Introduction to Asymptotic Methods

Maxim Efremov
Institut für Quantenphysik, Universität Ulm

Sheet 3

Exercise 9
Show that there is one real root of equation

\[ 1 + \sqrt{x^2 + \varepsilon} = e^x \]

for the small and positive values of \( \varepsilon \). Find the two leading terms of the approximate solution of this equation. (2 Points)

Find the approximate solutions to the roots of equation

\[ 1 + (x^n + \varepsilon)^{\frac{1}{n}} = e^x \]

as \( \varepsilon \to 0 \) for \( n = 1, 2, 3, \ldots \). (3 Points)

Exercise 10
Let \( x \tan(x) = u \) be the transcendental equation for \( x \in \left(0, \frac{\pi}{2}\right) \) and \( u > 0 \). Obtain the coefficients \( a_n \) with \( n = 0, 1, 2, \ldots \) of the asymptotic expansion

\[ x(u) = a_0 + \frac{a_1}{u} + \frac{a_2}{u^2} + \frac{a_3}{u^3} + \mathcal{O}\left(\frac{1}{u^4}\right) \]

for the root \( x(u) \) of this equation as \( u \to +\infty \). (2 Points)

Exercise 11
The Lambert function \( W_0(x) \) is the solution of the equation \( W_0 e^{W_0} = x \) defined on the set \( x \in [-e^{-1}, +\infty) \), such as \( W_0(0) = 0 \) and \( W_0(x) \geq -1 \). Moreover, the function \( W_0(x) \) has the branch point at \( x = e^{-1} \). By using the Lagrange-Bürmann formula, find the coefficients \( a_n \) (with \( n = 1, 2, 3 \)) of the asymptotic expansion for the function \( W_0(x) \) as \( x \to -e^{-1} \)

\[ W_0(x) = -1 + a_1\xi + a_2\xi^2 + a_3\xi^3 + \ldots \]

with \( \xi = \sqrt{2(1 + ex)} \to 0 \). (3 Points)

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