hat{\vec{p}} - e\vec{A}\,(\vec{r})\right]^2 \psi_E(\vec{r}) = E\psi_E(\vec{r}) \tag{1}$$

for the wave function $\psi_E(\vec{r})$. Here $\hat{\vec{p}} = -i\hbar\frac{\partial}{\partial\vec{r}}$ is the momentum operator and $\vec{A}(\vec{r})$ is the vector potential, that is $\vec{B} \equiv \nabla \times \vec{A}$.

a) By taking the vector potential in the form

$$\vec{A}_1 = \{-By, 0, 0\}\ ,$$

such as $\nabla \times \vec{A}_1 = \{0, 0, B\}$ with $\mathrm{div}\vec{A}_1 = 0$ (the Coulomb gauge), find the solutions ψ_E of Eq. (1) and the corresponding energies E in terms of the wave functions and spectrum of the linear harmonic oscillator.

(3 points)

b) Solve the classical equations of motion governed by the corresponding Hamiltonian function

$$\mathcal{H}\,(\vec{r}, \vec{p}) \equiv \frac{1}{2m}\left[\vec{p} - e\vec{A}_1\,(\vec{r})\right]^2 \tag{2}$$

with the initial condition given by $\vec{r}(t = 0) = \vec{r}_0$ and $\vec{p}(t = 0) = \vec{p}_0$.

(3 points)

c) Proof that the vector potential $\vec{A}_2 \equiv \{0, Bx, 0\}$ describes the same constant homogeneous magnetic field $\vec{B} \equiv \{0, 0, B\}$ with $\mathrm{div}\vec{A}_2 = 0$ (the Coulomb gauge) and find the gauge transformation function $\Lambda(\vec{r})$, such as

$$\vec{A}_2 = \vec{A}_1 + \nabla\Lambda\,.$$

(1 point)

d) What is the relation between the wave functions $\psi_E^{(1,2)}(\vec{r})$, which are solutions of Eq. (1) for the same energy E with the vector potentials $\vec{A}_1$ and $\vec{A}_2$, accordingly?

(1 point)