THEORIE DER KONDENSIERTEN MATERIE:
Quantum mechanics on macroscopic scales

Präsenzübungen SS 2012, Prof. J. Ankerhold and Prof. M. Blencowe

Exercise 1: Harmonic oscillator in occupation number representation

The Hamiltonian of a harmonic oscillator can be expressed as

$$H = \frac{p^2}{2m} + \frac{m}{2}\omega^2 x^2$$

with position operator x and momentum operator p. Their commutation relation is given by $[x, p] = \mathrm{i}\hbar$. Creation and annihilation operators are defined as

$$a = \frac{x}{\sqrt{2\hbar/(m\omega)}} + \mathrm{i}\frac{p}{\sqrt{2\hbar m\omega}}\,, \qquad a^\dagger = \frac{x}{\sqrt{2\hbar/(m\omega)}} - \mathrm{i}\frac{p}{\sqrt{2\hbar m\omega}}\,.$$

a) Show that $H = (a^\dagger a + a a^\dagger)\frac{\hbar\omega}{2}\,.$

b) Find the commutator of a and $a^\dagger$ and show that

$$H = (a^\dagger a + \frac{1}{2})\hbar\omega\,.$$

c) Find the Heisenberg equations of motion for the operators a and $a^\dagger$.

d) The ket $|n\rangle$ denotes an eigenstate of the occupation number operator $\hat{n} = a^\dagger a$ with $\hat{n}|n\rangle = n|n\rangle$. Use the commutator $[a^\dagger, \hat{n}]$ to show that $a^\dagger|n\rangle \propto |n+1\rangle$.

e) Use normalization, $\langle n|n\rangle = 1$ (for all n), and the fact that a annihilates the ground state, $a|0\rangle = 0$, to express excited states as

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle\,.$$

Exercise 2: Free Fermions

a) Use the antisymmetry properties of the state $|n_1, \ldots, n_r, \ldots . n_{r'}, \ldots\rangle$ to show that creation operators anticommute, $[a_r^\dagger, a_{r'}^\dagger]_+ = 0$.

b) In the same manner find $[a_r, a_{r'}^\dagger]_+$.

Exercise 3: Shifted Fermi-Oscillator

Consider the so-called shifted Fermi-oscillator with Hamiltonian,

$$H = \frac{\hbar\omega}{2}\left(a^\dagger a - a a^\dagger\right) - \hbar\gamma(a^\dagger + a)\,, \qquad (1)$$

where a and $a^\dagger$ are fermionic operators.

Find the stationary Schrödinger-equation und solve with the ansatz $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$ (why is this the most general ansatz?). Determine E, c_0 und c_1.

Hint: You will find two coupled equations for c_0 and c_1, which have a non-trivial solution, if the determinant of the coefficient matrix vanishes.