Theoretical Quantum Optics

Sheet 6 **SS 2017**
Date of issue: 26-05-2017 Discussion: 09-06-2017

Exercise 10 *The harmonic oscillator in the thermal state*

The density operator describing a harmonic oscillator with the frequency ω, which is in thermal equilibrium with a heat bath of temperature T, reads

$$\hat{\rho}_{HO} = \sum_{n=0}^{\infty} w_n |n\rangle\langle n| \, , \; w_n = N \exp\left(-\frac{E_n}{k_B T}\right) , \; E_n = \hbar\omega\left(n + \frac{1}{2}\right) .$$

a) Find the normalization constant N from the condition $\sum_{n=0}^{\infty} w_n = 1$.

(1 point)

b) Calculate $\mathrm{Tr}\hat{\rho}_{HO}^2$ and find the condition under which $\hat{\rho}_{HO}$ describes a pure state. Sketch $\mathrm{Tr}\hat{\rho}_{HO}^2$ as a function of the temperature T.

(2 points)

c) In the coordinate representation the eigenfunctions $\frac{1}{\sqrt{x_{HO}}}\psi_n(x/x_{HO}) \equiv \langle x|n\rangle$ of the harmonic oscillator are determined by the Hermite polynomials $H_n(\xi)$ as

$$\psi_n(\xi) = \frac{1}{(2^n\sqrt{\pi}n!)^{\frac{1}{2}}} \exp\left(-\frac{\xi^2}{2}\right) H_n(\xi) ,$$

where $x_{HO} = \sqrt{\frac{\hbar}{m\omega}}$ is the characteristic length scale of the harmonic oscillator. Find $\rho_{HO}(x, x') = \langle x| \hat{\rho}_{HO} |x'\rangle$ in the coordinate representation.

(3 points)

Hint: Use the Mehler's formula

$$\sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n \frac{1}{n!} H_n(x)H_n(y) = \frac{1}{\sqrt{1-z^2}} \exp\left\{\frac{2zxy - z^2(x^2+y^2)}{1-z^2}\right\} .$$

d) Calculate $\rho_{HO}(p, p') = \langle p| \hat{\rho}_{HO} |p'\rangle$ in the momentum representation.

(2 points)

e) Obtain the variances Δx^2 and Δp^2 and compare the obtained result with the Heisenberg uncertainty.

(4 points)