Theoretical Quantum Optics

Sheet 2 **SS 2017**
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Exercise 2 *Electron in a time-dependent magnetic field*

Let us consider the dynamics of an electron in a time-dependent magnetic field $\vec{B}(t)$, described by the Schrödinger equation with the Hamiltonian

$$\hat{H} \;=\; -\mu_B \hat{\vec{\sigma}} \cdot \vec{B}(t) \; .$$

Here $\mu_B \hat{\vec{\sigma}}$ is the magnetic moment operator with $\mu_B = \frac{e\hbar}{2m_e}$ (the Bohr magneton) and

$$\hat{\vec{\sigma}} \;\equiv\; \hat{\sigma}_x \vec{e}_x + \hat{\sigma}_y \vec{e}_y + \hat{\sigma}_z \vec{e}_z \; ,$$

with $\hat{\sigma}_x$, $\hat{\sigma}_y$ and $\hat{\sigma}_z$ being the conventional Pauli matrices having the form

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

We define the spin-up state $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as the eigenvector of $\hat{\sigma}_z$, that is $\hat{\sigma}_z |\uparrow\rangle = |\uparrow\rangle$, corresponding to the eigenvalue $+1$. The spin-down state $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is defined as the eigenvector of $\hat{\sigma}_z$, corresponding to the eigenvalue -1.

a) Solve the Schrödinger equation

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\psi\rangle \;=\; \hat{H} |\psi\rangle$$

for an electron in the time-dependent magnetic field given by

$$\vec{B}(t) \;=\; B_0 \begin{pmatrix} \sin\theta \cos(\omega t) \\ \sin\theta \sin(\omega t) \\ \cos\theta \end{pmatrix} \quad \text{with } 0 \le \theta \le \pi,$$

where ω is the rotation frequency of the magnetic field around the z-axis; θ is the angle between $\vec{B}$ and the z-axis.

(3 points)

b) Calculate the time-dependent probability $w_\uparrow(t)$ to find the electron in the spin-up state $|\uparrow\rangle$, if it is initially in the spin-down state, that is $|\psi(t=0)\rangle = |\downarrow\rangle$.

(1 point)

Exercise 3 *Electron in a constant magnetic field*

Here we study the same system as in the Exercise 2, but for the case $\omega = 0$, when the magnetic field $\vec{B}$ is constant and has the form

$$\vec{B} = \begin{pmatrix} B_x \\ 0 \\ B_z \end{pmatrix} = \begin{pmatrix} B_0 \sin\theta \\ 0 \\ B_0 \cos\theta \end{pmatrix} .$$

The general solution

$$|\psi(t)\rangle \quad = \quad \hat{U}(t)\,|\psi(0)\rangle$$

of the Schrödinger equation can be presented in terms of the time-evolution operator

$$\hat{U}(t) \quad = \quad e^{-\frac{i}{\hbar}\hat{H}t} = e^{it(\omega_x\hat{\sigma}_x + \omega_z\hat{\sigma}_z)}$$

with $\omega_x \equiv \frac{\mu_B B_x}{\hbar}$ and $\omega_z \equiv \frac{\mu_B B_z}{\hbar}$.

a) Using the anticommutator for the Pauli matrices

$$\{\hat{\sigma}_i, \hat{\sigma}_j\} \quad = \quad 2\delta_{ij}\mathbb{1}_2 \,,$$

show, that the identity

$$e^{i\alpha\vec{n}\cdot\hat{\vec{\sigma}}} \quad = \quad \mathbb{1}_2 \cos\alpha + i\vec{n}\cdot\hat{\vec{\sigma}}\sin\alpha \,, \tag{1}$$

is valid for any parameter α and the three-dimensional unit vector $\vec{n}$.

(3 points)

b) With the help of Eq. (1), recast the time-evolution operator $\hat{U}(t)$ as a linear superposition of the Pauli matrices and the identity matrix $\mathbb{1}_2$.

(2 points)

c) Find $|\psi(t)\rangle$ for the initial condition $|\psi(0)\rangle = |{\downarrow}\rangle$ and determine the time-dependent probability $w_{\uparrow}(t)$ to find the system in the spin-up state $|{\uparrow}\rangle$. Compare your result to those obtained in the Exercise 2.

(2 points)

d) Calculate

$$\vec{\mu}(t) \quad = \quad \langle\psi(t)|\,\mu_B\hat{\vec{\sigma}}\,|\psi(t)\rangle$$

and interpret your result.

(4 points)