Exercise 7  \hspace{0.5cm} \textit{Population and coherence in terms of the density matrix}

We consider the density matrix $\hat{\rho}$ and an orthogonal basis $|u_n\rangle$ ($n = 1, 2, ...$) of the state space.

a) By using the Cauchy-Schwarz inequality, prove the relation

$$\rho_{nm} \rho_{mm} \geq |\rho_{nm}|^2 \quad (1)$$

for $\rho_{nm} \equiv \langle u_n | \hat{\rho} | u_m \rangle$. Inequality (1) reveals that for a density matrix $\hat{\rho}$, the coherence ($\rho_{nm} \neq 0$ for $n \neq m$) can only occur between the states $|u_n\rangle$ and $|u_m\rangle$ whose populations ($\rho_{nn}$ and $\rho_{mm}$) are non-zero.

(1 point)

b) Moreover, prove that Inequality (1) results in the criteria

$$\text{Tr} \rho^2 \leq 1$$

for the system to be in a pure ($\text{Tr} \rho^2 = 1$) or in a mixed ($\text{Tr} \rho^2 < 1$) state.

(1 point)

Exercise 8  \hspace{0.5cm} \textit{Two spin-1/2 particles and Werner states}

We consider a system of two spin-1/2 particles. A basis of this bipartite system is given, for example, in terms of the Bell states

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}} (|\downarrow\rangle_1 |\downarrow\rangle_2 + |\uparrow\rangle_1 |\uparrow\rangle_2) \quad ,$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}} (|\downarrow\rangle_1 |\downarrow\rangle_2 - |\uparrow\rangle_1 |\uparrow\rangle_2) \quad ,$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}} (|\downarrow\rangle_1 |\uparrow\rangle_2 + |\uparrow\rangle_1 |\downarrow\rangle_2) \quad ,$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}} (|\downarrow\rangle_1 |\uparrow\rangle_2 - |\uparrow\rangle_1 |\downarrow\rangle_2) \quad ,$$

where the spin-up state $|\uparrow\rangle_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the spin-down state $|\downarrow\rangle_i = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ form a basis of the state space for a single spin-1/2 particle $i \ (i = 1, 2)$. 
a) Find the two-particle states corresponding to the total spin \( S = 1 \) (triplet) and the two-particle state corresponding to the total spin \( S = 0 \) (singlet) in terms of the Bell states.

(1 point)

b) Present the Bell states as matrix by evaluating the tensor products, for example

\[
|\downarrow\rangle_1 |\downarrow\rangle_2 = |\downarrow\rangle_1 \otimes |\downarrow\rangle_2.
\]

(1 point)

c) We introduce a family of the Werner states

\[
\hat{\rho}(\alpha) = \alpha |\Phi^+\rangle \langle \Phi^+| + \frac{1}{4} (1 - \alpha) \mathbb{I}_4
\]

characterized by a real parameter \( \alpha \). Represent the Werner states as a 4x4-matrix in the basis \( \{|\uparrow\rangle_1, |\uparrow\rangle_2, |\downarrow\rangle_1, |\downarrow\rangle_2\} \).

(1 point)

d) Find the conditions for \( \alpha \), such that \( \hat{\rho}(\alpha) \) is a density matrix.

(2 points)

e) Calculate the parameter \( \alpha \) for which the Werner state describes a pure state.

(1 point)

f) Obtain the reduced density matrices \( \hat{\rho}_1 \) and \( \hat{\rho}_2 \) for the single particle states. Find the values of \( \alpha \), such that either \( \hat{\rho}_1 \) or \( \hat{\rho}_2 \) are not in a mixed state.

(2 points)

Exercise 9  

**Density matrix for a continuous basis**

A bipartite system, consisting of the two subsystems \( A \) and \( B \), is prepared in a pure state and characterized by the wave function \( \psi(\alpha, \beta) \), where \( \alpha \) and \( \beta \) are continuous variables of the subsystems \( A \) and \( B \), accordingly. We introduce the reduced density matrix

\[
\rho_A(\alpha, \alpha') = \int d\beta \psi^*(\alpha, \beta)\psi(\alpha', \beta)
\]

to describe only the subsystem \( A \).

a) Find the condition for the function \( \psi(\alpha, \beta) \), such that

- \( \text{Tr}(\rho_A) = 1 \) is fulfilled.
- the reduced density matrix \( \rho_A(\alpha, \alpha') \) describes a pure state.

(2 points)
b) We define the mean value $\langle \hat{f}_A \rangle$ of the operator $\hat{f}_A$, which only acts on the subsystem $A$, as

$$\langle \hat{f}_A \rangle \equiv \langle \psi | \hat{f}_A | \psi \rangle .$$

Represent $\langle \hat{f}_A \rangle$ in terms of $\rho_A(\alpha, \alpha')$. (1 point)

c) Write down the dynamical equation for $\rho_A(\alpha, \alpha', t)$. (2 points)