Theoretical Quantum Optics

Sheet 3 **SS 2017**

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Exercise 4 *The selection rules (electric dipole transitions)*

The quantum state $|n, l, m_l\rangle$ of an electron in a hydrogen atom is characterised by the values of the principal quantum number n ($n = 1, 2, \ldots$), the angular momentum quantum number l ($l = 0, 1, \ldots, n-1$) and the corresponding magnetic quantum number m_l ($m_l = -l, -l+1, \ldots, l-1, l$). The level scheme for the states with $n = 2$ is shown in Fig. 1, with the quantization axis directed along the z-axis.

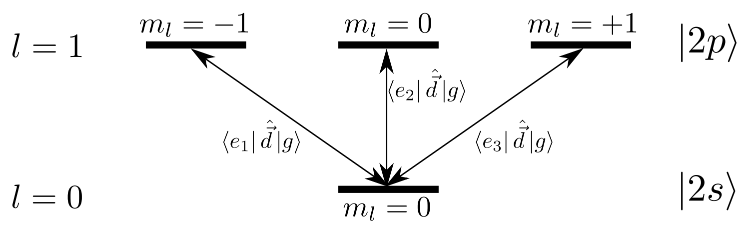

Figure 1: Level scheme of a hydrogen atom for $n = 2$.

The coordinate representation of these states reads

$$\psi_g(\vec{r}) = \langle \vec{r}|g\rangle = \langle \vec{r}|2,0,0\rangle = R_{2,0}(r) Y_{0,0}(\theta,\varphi) \;,$$

$$\psi_{e_1}(\vec{r}) = \langle \vec{r}|e_1\rangle = \langle \vec{r}|2,1,-1\rangle = R_{2,1}(r) Y_{1,-1}(\theta,\varphi) \;,$$

$$\psi_{e_2}(\vec{r}) = \langle \vec{r}|e_2\rangle = \langle \vec{r}|2,1,0\rangle = R_{2,1}(r) Y_{1,0}(\theta,\varphi) \;,$$

$$\psi_{e_3}(\vec{r}) = \langle \vec{r}|e_3\rangle = \langle \vec{r}|2,1,+1\rangle = R_{2,1}(r) Y_{1,+1}(\theta,\varphi) \;.$$

The radial part $R_{n,l}(r)$ of the electronic wave function is normalised as

$$\int_0^\infty \mathrm{d}r\, r^2 R_{nl}^2(r) = 1$$

and given by

$$\begin{aligned} R_{2,0}(r) &= \frac{1}{\sqrt{2a_B^3}}\left(1 - \frac{r}{2a_B}\right) e^{-\frac{r}{2a_B}} \;, \\ R_{2,1}(r) &= \frac{1}{2\sqrt{6a_B^3}} \frac{r}{a_B} e^{-\frac{r}{2a_B}} \;, \end{aligned}$$

where a_B denotes the Bohr radius.

The angular part of the wave function is given by the spherical harmonics $Y_{l,m_l}(\theta,\varphi)$, which are normalised and orthogonal to each other, that is

$$\int_0^\pi \mathrm{d}\theta \, \sin\theta \int_0^{2\pi} \mathrm{d}\varphi \, Y^*_{l,m_l}(\theta,\varphi) Y_{l',m_l'}(\theta,\varphi) \quad = \quad \delta_{l,l'}\delta_{m_l,m_l'} \; .$$

In particular, for the states under consideration

$$\begin{aligned} Y_{0,0}(\theta,\varphi) &= \sqrt{\frac{1}{4\pi}} \; , \\ Y_{1,0}(\theta,\varphi) &= i\sqrt{\frac{3}{4\pi}} \cos\theta \; , \\ Y_{1,\pm 1}(\theta,\varphi) &= \mp i\sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\varphi} \; . \end{aligned}$$

In the lecture we have defined the dipole matrix element for the transition between the ground $|g\rangle$ and the exited $|e\rangle$ states as

$$\vec{d}_{eg} \quad \equiv \langle e|\,\hat{\vec{d}}\,|g\rangle \equiv \textstyle\iiint \mathrm{d}\vec{r}\, \psi^*_e(\vec{r})(e\vec{r})\psi_g(\vec{r}) \; .$$

a) Calculate the dipole matrix elements

- $\vec{d}_{1g} = \langle e_1|\,\hat{\vec{d}}\,|g\rangle$ for $|e_1\rangle = |2,1,-1\rangle$
- $\vec{d}_{2g} = \langle e_2|\,\hat{\vec{d}}\,|g\rangle$ for $|e_2\rangle = |2,1,0\rangle$
- $\vec{d}_{3g} = \langle e_3|\,\hat{\vec{d}}\,|g\rangle$ for $|e_3\rangle = |2,1,+1\rangle$

for the transition between the ground state $|g\rangle = |2,0,0\rangle$ and a given exited state $|e_i\rangle$ with $i = 1,2,3$ (see Fig. 1).

(3 points)

b) What is the corresponding polarization of the electric field $\vec{E}$ needed to induce each transition?

(1 point)