Exercise 14  \textit{The Dark state in a Λ-system}

Let us consider three-level system consisting of the two low-energy states \(|g_{1,2}\rangle\) and the high-energy state \(|e\rangle\), see Fig. 1.

![Diagram of atomic states in the Λ-type three-level system](image)

Figure 1: Scheme of atomic states in the Λ-type three-level system. Here \(\Omega_{1,2} = \frac{1}{\hbar} \left| \vec{d}_{eg_{1,2}} \vec{E}_{1,2}(t) \right|\) are the Rabi frequencies and \(\Delta_{1,2} = \omega_{eg_{1,2}} - \omega_{1,2}\) are the corresponding detunings.

a) Within the rotating-wave approximation, derive the Hamiltonian, which describes the time dynamics of the Λ-system, Fig. 1, interacting with the two electric fields

\[\vec{E}_{1,2} = \vec{E}_{1,2}(t) \cos(\omega_{1,2} t + \varphi_{1,2}).\]

For simplicity, we can take \(E_e = 0\) and the diagonal matrix elements \(\vec{d}_{k,k} = 0\) with \(k = e, g_{1},\) or \(g_{2}\).

(2 points)

b) By making the linear transformation, reduce the Schrödinger equation for the coefficients \(c_e, c_{g1}, c_{g2}\) of the state vector

\[|\psi(t)\rangle = c_e(t) |e\rangle + c_{g1} |g_{1}\rangle + c_{g2} |g_{2}\rangle\]

to the form

\[i \frac{d}{dt} \begin{pmatrix} \tilde{c}_e \\ \tilde{c}_{g1} \\ \tilde{c}_{g2} \end{pmatrix} = \hat{H}_\Lambda \begin{pmatrix} \tilde{c}_e \\ \tilde{c}_{g1} \\ \tilde{c}_{g2} \end{pmatrix},\]

where

\[\hat{H}_\Lambda = \begin{pmatrix} 0 & -\frac{1}{2} \Omega_1 & -\frac{1}{2} \Omega_2 \\ -\frac{1}{2} \Omega_1 & -\Delta_1 & 0 \\ -\frac{1}{2} \Omega_2 & 0 & -\Delta_2 \end{pmatrix}\]

with \(\Delta_{1,2} \equiv \Delta_{1,2} + \varphi_{1,2}\).
Find the condition under which the Hamiltonian $\hat{H}_\Lambda$ has an eigenvalue independent of $\Omega_{1,2}$ and obtain the corresponding eigenvector in terms of $|e\rangle$, $|g_1\rangle$, and $|g_2\rangle$ state vectors.

(3 points)